Union of two arithmetic sequences.
Basic calculation formula.
(1)

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Abstract. Let A, B denote infinite arithmetic sequences in $\mathbb{N}$ with initial terms equal to 0. The union of A, B is strictly increasing sequence U, containing only all elements of A, B (without repetitions). We will derive the formula for the n-th element of the union $U(A,B)=(u_n)$ in the form: $u_n=f(n,a,b)$.

1 Purpose

Definition 1.1 Let $A_1...A_n$ denote infinite, the same monotonic arithmetic sequences in $\mathbb{R}$. Union of $A_1...A_n$ is strictly monotonic sequence U, containing only all the terms of the sequences $A_1...A_n$ (without repetition).

In this paper we will deal with the simplest case - the union of two arithmetic sequences $A=(a_i), B=(b_j)$. We will consider only such A, B, that meet the following conditions:

Conditions 1.1

1) A, B are composed of natural numbers: $\forall_{i,j\in\mathbb{N}}\ a_i, b_j\in\mathbb{N}$,
2) initial terms are equal to 0: $a_0=0,\ b_0=0$,
3) common differences are: $a\geq2,\ b\geq2\ \ a,b\in\mathbb{N}$,
4) Greatest common divisor of common diffirences: $GCD(a,b)<Min(a,b)$.
5) $b>a$*

Using condition 2): $a_i=ia,\ b_j=jb$. In 3) and 4) we immediately exclude trivial cases: for $a=0$ we have $U=B$, for $a=1$ $U=A$ and for $GCD(a,b)=a$ also $U=A$ (and *vice versa* the same for b).
(*) Condition 5) is not obligatory, but it segregates the properties and concepts used in this paper and will not matter in the end.

The goal is to derive the formula for the n-th union $U=(u_n)$ element in the form: $u_n=f(n,a,b)$ in accordance with the definition and conditions.

2 Notation and preliminary findings

2.1 Basic properties of the union

We will explain the terms we use in the example with a Table 1 (next page), containing the initial 34 elements of a union of sequences with common differences: $a=5, b=13$.

Table 1: Sample union

$\mathbf{U(5,13)}$			**c**			
R	**g**	**r**	**0**	**1**	**2**	**3**
0		0	0	5	10	
1		1	13	15	20	25
2	0	2	26	30	35	
3		3	39	40	45	50
4		4	52	55	60	
5		0	65	70	75	
6		1	78	80	85	90
7	1	2	91	95	100	
8		3	104	105	110	115
9		4	117	120	125	

The table was filled in the following way:
Conditions 2.1

1. Rows R and columns c are numerated from 0.
2. All the terms of the progression with a greater difference (i.e. B) were typed in the column c=0.
3. In the columns c>0 terms of the sequence A were written in rows, so their values in current row are larger then value in column 0 and less than the value in column 0 of the next row. It follows that the terms from A equal to some terms of B, were **omitted** when filling the table.

2.2 Notation

In this paper, we use a special kinds of notation for subsequences and their members:

- X^Y denotes such a subsequence of X, which all elements belong to the structure Y, where Y is a group, row, or column. Samples: U^G - subsequence of union terms belonging to a group. N^R - subsequence of indexes of union terms belonging to a row.
- $X^{Y,Z}$ denotes one element of X, which belong to the union of structures Y, Z (i.e. to $Y \bigcap Z$). Sample: $i^{R,c=1}$ is index of term of A in row R and column 1.
- X^Y_{min}=min(X^Y), X^Y_{max}=max(X^Y)

We will use short form notation for least common multiple and greatest common divisor of a, b:
ω=$LCM(a,b)$
Θ=$GCD(a,b)$

$|X|$ denote number of elements of finite progress X.

2.3 Groups

Definition 2.1 The group is a subsequence of U. Initial term of group is the common element of A, B and final term is an element of U, that precedes the next common element of A, B.

Terms of A, B are different except for zero and all multiples of the least common multiple a and b. Hence the common terms of A, B will take the following values: $\omega_g{=}g\omega$ where $g{\in}\mathbb{N}$ is number of group.

For each g, all terms of union in the interval $\langle\omega_g, \omega_{g+1}{-}1\rangle$ are named group G_g. In particular, the terms of union of the interval $\langle\omega_0, \omega_1{-}1\rangle$ are group G_0. In the Table 1 the numbers of groups g are in the second column.

For each group there are obvious relationships:

- Number of rows (see section 2.4) in a group: $|R^G|{=}|r|{=}\frac{\omega}{b}$. With the general characteristics of LCM and GCD we have $\omega{=}\frac{ab}{\Theta}$, so $|r|{=}\frac{a}{\Theta}$.
- Number of terms of sequence B in each group: $|B^G|{=}|r|{=}\frac{a}{\Theta}$.
- Number of terms of sequence A in each group: $|A^G|{=}\frac{\omega}{a}{=}\frac{b}{\Theta}$.
- Number of terms of union U in each group:

$$|U^G|{=}|B^G|{+}|A^G|{-}1{=}\frac{a}{\Theta}{+}\frac{b}{\Theta}{-}1{=}\frac{a{+}b{-}\Theta}{\Theta} \tag{1}$$

2.4 Rows

Rows will be marked with r or R. When considering only one group (in particular the G_0 group), we will use relative rows numbers with lowercase letter r, where r can only accept values from 0 to $\frac{a}{\Theta}{-}1$. The uppercase letter R will be the absolute number of any row, referring to the entire table. The following identitys holds:

$$r{=}Mod\,(R, |r|)\,{=}Mod\left(R, \frac{a}{\Theta}\right){=}R{-}\frac{a}{\Theta}\left\lfloor\frac{R\Theta}{a}\right\rfloor \tag{2}$$

$$R{=}g|r|{+}r{=}g\frac{a}{\Theta}{+}r \tag{3}$$

2.5 Columns

Columns, marked with c, are numbered from 0. In the column 0 there are subsequent terms of sequence B: $b_j{=}b_R{=}Rb$. This column is always filled. Number of columns $c{>}0$ depends on the values of b and a. Based on the Table 1, we conclude that the columns must be at least $\left\lfloor\frac{b}{a}\right\rfloor$. But, in some cases, they may be one more. This will happen when $0{<}a^R_{min}{-}b_R{<}b{-}a\left\lfloor\frac{b}{a}\right\rfloor$ (where a^R_{min} is the smallest value a_i in row R). It follows that all of the columns from 1 to $\left\lfloor\frac{b}{a}\right\rfloor$ are filled. The last column, always numbered $\left\lfloor\frac{b}{a}\right\rfloor{+}1$, contains only the numbers of the A sequence, which satisfy the condition $bR{+}\left\lfloor\frac{b}{a}\right\rfloor a{<}ai{<}b\,(R{+}1)$. This column is never completly full because at least the initial ($r{=}0$) and the last row $\left(r{=}\frac{a}{\Theta}{-}1\right)$ of each group must be empty cells.

2.6 Indexes

As for the rows: when considering only one group, in particular the G_0 group, we will use the relative indices of the union indexes with lower case n, where n can take values from 0 to $|U^G|-1$ only (see (1)). The capital letter N will be the absolute index of any union element, referring to the entire table. Indexes i, j are always absosute.

2.7 Other properties

1. For the given coprime a, b and any multiple of them, i.e. for $a'=\Theta a$, $b'=\Theta b$ the Table 1 layout for each group is identical, the number of rows in the group and the number of columns is the same, and in the last column the numbers always appear in the same positions as in the table for a and b. Hence in the formulas there is Θ.

2. Numbers in G_g group are greater than the numbers in G_0 at the same positions by $g\omega$, i.e. for $R\in G_g$: $u\left(g,R,c\right)=u\left(0,R-g\dfrac{a}{\Theta},c\right)=u(r,c)+g\omega$. Therefore, knowing only the properties of the G_0, we can deduce all the properties of the whole union.

3 Deriving formula for the union

The final formula $u_N=f(N)$ will be derived in several steps:

1. We derive an indirect formula for union terms in group G_0, dependent on rows and columns numbers: $u(n,r,c)=f(n,r,c)$.

2. For each of the numbers $u(n,r,c)$ we will calculate the row number $r=r(n)$.

3. For any group this relative formula will be converted to an absolute row number: $R=R(N)$.

4. We specify the column number that determines the affiliation of the union element to sequence A or B: $C=C(N)$.

5. We get the final result in the form: $u_N=f\left(N,R(N),C(N)\right)=f(N)$.

3.1 Step 1

We work within the G_0 group. All numbers in the column $c=0$ are multiples of b. The remaining columns contain multiples of a.

$$u(i,j,c)=\begin{cases} bj & \text{for: } c=0 \\ ai & \text{for: } c>0 \end{cases} \tag{4}$$

For $c=0$ $bj=br$. For $c>0$ multiple will be equal to the index i of the term a_i. Because $n=r+i$, ie $i=n-r$, then:

$$u(n,r,c)=\begin{cases} br & \text{for: } c=0 \\ a(n-r) & \text{for: } c>0 \end{cases} \tag{5}$$

3.2 Step 2

For each u_n index n is equal to number of union elements less then u_n. If u_n is in column 0, then n is equal to the number of union terms in rows from 0 to $r(n){-}1$. Among them are r terms from B and $\left\lfloor \frac{b_r}{a} \right\rfloor = \left\lfloor \frac{rb}{a} \right\rfloor$ from A, which together give:

$$n{=}r{+}\left\lfloor \frac{rb}{a} \right\rfloor{=}\left\lfloor \frac{r(a{+}b)}{a} \right\rfloor \qquad \text{for } c{=}0$$

We write the right floor by definition, taking into account that its internal fraction can not be integer (Conditions 1.1 p.4), so both inequalities will be strict inequalities:

$$n{<}\frac{r(a{+}b)}{a}{<}n{+}1$$

$$\frac{na}{a{+}b}{<}r{<}\frac{(n{+}1)a}{a{+}b}$$

Since left inequality is strict and the difference between right and left expressions is less than 1, then:

$$r{=}\left\lfloor \frac{(n{+}1)a}{a{+}b} \right\rfloor \tag{6}$$

3.3 Step 3

The formula (6) is derived in G_0, but using only relative values of r and n is correct in each group. To determine the relation $R(N)$:

i) any index N from a certain group G_g we will convert to its equivalent n

ii) from (6) we will calculate $r(n)$

iii) next, we will recalculate the absolute row number R from G_g, resulting in the desired dependence.

i)
For any N: $n{=}Mod\left(N, |U^G|\right)$. By substituting (1) we have:

$$n{=}Mod\left(N, \frac{a{+}b{-}\Theta}{\Theta}\right)$$

We write the Mod from the definition:

$$n{=}N{-}\frac{a{+}b{-}\Theta}{\Theta}\left\lfloor \frac{N\Theta}{a{+}b{-}\Theta} \right\rfloor$$

We count group number from the dependence:

$$g(N){=}\left\lfloor \frac{N}{|U^G|} \right\rfloor$$

We substitute (1):

$$g{=}\left\lfloor \frac{N\Theta}{a{+}b{-}\Theta} \right\rfloor \tag{7}$$

hence:

$$n = N - \frac{a+b-\Theta}{\Theta} g \tag{8}$$

ii)
We replace n in (6) with (8):

$$r = \left\lfloor \frac{\left(N - \frac{a+b-\Theta}{\Theta} g + 1\right) a}{a+b} \right\rfloor \tag{9}$$

iii)

Now we insert (9) into (3):

$$R = \left\lfloor \frac{\left(N - \frac{a+b-\Theta}{\Theta} g + 1\right) a}{a+b} \right\rfloor + g\frac{a}{\Theta}$$

The right component is integer so we can put it to the floor:

$$\begin{aligned}
R &= \left\lfloor \frac{\left(N - \frac{a+b-\Theta}{\Theta} g + 1\right) a}{a+b} + g\frac{a}{\Theta} \right\rfloor \\
&= \left\lfloor \frac{a}{a+b} \left(N - \frac{1}{\Theta}(a+b-\Theta)g + 1\right) + g\frac{a}{\Theta} \right\rfloor \\
&= \left\lfloor \frac{a}{a+b}(N+1) - \frac{a}{a+b}\frac{1}{\Theta}(a+b-\Theta)g + g\frac{a}{\Theta} \right\rfloor \\
&= \left\lfloor \frac{a}{a+b}(N+1) + g\left(\frac{a}{\Theta} - \frac{a}{a+b}\frac{1}{\Theta}(a+b-\Theta)\right) \right\rfloor \\
&= \left\lfloor \frac{a}{a+b}(N+1) + \frac{a}{a+b} g \left(\frac{a+b}{\Theta} - \frac{a+b-\Theta}{\Theta}\right) \right\rfloor \\
&= \left\lfloor \frac{a}{a+b}(N+1) + \frac{a}{a+b} g \cdot (1) \right\rfloor \\
&= \left\lfloor \frac{a}{a+b}(N+1+g) \right\rfloor \\
&= \left\lfloor \frac{a}{a+b} \left\lfloor \frac{N\Theta}{a+b-\Theta} + N + 1 \right\rfloor \right\rfloor \quad \text{after substitution } g \text{ from (7)}
\end{aligned}$$

Ultimately:

$$R(N) = \left\lfloor \frac{a}{a+b} \left\lfloor \frac{N(a+b)}{a+b-\Theta} + 1 \right\rfloor \right\rfloor \tag{10}$$

3.4 Step 4

We don't need to find a specific column number, but to distinguish whether the union element is in column 0 or outside. So, the formula will look like:

$$C(N) = \begin{cases} 0 & \text{for: } c=0 \\ 1 & \text{for: } c>0 \end{cases}$$

We can construct it easily, noting that the union element of index $N{>}0$ occurs in column 0 only when, the row has changed at the same time.

$$C(N)=\begin{cases}0 & \text{for: } R(N)-R(N-1)=1\\ 1 & \text{for: } R(N)-R(N-1)=0\end{cases}\quad \text{for: } N>0$$

Let's say the same thing in unconditional form ($N{>}0$):

$$C(N)=1-(R(N)-R(N-1))=1+R(N-1)-R(N)$$

After substituting (10) we obtain:

$$C(N)=1+\left\lfloor\frac{a}{a+b}\left\lfloor\frac{(N-1)(a+b)}{a+b-\Theta}+1\right\rfloor\right\rfloor-\left\lfloor\frac{a}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}+1\right\rfloor\right\rfloor \tag{11}$$

The above formula for $N{=}0$ gives:

$$C(0)=1+\left\lfloor\frac{a}{a+b}\left\lfloor\frac{-\Theta}{a+b-\Theta}+1\right\rfloor\right\rfloor-\left\lfloor\frac{a}{a+b}\right\rfloor=1+\left\lfloor\frac{a}{a+b}\cdot(-1)\right\rfloor-\left\lfloor\frac{a}{a+b}\right\rfloor=1-1-0=0$$

i.e. (11) is correct for $N{>}0$ and for $N{=}0$, so it is true for every N.

3.5 Step 5

The formula (5) for relative values will be transformed into formula for absolute values. The first member, for $c{=}0$ ($C{=}0$), will have the form: bR. For $c{>}0$ ($C{=}1$), the common elements for sequences A and B from the column $c{=}0$ should be taken into account at the beginning of each group. This means, that for absolute indexes we must omit these common elements from A by adding a G-number: $i{=}N{-}R{+}g$. After this correction we receive:

$$u(N,g,R,C)=\begin{cases}bR & \text{for: } C=0\\ a(N-R+g) & \text{for: } C=1\end{cases} \tag{12}$$

We write the lower formula (for $C{=}1$):

$$u(N,g,R,1)=a(N-R+g)$$

We substitute (10) and (7) and continue for $C{=}1$:

$$u(N)=a\left(N-\left\lfloor\frac{a}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}+1\right\rfloor\right\rfloor+\left\lfloor\frac{N\Theta}{a+b-\Theta}\right\rfloor\right)$$

We place the first and last member into the middle floor by changing the sign:

$$u(N)=-a\left\lfloor\frac{a}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}+1\right\rfloor-\left\lfloor\frac{N\Theta}{a+b-\Theta}\right\rfloor-N\right\rfloor$$

We place the final N into the preceding floor:

$$\begin{aligned}u(N)&=-a\left\lfloor\frac{a}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}+1\right\rfloor-\left\lfloor\frac{N\Theta+N(a+b-\Theta)}{a+b-\Theta}\right\rfloor\right\rfloor\\&=-a\left\lfloor\frac{a}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}+1\right\rfloor-\left\lfloor\frac{N(a+b)}{a+b-\Theta}\right\rfloor\right\rfloor\end{aligned}$$

We exclude number 1 from the first floor:

$$\begin{aligned} u(N) &= -a\left\lfloor \frac{a}{a+b} + \frac{a}{a+b}\left\lfloor \frac{N(a+b)}{a+b-\Theta}\right\rfloor - \left\lfloor \frac{N(a+b)}{a+b-\Theta}\right\rfloor \right\rfloor \\ &= -a\left\lfloor \frac{a}{a+b} + \left(\frac{a}{a+b}-1\right)\left\lfloor \frac{N(a+b)}{a+b-\Theta}\right\rfloor \right\rfloor \\ &= -a\left\lfloor \frac{a}{a+b} - \frac{b}{a+b}\left\lfloor \frac{N(a+b)}{a+b-\Theta}\right\rfloor \right\rfloor \\ &= -a\left\lfloor -\frac{b}{a+b}\left(\left\lfloor \frac{N(a+b)}{a+b-\Theta}\right\rfloor - \frac{a}{b}\right) \right\rfloor \end{aligned}$$

Using the dependency $-\lfloor -x\rfloor = \lceil x\rceil$ we substitute the floor for the ceil:

$$u(N) = a\left\lceil \frac{b}{a+b}\left(\left\lfloor \frac{N(a+b)}{a+b-\Theta}\right\rfloor - \frac{a}{b}\right) \right\rceil \qquad \text{for} \quad C=1 \tag{13}$$

Now, after substituting the formulas (10), (11) and (13) to (12), we obtain the final, complete formula for the N-th union element:

$$u_N = \begin{cases} b\left\lfloor \frac{a}{a+b}\left\lfloor \frac{N(a+b)}{a+b-\Theta}+1\right\rfloor \right\rfloor & \text{for} \quad C=0 \\ a\left\lceil \frac{b}{a+b}\left(\left\lfloor \frac{N(a+b)}{a+b-\Theta}\right\rfloor - \frac{a}{b}\right) \right\rceil & \text{for} \quad C=1 \end{cases} \tag{14}$$

$$\text{where} \quad C = 1 + \left\lfloor \frac{a}{a+b}\left\lfloor \frac{(N-1)(a+b)}{a+b-\Theta}+1\right\rfloor \right\rfloor - \left\lfloor \frac{a}{a+b}\left\lfloor \frac{N(a+b)}{a+b-\Theta}+1\right\rfloor \right\rfloor$$

According to (4):

$$j(N) = \left\lfloor \frac{a}{a+b}\left\lfloor \frac{N(a+b)}{a+b-\Theta}+1\right\rfloor \right\rfloor \qquad \text{for} \quad C=0 \tag{15}$$

$$i(N) = \left\lceil \frac{b}{a+b}\left(\left\lfloor \frac{N(a+b)}{a+b-\Theta}\right\rfloor - \frac{a}{b}\right) \right\rceil \quad \text{for} \quad C=1 \tag{16}$$

Now we can write formula (14) in shorter unconditional form:

$$u_N = Cai(N) + (1-C)bj(N) \tag{17}$$

Remark 1 If $a>b$ all the terms of the sequence A will be typed in the column $c=0$ of Table 1, then all the above reasoning can be repeated, obtaining the same result. Then, for $a>b$ the table will have only two columns and for $a=b$ formula (14) simplifies to $u_N = aN$.

Remark 2 Papers continuing the topic: http://vixra.org/author/waldemar_zielinski. They appear rather rarely, but they do appear...