A short disproof of the Riemann hypothesis

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Abstract

This paper disproves the Riemann hypothesis by analyzing the integral representation of the Riemann zeta function that converges absolutely in the root-free region.

1 Introduction

The starting point is the definition of the Riemann zeta function $\zeta(s)$ by the use of the Euler product over all primes p, as well as by the use of the classical Möbius function $\mu(n)$:

$$\frac{1}{\zeta(s)} = \prod_{p=2}^{\infty} 1 - \frac{1}{p^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \qquad , \Re(s) > 1 \tag{1}$$

The series in eq. (1) converges absolutely on $\Re(s) > 1$. However, it converges only conditionally on $\Re(s) \leq 1$.

On the other hand, the series $\sum_{n=1}^{\infty} \mu(n)/n^s$ from eq. (1) can be rewritten as the Riemann-Stieltjes integral:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = s \int_1^{\infty} \frac{M(a)}{a^{s+1}} da \qquad , \Re(s) > R \tag{2}$$

Here, in eq. (2), $M(a)$ is the Mertens function $M(a) = \sum_{n=1}^{a} \mu(n)$, and R is the largest real part of zeta roots: $R = \max\{\Re(\rho) : \zeta(\rho) = 0\}$.

The interesting feature of the integral in eq. (2) is that it converges absolutely on $\Re(s) > R$ as soon as it converges, or in other words, as soon as the Mertens function behaves asymptotically as $M(a) = O(a^{R+\delta})$ for every $\delta > 0$. Since integral in eq. (2) converges absolutely in the root-free region of the critical strip, it represents an analytic function in that region. And so, by the uniqueness of the analytic continuation, we find that eq. (2) stands true on the half-plane $\Re(s) > R$.

Since integral in eq. (2) converges absolutely on $\Re(s) > R$, it can be Mellin inverted in that region:

$$M(a) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{a^s}{s\zeta(s)} ds \qquad , \Re(s) > R \tag{3}$$

Here, b is any real number from the region $\Re(s) > R$ of course.

Since we know that the Mellin inverse (3) exists, the theory of Mellin transforms tells us that the function

$$\frac{1}{s\zeta(s)} \qquad , \Re(s) > R \tag{4}$$

must be absolutely integrable along any line parallel to the imaginary axis on its region of convergence $\Re(s) > R$.

The results listed so far are all very well known. We need one more very well known result, though. Namely, let ρ denote a zeta function nontrivial root. Then the sum $\sum 1/|\rho|$ over all nontrivial zeta roots is not bounded, but diverges to infinity. One can prove this easily from the fact that there are asymptotically $T \log T$ nontrivial zeta roots up to hight T. If all the nontrivial roots ρ were located approximately at hight T, then $|\rho| \approx T$ and $1/|\rho|$ is then as small as possible, $1/|\rho| \approx 1/T$. Thus $\sum 1/|\rho| \geq T \log T/T = \log T$, and this is not bounded as T grows without bounds.

2 A short sketch of this disproof

This disproof is fairly simple. One considers the Mellin transform

$$M(a) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{a^s}{s\zeta(s)} ds \qquad , \Re(s) > R$$

As with all Mellin transforms, it's in the form

$$f(a) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} a^s F(s) ds \qquad , \Re(s) > R$$

We know from the theory of Mellin transforms that $F(s)$ is absolutely integrable along the contour of integration on the fundamental strip $\Re(s) > R$. Hence, $1/s\zeta(s)$ is absolutely integrable along the contour of integration on $\Re(s) > R$. This condition of absolute integrability of $1/s\zeta(s)$ reads

$$-i \int_{b-i\infty}^{b+i\infty} \frac{ds}{|s||\zeta(s)|} < \infty$$

We now define the contour of integration to be arbitrarily close to the nontrivial zeta root. And then we only pay attention to arbitrarily small parts of the contour of integration that are next to nontrivial roots. These arbitrarily small parts of the contour of integration are in an ε-neighborhood of nontrivial roots, and hence we know the zeta function along such parts of the contour of integration behaves as $\zeta(\rho+\varepsilon) = c_\rho \varepsilon^n$. This simplifies calculations considerably. So, instead of calculating the entire absolutely convergent integral along the entire contour of integration, we just compute the parts of the integral that are in the ε-neighborhoods of nontrivial zeta roots, not really aiming at computing the entire integral. Since the entire integral converges absolutely along the entire contour of integration, we find that the part along the contours lying in ε-neighborhoods of nontrivial zeta roots must converge as well. More precisely, the result is bounded. However, since $\zeta(\rho+\varepsilon) = c_\rho \varepsilon^n$, the result depends on ε. The arbitrarily small quantity ε is a free parameter, it has no fixed magnitude. Hence, the partial integral could grow arbitrarily large if it was dominated by the $1/\varepsilon$ term. Hence, the result cannot depend on $1/\varepsilon$. This demonstrates that all of the nontrivial zeta roots closest to the line $\Re(s) = 1$ are simple. Finally, we arrive at the fact that the partial absolutely convergent integral of $1/s\zeta(s)$ is proportional to the infinite sum $\sum 1/\rho_R$ that runs over all such nontrivial roots ρ_R that are closest to the root-free region. Hence, since the entire absolutely convergent integral is bounded, so is its part. This means that $\sum 1/\rho_R$ is bounded. However, the sum $\sum 1/\rho$ over all nontrivial zeta roots ρ is not bounded. Therefore, not all nontrivial zeta roots ρ are located on a single line. This disproves the Riemann hypothesis then, since the Riemann hypothesis states that all the nontrivial zeta roots are located on a single line $\Re(s) = 1/2$.

3 Analysis

We start the analysis by defining $b = R+\varepsilon$. In other words, we shift the contour of integration in eqs. (3) and (4) as close to the zeta roots as possible, with ε being an arbitrarily small strictly positive real number, as depicted in Figure 1.

Since function $1/s\zeta(s)$ from eq. (4) being absolutely integrable on $\Re(s) > R$ along any line parallel to the imaginary axis, we pay attention to this absolutely convergent integral:

$$-i \int_{R+\varepsilon-i\infty}^{R+\varepsilon+i\infty} \frac{ds}{|s||\zeta(s)|} = \int_{-\infty}^{+\infty} \frac{d\tau}{|R+\varepsilon+i\tau||\zeta(R+\varepsilon+i\tau)|} < \infty \qquad (5)$$

Next, rewrite the last integral of eq. (5) in the form of its Riemannean sum:

$$\lim_{N\to\infty} \sum_{k=-N}^{k=N} \frac{\Delta\tau_k}{|R+\varepsilon+i\tau_k||\zeta(R+\varepsilon+i\tau_k)|} < \infty \qquad (6)$$

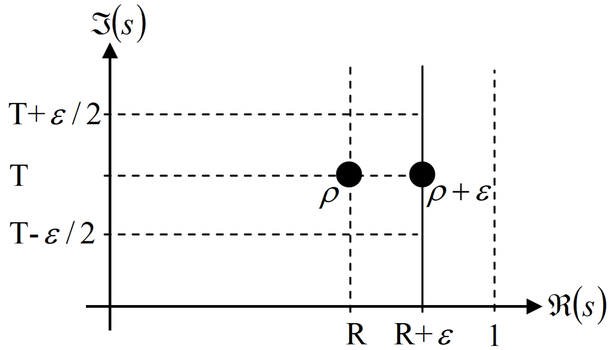

Figure 1: *Root* $\rho = R + iT$ *and the integration contour* $\Re(s) = b = R + \varepsilon$

We notice here that all the summands in the Riemannean sum (6) are strictly positive.

Next, consider only the small part of the contour of integration in the ε-neighborhood of an arbitrary zeta root $\rho = R + iT$, not the entire contour of integration $\Re(s) = b$, as depicted in Figure 1. In other words, consider just one summand of the Riemannean sum (6), the one evaluated at the point of the contour of integration $\rho + \varepsilon$ closest to the zeta root $\rho = R + i\tau_k$, letting two arbitrarily small quantities $\Delta\tau_k$ and ε being of the same magnitude, without the loss of generality, $\Delta\tau_k = \varepsilon$:

$$\frac{\varepsilon}{|\rho + \varepsilon||\zeta(\rho + \varepsilon)|} \tag{7}$$

There's a zeta root at ρ by assumption. So, zeta behaves on the contour as $\zeta(\rho + \varepsilon) = c_\rho \varepsilon^n$, with $c_\rho \neq 0$ and with $n \in \mathbb{N}$ being the order of the root ρ. Hence, the term (7) reads

$$\frac{\varepsilon^{1-n}}{|\rho||c_\rho|} \tag{8}$$

We have neglected the arbitrarily small quantity ε in $|\rho + \varepsilon|$ because the impact of ε is arbitrarily small and therefore negligible in it.

If $n \neq 1$ in eq. (8), then the term (8) can be arbitrarily large, because ε is arbitrarily small. However, (8) cannot be arbitrarily large, because then the integrals (3) and (6) would be arbitrarily large. However, the integral (6) converges to a value that is not arbitrarily large, and it consists of strictly positive Riemannean summands, so no other part of the Riemannean sum could possibly cancel the arbitrarily large part out, making the sum bounded. Therefore, one concludes $n = 1$, and hence all the nontrivial zeta roots closest to the line $\Re(s) = 1$ are simple.

Thus, with $n = 1$, the term (8) becomes

$$\frac{1}{|\rho||c_\rho|} \tag{9}$$

This analysis holds for any nontrivial zeta root ρ_R with $\Re(\rho_R) = R$. Hence, the sum

$$\sum_{\Re\rho=R} \frac{1}{|\rho||c_\rho|} < \infty \tag{10}$$

must converge, because this sum is a part of the Riemannean sum given by the convergent integral (6) of positive terms.

One notices that $c_\rho \neq \infty$ for any nontrivial zeta root ρ. Hence, $1/|c_\rho| > 0$.

Furthermore, there exists $0 < \alpha < \infty$ such that $1/|c_\rho| \geq \alpha$ for all zeta roots ρ, because $1/|c_\rho| > 0$ and because $\mathbb{R}$ is dense. In other words, the set $\{1/|c_\rho|\}$ has a minimum, say α, not equal to zero: $0 < \alpha = \min\{1/|c_\rho| : \zeta(\rho) = 0\}$. With this α, one finds $1/|c_\rho| \geq \alpha$.

Thus, one concludes:

$$\sum_{\Re(\rho)=R} \frac{1}{|\rho||c_\rho|} \geq \alpha \sum_{\Re(\rho)=R} \frac{1}{|\rho|} \qquad , 0 < \alpha < \infty \tag{11}$$

Since $\sum 1/|\rho c_\rho|$ given by eq. (10) is bounded, so is $\sum 1/|\rho|$, as seen by inspecting eqs. (10) and (11). Therefore, one concludes that the sum $\sum 1/|\rho_R|$ over all the nontrivial zeta roots ρ_R with $\Re(\rho_R) = R$ is bounded.

However, it's a known fact that the sum $\sum 1/|\rho|$ over all nontrivial zeta roots is not bounded. Therefore, ρ_R are not all the nontrivial zeta roots. There have to be more nontrivial roots elsewhere away from the line $\Re(s) = R$ in the critical strip $0 < \Re(s) < R$. In other words, not all nontrivial zeta roots are located on a single line.

Since the Riemann hypothesis states that all the nontrivial zeta roots are located on the single line $\Re(s) = 1/2$, this disproves the Riemann hypothesis.

4 Conclusion

This paper disproves the Riemann hypothesis by analyzing the integral representation of the Riemann zeta function that converges absolutely in the root-free region.