Projection of a Vector upon a Plane from an Arbitrary Angle, via Geometric (Clifford) Algebra

December 19, 2017

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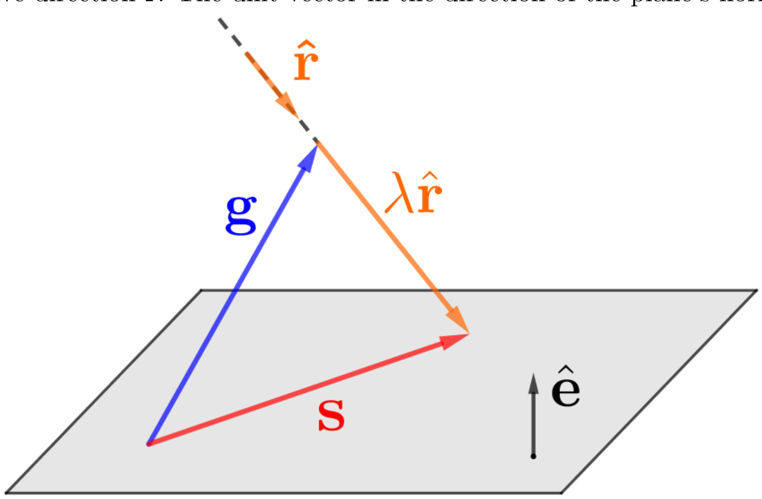
Abstract

We show how to calculate the projection of a vector, from an arbitrary direction, upon a given plane whose orientation is characterized by its normal vector, and by a bivector to which the plane is parallel. The resulting solutions are tested by means of an interactive GeoGebra construction.

Vector $\mathbf{s}$ is the “shadow” of vector $\mathbf{g}$ cast upon the plane by “rays of the Sun” that have direction $\hat{\mathbf{r}}$. The unit vector in the direction of the plane's normal is



$\hat{\mathbf{e}}$.

“Calculate the vector $\mathbf{s}$*, which is the “shadow” of vector* $\mathbf{g}$ *cast upon the plane by “rays of the Sun” that have direction* $\hat{\mathbf{r}}$*. The unit vector in the direction of the plane's normal is* $\hat{\mathbf{e}}$*.”*

Contents

1 Introduction

In this document, we will solve—numerically as well as symbolically—a problem of a type that can take the following concrete form, with reference to Fig.1:

> "A pole (not necessarily vertical) casts a shadow onto the perfectly flat plaza into which it is set. With respect to a right-handed orthonormal reference frame with basis vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$, the direction of the Sun's rays is $\hat{\mathbf{r}} = \hat{\mathbf{a}}r_a + \hat{\mathbf{b}}r_b + \hat{\mathbf{c}}r_c$. The vector $\mathbf{g}$ from the pole's base to the pole's tip, is $\mathbf{g} = \hat{\mathbf{a}}g_a + \hat{\mathbf{b}}g_b + \hat{\mathbf{c}}g_c$, and the upward-pointing unit vector normal to the plane is $\hat{\mathbf{e}} = \hat{\mathbf{a}}e_a + \hat{\mathbf{b}}e_b + \hat{\mathbf{c}}e_c$. Calculate $\mathbf{s}$, the vector from the base of the pole to the tip of the pole's shadow."

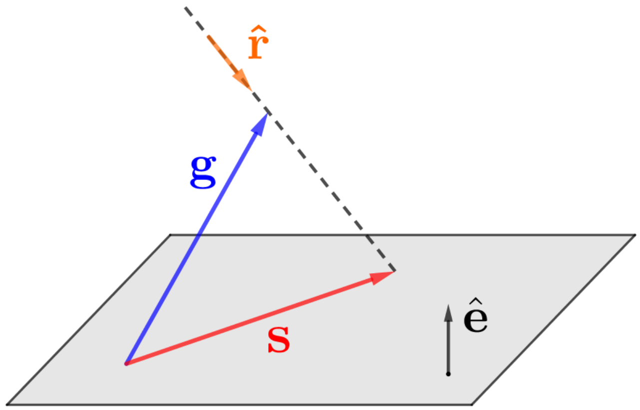


Figure 1: Vector $\mathbf{s}$ is the "shadow" of vector $\mathbf{g}$ cast upon the plane by "rays of the Sun" that have direction $\hat{\mathbf{r}}$. The unit vector in the direction of the plane's normal is $\hat{\mathbf{e}}$.

2 Formulating the Problem in Geometric-Algebra (GA) Terms, and Devising a Solution Strategy

2.1 Initial Observations

Let's begin by making a few observations that might be useful:

1. By saying "the direction of the Sun's rays is $\hat{\mathbf{r}} = \hat{\mathbf{a}}r_a + \hat{\mathbf{b}}r_b + \hat{\mathbf{c}}r_c$", we assumed that all of the Sun's rays are parallel. We'll use that assumption throughout this document.
2. The tip of the shadow is at the point where a ray that just grazes the tip of the pole intersects the surface of the plaza.
3. Therefore, the vector from the tip of the pole to the tip of the shadow is some scalar multiple of $\hat{\mathbf{r}}$. We'll call that scalar multiple $\lambda\hat{\mathbf{r}}$, and add it to our earlier diagram to produce Fig. 2.
4. From Fig. 2, we can see that $\mathbf{s} = \mathbf{g} + \lambda\hat{\mathbf{r}}$.

2.2 Recalling What We've Learned from Solving Similar Problems Via GA

Let's also refresh our memory about techniques that we may have used to solve other problems via GA:

1. Problems involving projections onto a plane are usually solved by using the appropriately-oriented bivector that is parallel to the plane, rather than

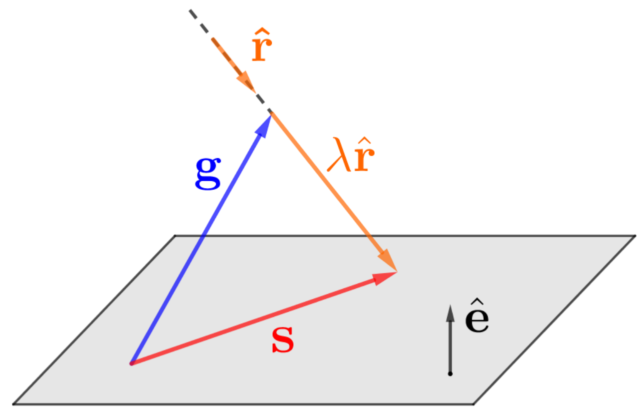


Figure 2: The same situation as in Fig. 1, but noting that the vector from the tip of $\mathbf{g}$ to the tip of $\mathbf{s}$ is a scalar multiple ("λ") of $\hat{\mathbf{r}}$.

by using the vector that is perpendicular to it. The Appendix (Section 5) shows how to find the required bivector, given said vector.

2. In a GA equation with two unknowns, such as the equation $\mathbf{s} = \mathbf{g} + \lambda\hat{\mathbf{r}}$ at the end of the preceding list, a common strategy is to eliminate one of the unknowns by using either the "dot" product or the 'wedge" product ("$\wedge$") with a known quantity. Examples of this strategy are given in Ref. [2], and in Ref. [3], pp. 39-47.

2.3 Further Observations, and Identifying a Strategy

Guided by Sections 2.1 and 2.2, we might realize that the vector $\mathbf{s}$ is perpendicular to $\hat{\mathbf{e}}$. Thus, one method of solving the equation $\mathbf{s} = \mathbf{g} + \lambda\hat{\mathbf{r}}$ is to eliminate $\mathbf{s}$ by "dotting" both sides with $\hat{\mathbf{e}}$, thereby obtaining an equation that from which we can obtain an expression for λ in terms of $\mathbf{g}$, $\hat{\mathbf{e}}$, and $\hat{\mathbf{r}}$. That expression can then be substituted for λ in the original eqation ($\mathbf{s} = \mathbf{g} + \lambda\hat{\mathbf{r}}$) to find $\mathbf{s}$.

The same observations that led us to the first strategy also lead us to see that $\mathbf{s}$ is parallel to the plane of the plaza. Therefore, $\mathbf{s}$'s product "$\wedge$" with the bivector that's parallel to that plane is zero. That is, if we denote said bivector by the symbol "$\mathbf{T}$", then $\mathbf{s} \wedge \mathbf{T} = 0$. Using this observation, we also arrive at an equation for λ—and thus for $\mathbf{s}$—but this time in terms of $\mathbf{g}$, $\hat{\mathbf{r}}$, and $\mathbf{T}$.

We'll use both approaches in this document.

3 Solutions for s

We'll begin with the solution that uses the normal vector $\hat{\mathbf{e}}$.

3.1 Solution via the Inner Product with $\hat{\mathbf{e}}$

Taking up the first of the solution strategies that we identified in Section 2.3, we write

$$\begin{aligned} \mathbf{s} &= \mathbf{g} + \lambda\hat{\mathbf{r}}; \\ \underbrace{\mathbf{s}\cdot\hat{\mathbf{e}}}_{=0} &= (\mathbf{g} + \lambda\hat{\mathbf{r}})\cdot\hat{\mathbf{e}}; \\ \therefore \quad \lambda &= -\frac{\mathbf{g}\cdot\hat{\mathbf{e}}}{\hat{\mathbf{r}}\cdot\hat{\mathbf{e}}}. \end{aligned} \tag{3.1}$$

Question: Does our expression for λ make sense?

Let's pause for a moment to examine that result before proceeding. Does it make sense? The geometric interpretation of that result is that $|\lambda|$ is the ratio of the lengths of the projections of $\mathbf{g}$ and $\hat{\mathbf{r}}$ upon $\hat{\mathbf{e}}$. So far, so good—a study of Fig. 2 confirms that $|\lambda|$ must indeed be equal to that ratio. Examining Fig. 2 further, we see (1) that no shadow will be produced unless λ is positive, and (2) that no shadow will be produced unless the projections of $\mathbf{g}$ and $\hat{\mathbf{r}}$ are oppositely directed. Eq. (3.1) is consistent with those observations: λ is positive only when $\mathbf{g}\cdot\hat{\mathbf{e}}$ and $\hat{\mathbf{r}}\cdot\hat{\mathbf{e}}$ are opposite in sign, and that difference in sign occurs only when $\hat{\mathbf{e}}$ and $\hat{\mathbf{r}}$ are oppositely directed.

Now that we've assured ourselves that our expression for λ makes sense, we continue by making the substitutions $\hat{\mathbf{r}} = \hat{\mathbf{a}}r_a + \hat{\mathbf{b}}r_b + \hat{\mathbf{c}}r_c$, $\mathbf{g} = \hat{\mathbf{a}}g_a + \hat{\mathbf{b}}g_b + \hat{\mathbf{c}}g_c$, and $\hat{\mathbf{e}} = \hat{\mathbf{a}}e_a + \hat{\mathbf{b}}e_b + \hat{\mathbf{c}}e_c$:

$$\begin{aligned} \lambda &= -\frac{\left(\hat{\mathbf{a}}g_a + \hat{\mathbf{b}}g_b + \hat{\mathbf{c}}g_c\right)\cdot\left(\hat{\mathbf{a}}e_a + \hat{\mathbf{b}}e_b + \hat{\mathbf{c}}e_c\right)}{\left(\hat{\mathbf{a}}r_a + \hat{\mathbf{b}}r_b + \hat{\mathbf{c}}r_c\right)\cdot\left(\hat{\mathbf{a}}e_a + \hat{\mathbf{b}}e_b + \hat{\mathbf{c}}e_c\right)} \\ &= -\frac{g_ae_a + g_be_b + g_ce_c}{r_ae_a + r_be_b + r_ce_c}. \end{aligned} \tag{3.2}$$

Now, we substitute that expression for λ in our original equation, then simplify:

$$\begin{aligned} \mathbf{s} &= \mathbf{g} + \lambda\hat{\mathbf{r}} \\ &= \hat{\mathbf{a}}g_a + \hat{\mathbf{b}}g_b + \hat{\mathbf{c}}g_c - \left[\frac{g_ae_a + g_be_b + g_ce_c}{r_ae_a + r_be_b + r_ce_c}\right]\left(\hat{\mathbf{a}}r_a + \hat{\mathbf{b}}r_b + \hat{\mathbf{c}}r_c\right). \end{aligned}$$

By expanding the product on the right-hand side, then rearranging, the result is

$$\begin{aligned} \mathbf{s} = \quad & \hat{\mathbf{a}}\left[\frac{g_a\left(r_be_b + r_ce_c\right) - r_a\left(g_be_b + g_ce_c\right)}{r_ae_a + r_be_b + r_ce_c}\right] \\ & + \hat{\mathbf{b}}\left[\frac{g_b\left(r_ae_a + r_ce_c\right) - r_b\left(g_ae_a + g_ce_c\right)}{r_ae_a + r_be_b + r_ce_c}\right] \\ & + \hat{\mathbf{c}}\left[\frac{g_c\left(r_ae_a + r_be_b\right) - r_c\left(g_ae_a + g_be_b\right)}{r_ae_a + r_be_b + r_ce_c}\right]. \end{aligned} \tag{3.3}$$

3.2 Solution via the Outer Product with $\mathbf{T}$

In this section, we'll write $\mathbf{T}$ as $\mathbf{T} = \hat{\mathbf{a}}\hat{\mathbf{b}}\tau_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\tau_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}\tau_{ac}$ in order to arrive at a solution in which the plane of the plaza is expressed in that way. The Appendix

(5) shows how to find $\mathbf{T}$ in terms of the components of $\hat{\mathbf{e}}$.

We indicated in Section 2.3 that because $\mathbf{s}$ is parallel to the plaza (and therefore to $\mathbf{T}$), $\mathbf{s} \wedge \mathbf{T} = 0$. Using that fact, we arrive at a preliminary version of λ as follows:

$$\begin{aligned}
\mathbf{s} &= \mathbf{g} + \lambda\hat{\mathbf{r}}; \\
\underbrace{\mathbf{s} \wedge \mathbf{T}}_{=0} &= (\mathbf{g} + \lambda\hat{\mathbf{r}}) \wedge \mathbf{T}; \\
\lambda\hat{\mathbf{r}} \wedge \mathbf{T} &= -\mathbf{g} \wedge \mathbf{T} \\
\therefore \quad \lambda &= -\left(\mathbf{g} \wedge \mathbf{T}\right)\left(\hat{\mathbf{r}} \wedge \mathbf{T}\right)^{-1}.
\end{aligned} \tag{3.4}$$

Now, we need to calculate $\mathbf{g} \wedge \mathbf{T}$ and $\left(\hat{\mathbf{r}} \wedge \mathbf{T}\right)^{-1}$. To find the former, we use Macdonald's ([4], p. 111) definition of the product "$\wedge$". See also the list of formulas in Reference [2], pp. 2-4.

$$\begin{aligned}
\mathbf{g} \wedge \mathbf{T} &= \langle \mathbf{g}\mathbf{T} \rangle_3 \\
&= \langle \Big(\hat{\mathbf{a}} g_a + \hat{\mathbf{b}} g_b + \hat{\mathbf{c}} g_c\Big) \Big(\mathbf{T} = \hat{\mathbf{a}}\hat{\mathbf{b}}\tau_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}\tau_{bc}\Big) \rangle_3 \\
&= \hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}\left(\tau_{ab} g_c + \tau_{bc} g_a - \tau_{ac} g_b\right).
\end{aligned}$$

Similarly, $\hat{\mathbf{r}} \wedge \mathbf{T} = \hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}\left(\tau_{ab} r_c + \tau_{bc} r_a - \tau_{ac} r_b\right)$. We recognize the product $\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$ as I_3: the unit pseudoscalar for $\mathbb{G}_3$. Its multiplicative inverse (I_3^{-1}) is $-I_3$, $= -\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$. Therefore, multiplicative inverse of $\hat{\mathbf{r}} \wedge \mathbf{T}$ is

$$\begin{aligned}
\left(\hat{\mathbf{r}} \wedge \mathbf{T}\right)^{-1} &= \frac{I_3^{-1}}{\left|\hat{\mathbf{r}} \wedge \mathbf{T}\right|^2} \\
&= -\frac{\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}}{\left(\tau_{ab} r_c + \tau_{bc} r_a - \tau_{ac} r_b\right)^2}.
\end{aligned}$$

Using that result, and our expression for $\hat{\mathbf{r}} \wedge \mathbf{T}$, Eq. (3.4) becomes

$$\begin{aligned}
\lambda &= -\left[\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}\left(\tau_{ab} r_c + \tau_{bc} r_a - \tau_{ac} r_b\right)\right]\left[-\frac{\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}\left(\tau_{ab} r_c + \tau_{bc} r_a - \tau_{ac} r_b\right)}{\left(\tau_{ab} r_c + \tau_{bc} r_a - \tau_{ac} r_b\right)^2}\right] \\
&= -\frac{\tau_{ab} g_c + \tau_{bc} g_a - \tau_{ac} g_b}{\tau_{ab} r_c + \tau_{bc} r_a - \tau_{ac} r_b}.
\end{aligned} \tag{3.5}$$

Substituting this expression for λ in $\mathbf{s} = \mathbf{g} + \lambda\hat{\mathbf{r}}$, we obtain

$$\begin{aligned}
\mathbf{s} = \quad & \hat{\mathbf{a}}\left[\frac{\tau_{ab}\left(g_a r_c - g_c r_a\right) + \tau_{ac}\left(g_b r_a - g_a r_b\right)}{\tau_{ab} r_c + \tau_{bc} r_a - \tau_{ac} r_b}\right] \\
&+ \hat{\mathbf{b}}\left[\frac{\tau_{ab}\left(g_b r_c - g_c r_b\right) + \tau_{bc}\left(g_b r_a - g_a r_b\right)}{\tau_{ab} r_c + \tau_{bc} r_a - \tau_{ac} r_b}\right] \\
&+ \hat{\mathbf{c}}\left[\frac{\tau_{bc}\left(g_c r_a - g_a r_c\right) + \tau_{ac}\left(g_b r_c - g_c r_b\right)}{\tau_{ab} r_c + \tau_{bc} r_a - \tau_{ac} r_b}\right].
\end{aligned} \tag{3.6}$$

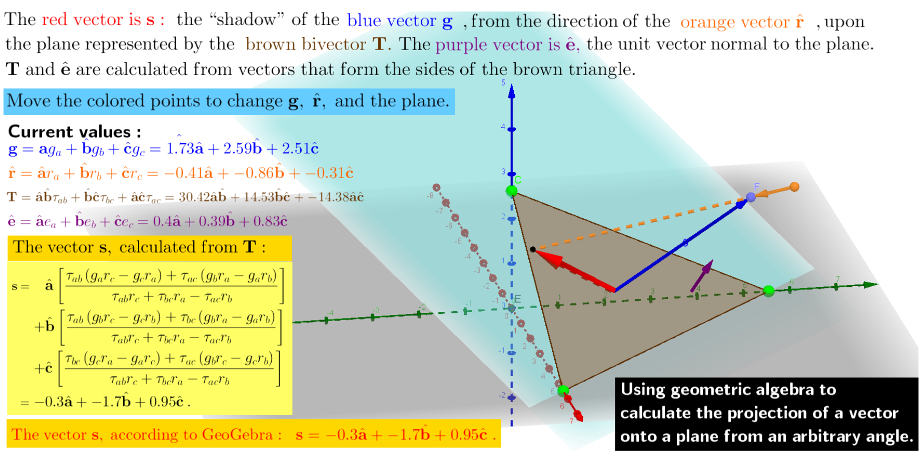


Figure 3: Screen shot (Ref. [5]) of an interactive GeoGebra worksheet that calculates the vector $\mathbf{s}$, and compares the result to the vector $\mathbf{s}$ that was obtained by construction.

4 Testing the Formulas that We've Derived

Fig. 3 shows an interactive GeoGebra worksheet (Reference [5]) that calculates the vector $\mathbf{s}$, and compares the result to the vector $\mathbf{s}$ that was obtained by construction. The worksheet calculates λ from $\hat{\mathbf{e}}$ as well as from $\mathbf{T}$, but shows the numerical calculation only for $\mathbf{T}$ because of space limitations.

References

[1] J. A. Smith, 2017a, "Formulas and Spreadsheets for Simple, Composite, and Complex Rotations of Vectors and Bivectors in Geometric (Clifford) Algebra", http://vixra.org/abs/1712.0393.

[2] J. A. Smith, 2017b, "Some Solution Strategies for Equations that Arise in Geometric (Clifford) Algebra", http://vixra.org/abs/1610.0054 .

[3] D. Hestenes, 1999, *New Foundations for Classical Mechanics*, (Second Edition), Kluwer Academic Publishers (Dordrecht/Boston/London).

[4] A. Macdonald, *Linear and Geometric Algebra* (First Edition) p. 126, CreateSpace Independent Publishing Platform (Lexington, 2012).

[5] J. A. Smith, 2017c, "Projection of Vector on Plane via Geometric Algebra" (a GeoGebra construction), https://www.geogebra.org/m/ykzkbQJq.

5 Appendix: Calculating the Bivector of a Plane Whose Normal is the Vector $\hat{\mathbf{e}}$

As may be inferred from a study of References [3] (p. (56, 63) and [4] (pp. 106-108) , the bivector $\mathbf{T}$ that we seek is the one whose dual is $\hat{\mathbf{e}}$. That is, $\mathbf{Q}$ must satisfy the condition

$$\begin{aligned} \hat{\mathbf{e}} &= \mathbf{Q}I_3^{-1}; \\ \therefore \mathbf{Q} &= \hat{\mathbf{e}}I_3. \end{aligned} \tag{5.1}$$

Although we won't use that fact here, I_3^{-1} is I_3's negative: $I_3^{-1} = -\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$.

where I_3 is the right-handed pseudoscalar for $\mathbb{G}^3$. That pseudoscalar is the product, written in right-handed order, of our orthonormal reference frame's basis vectors: $I_3 = \hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$ (and is also $\hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}\hat{\mathbf{a}}\hat{\mathbf{b}}$). Therefore, writing $\mathbf{Q}$ as $\mathbf{Q} = \hat{\mathbf{a}}e_a + \hat{\mathbf{b}}e_b + \hat{\mathbf{c}}e_c$,

To make this simplification, we use the following facts:

- The product of two perpendicular vectors (such as $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$) is a bivector;
- Therefore, for any two perpendicular vectors $\mathbf{p}$ and $\mathbf{q}$, $\mathbf{qp} = -\mathbf{qp}$; and
- (Of course) for any unit vector $\hat{\mathbf{p}}$, $\hat{\mathbf{p}}\hat{\mathbf{p}} = 1$.

$$\begin{aligned} \mathbf{Q} &= \hat{\mathbf{e}}I_3 \\ &= \left(\hat{\mathbf{a}}e_a + \hat{\mathbf{b}}e_b + \hat{\mathbf{c}}e_c\right)\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}} \\ &= \hat{\mathbf{a}}\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}e_a + \hat{\mathbf{b}}\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}e_b + \hat{\mathbf{c}}\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}e_c \\ &= \hat{\mathbf{a}}\hat{\mathbf{b}}e_c + \hat{\mathbf{b}}\hat{\mathbf{c}}e_a - \hat{\mathbf{a}}\hat{\mathbf{c}}e_b. \end{aligned} \tag{5.2}$$

In writing that last result, we've followed [4]'s convention (p. 82) of using $\hat{\mathbf{a}}\hat{\mathbf{b}}$, $\hat{\mathbf{b}}\hat{\mathbf{c}}$, and $\hat{\mathbf{a}}\hat{\mathbf{c}}$ as our bivector basis. Examining Eq. (5.2) we can see that if we write $\mathbf{Q}$ in the form $\mathbf{Q} = \hat{\mathbf{a}}\hat{\mathbf{b}}q_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}q_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}q_{ac}$, then

$$q_{ab} = e_c, \quad q_{bc} = e_a, \quad q_{ac} = -e_c. \tag{5.3}$$