POSITIVITY OF LI COEFFICIENTS FOR $n > 10^{24}$

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ABSTRACT. We investigate Riemann's Zeta function, as $(s-1)\zeta(s)$, under the Möbius transformation $s = \frac{1}{1-z}$ which maps the half plane right to the critical strip to the unit disk. Application of a generalized Poisson-Jensen formula (due to Nevanlinna) shows that the investigated function has only a finite number of zeros in the interior of the unit disk. We show, that the Li coefficients $\lambda_n = \sum_\rho (1-(1-1/\rho)^n)$ are positive for $n > 10^{24}$, and discuss consequences.

1. INTRODUCTION

In 1997, Xian-Jin Li derived an unexpected criterion equivalent to the Riemann Hypothesis [13]. Hi showed that the positivity of the sequence of (real) numbers, the so-called Li coefficients

$$\lambda_n = \frac{1}{\Gamma(n)} \frac{d^n}{ds^n} [s^{n-1} log(\xi(s))]_{s=1}, \tag{1.1}$$

where $\xi(s)$ is defined as

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

is equivalent to the Riemann Hypothesis.

Li derived his results in an interesting setting. He applied to $\xi(s)$ the Möbius transformation $z = 1 - \frac{1}{s}$, which maps the critical line to the unit circle, and the half-plane right to the critical line to the unit disk. In other words, he investigated

$$\phi(z) = \xi\left(\frac{1}{1-z}\right).$$

The λ_n are then the Taylor-coefficients of the logarithmic derivative of $\phi(z)$

$$\frac{\phi'(z)}{\phi(z)} = \sum_{n=0}^{\infty} \lambda_{n+1} z^n. \tag{1.2}$$

By applying the Hadamard product to $\phi(z)$, Li derived an alternative representation of the λ_n:

$$\lambda_n = \sum_\rho \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right] \tag{1.3}$$

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Here, the sum runs over the nontrivial zeros of $\zeta(s)$, paired and counted with multiplicity, and $\sum_\rho$ has to be understood as $lim_{T\to\infty}\sum_{\rho:|\rho|<T}$.

In a largely unknown work of 1992 [11] , J.B. Keiper obtained similar results. Especially he defined coefficients similar to Li's (Keiper's coefficients equal $\frac{1}{n}\lambda_n)$), but did not state as clearly as Li the equivalance of the positivity of the sequence of λ_n to Riemann Hypothesis.

In 1999, E. Bombieri and J. Lagarias [1] further investigated Li's criterion and could show, that it is closely connected to the "explicit formulae" of prime number theory, and implicitly gave a splitting of Li coefficients into ([1], Theorem 2)

$$\begin{aligned}\lambda_n = &\left[1-(log(4\pi)+\gamma)\frac{n}{2}-\sum_{j=2}^{n}(-1)^{j-1}\binom{n}{k}(1-2^{-j})\zeta(j)\right]\\ &-\sum_{j=1}^{n}\binom{n}{k}\eta_{j-1}.\end{aligned}\tag{1.4}$$

Here η_n denote coefficients which are closely connected to the Stieltjes constants γ_n ($\gamma_0=\gamma$, the Euler constant)

$$\gamma_n := \frac{(-1)^n}{n!}\lim_{x\to\infty}\left(\sum_{k\le x}\frac{1}{k}(log(k))^n-\frac{(log(x))^{n+1}}{n+1}\right).\tag{1.5}$$

$$\eta_n := \frac{(-1)^n}{n!}\lim_{x\to\infty}\left(\sum_{k\le x}\frac{\Lambda(k)}{k}(log(k))^n-\frac{(log(x))^{n+1}}{n+1}\right),\tag{1.6}$$

where $\Lambda(k)$ is the von Mangoldt function, defined as

$$\Lambda(k)=\begin{cases} log(p) & : \text{ k is a prime } p \text{ or any power of a prime, } p^n\\ 0 & : \text{ otherwise}\end{cases}$$

We have the corresponding series representations of $\zeta(s)$ resp. $\zeta'(s)/\zeta(s)$ (see e.g. [1]) :

$$\zeta(s)=\frac{1}{s-1}+\sum_{k=0}^{\infty}\gamma_k(s-1)^k \text{ and}\tag{1.7}$$

$$-\frac{\zeta'(s)}{\zeta(s)}=\frac{1}{s-1}+\sum_{k=0}^{\infty}\eta_k(s-1)^k\tag{1.8}$$

In his computation of the Li coefficients for $n\le 3300$ in [14], K. Maslanka coined the handy terms 'trend' and 'oscillations' for the first and second part of (1.4), and denoted them by $\bar\lambda_n$ and $\tilde\lambda_n$ respectively

$$\lambda_n =: \bar\lambda_n+\tilde\lambda_n.\tag{1.9}$$

His calculations showed that the trend part $\bar\lambda_n$ is strictly growing with n, while the oscillatory part $\tilde\lambda_n$ is <20 in absolute value, over the range of his computations.

So an apparent strategy for proving Riemann Hypothesis is to show that $\bar{\lambda}_n \geq |\tilde{\lambda}_n|$ for all n. Maslanka was well aware of this strategy, and gave approximative expressions for $\bar{\lambda}_n$ ([14], (2.7)).

A. Voros [20] could further precise the behavior of $\bar{\lambda}_n$, giving the asymptotic expansion ([20], (24))

$$\bar{\lambda}_n \sim \frac{n}{2}(log(n) - 1 - \gamma - log(2\pi)) + \frac{3}{4} - \sum_{k=1}^{\infty} \frac{B_{2k}}{4k} n^{1-2k} \qquad (n \to \infty) \tag{1.10}$$

(B_{2k} being the Bernoulli numbers). For our purpose it is enough to have the lower bounds on $\bar{\lambda}_n$ given by Coffey ([3], Theorem 2)

$$\bar{\lambda}_n \geq \frac{n}{2} log(n) + (\gamma - 1)\frac{n}{2} + \frac{1}{2}. \tag{1.11}$$

Regarding the oscillations $\tilde{\lambda}_n$, less progress has been made. The deceptively simple expression

$$\tilde{\lambda}_n = -\sum_{j=1}^{n} \binom{n}{k} \eta_{j-1} \tag{1.12}$$

appears to be rather intractable. Coffey in [3] could give upper bounds for $|\eta_n|$ and proved the alternation in sign for η_n [4]. Maslanka, in [14] and [15], gave explicit expressions in terms of the Stieltjes constants (1.6), not allowing for an easy estimation of $|\tilde{\lambda}_n|$, however.

In this paper we will give an upper bound to $|\tilde{\lambda}_n|$, and prove the following

Theorem 1.1. *The Li coefficients λ_n are nonnegative for all $n > 10^{24}$*

Before, we need some other results however.

2. Required results from Nevanlinna theory

Let

$$c_k(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} log|f(re^{i\theta})| e^{-ik\theta} d\theta \tag{2.1}$$

be the Fourier coefficients of a meromorphic function $log|f(re^{i\theta})|$, i.e.

$$log|f(re^{i\theta})| = \sum_{k=-\infty}^{\infty} c_k(r, f) e^{ik\theta}, \tag{2.2}$$

From [19] we then have the following

Theorem 2.1. *If $f(z)$ is meromorphic in $|z| \leq R$, with $f(0) \neq 0, \infty$, and $\{z_n\}$ the sequence of zeros, $\{w_n\}$ the sequence of poles of f, and if $log(f(z)) = \sum_{k=0}^{\infty} \alpha_k z^k$ near $z = 0$, then for $0 < r \leq R$, we have*

$$log|f(re^{i\theta})| = \sum_{k=-\infty}^{\infty} c_k(r, f) e^{ik\theta}, \tag{2.3}$$

where the $c_k(r,f)$ *are given by*

$$c_0(r,f) = log|f(0)| + \sum_{|z_n|\le r} log\frac{r}{|z_n|} - \sum_{|w_n|\le r} log\frac{r}{|w_n|} \tag{2.4}$$

For $k = 1, 2, 3, ...,$

$$\begin{aligned} c_k(r,f) =& \frac{1}{2}\alpha_k r^k + \frac{1}{2k}\sum_{|z_n|\le r}\left[\left(\frac{r}{z_n}\right)^k - \left(\frac{\bar{z}_n}{r}\right)^k\right] \\ &- \frac{1}{2k}\sum_{|z_n|\le r}\left[\left(\frac{r}{w_n}\right)^k - \left(\frac{\overline{w}_n}{r}\right)^k\right] \end{aligned} \tag{2.5}$$

For $k = 1, 2, 3, ...,$

$$c_k(r,f) = \overline{c_k(r,f)}. \tag{2.6}$$

Remark 2.2. The theorem is also called *Fundamental Theorem of Nevanlinna Theory*, or *Poisson-Jensen-Nevanlinna formula*. For a proof see e.g. [19] (Lemma 4.2), [9] (Theorem 2.4), or Nevanlinna's original work [17] .

In addition to Theorem 2.1, we need some standard concepts from *Nevanlinna theory*. Define

$$N(r,f) := \int_0^r \frac{n(t,f)}{t}dt, \tag{2.7}$$

where $n(r,f)$ denotes the number of poles of f in the disk $\{z : |z| \le r\}$, and

$$m(r,f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} log^+|f(re^{i\theta})|d\theta, \tag{2.8}$$

where $log^+(x) = max(log(x), 0)$. The *Nevanlinna characteristic*, a means to measure the growth of the function f, is then defined as

$$T(r,f) := m(r,f) + N(r,f). \tag{2.9}$$

A basic result of Nevanlinna theory is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|log|f(re^{i\theta})|\right| d\theta \le 2T(r,f) + log|f(0|. \tag{2.10}$$

3. Required results from the theory of zeta-function

K. Ford in [8] gives the following explicit estimate for the modulus of $\zeta(s)$ in the critical strip:

Theorem 3.1. *The inequality*

$$|\zeta(\sigma+it)| \le At^{B(1-\sigma)^{3/2}}(log(t))^{2/3} \qquad (t \ge 3, \frac{1}{2} \le \sigma \le 1) \tag{3.1}$$

holds with $B = 4.45$ *and* $A = 76.2$.

Fords work is an improvement of former results of Richert [18] and Cheng [5].

We also will need some estimations of the Stieltjes constants (1.5). Nan-You and Williams [16] give the following bounds:

$$|\gamma_n| \leq \begin{cases} \dfrac{2(2n)!}{n^{n+1}(2\pi)^n n!} & n \text{ odd} \\ \dfrac{4(2n)!}{n^{n+1}(2\pi)^n n!} & n \text{ even} \end{cases}$$

We adjust these estimation a bit for our needs. Taking ([6] (Lemma 12))

$$\frac{(2n)!}{n!} \leq \sqrt{2}\left(\frac{4n}{e}\right)^n \text{ for } k \geq 1$$

we get

$$|\gamma_n| \leq \frac{4(2n)!}{n^{n+1}(2\pi)^n n!} = \frac{4(2n)!}{n(\frac{5\pi n}{9})^n(\frac{18}{5})^n n!} \leq \frac{4\sqrt{2}}{n(\frac{18}{5})^n}\left(\frac{36}{5e\pi}\right)^n \leq \frac{1}{3.6^{n+1}} \tag{3.2}$$

for $n \geq 7$.

4. On the positivity of Li coefficients

It is natural to investigate the function $(s-1)\zeta(s)$ as this has the same relation to $\tilde{\lambda}_n$ as $\xi(s)$ has to λ_n, e.g.

$$\tilde{\lambda}_n = \frac{1}{\Gamma(n)}\frac{d^n}{ds^n}[s^{n-1}log((s-1)\zeta(s))]_{s=1}, \tag{4.1}$$

cf. (1.1). Again we consider this function under the Möbius transformation $z = 1 - \frac{1}{s}$, i.e. we are interested in

$$f(z) = \frac{z}{1-z}\zeta\left(\frac{1}{1-z}\right). \tag{4.2}$$

$f(z)$ is analytic in the unit disk. Similar to (1.2) we also have

$$\frac{f'(z)}{f(z)} = \frac{d}{dz}log(f(z)) = \sum_{k=1}^{\infty}\tilde{\lambda}_{n+1}z^n. \tag{4.3}$$

A necessary and sufficient condition for Riemann Hypothesis being true is that $f'(z)/f(z)$ is analytic in the unit disk. So we cannot assume regularity. We can safely say however, that $f'(z)/f(z)$ is analytic near the origin (actually it is analytic at least in $\{z : |z| \leq 1 - 10^{-23}\}$ from what we know numerically about the zeros of $\zeta(s)$, see also the discussion in Remark 4.4). We therefore can integrate the series (4.3) term-by-term, to get

$$log(f(z)) = \sum_{k=1}^{\infty}\frac{\tilde{\lambda}_{n+1}}{n+1}z^{n+1} = \sum_{k=0}^{\infty}\frac{\tilde{\lambda}_n}{n}z^n, \tag{4.4}$$

which is also analytic near the origin.

Next we want to get some information on the growth of $f(z)$ in the unit disk. As $f(z)$ is unbounded, we will investigate $(z-1)^5 f(z) = z(1-z)^4\zeta\left(\frac{1}{1-z}\right)$. First we show

Theorem 4.1. *On the critical line,*

$$\left|\frac{\zeta(s)}{s^4}\right| < 23.4080. \tag{4.5}$$

Remark 4.2. Apparently, the desired result would be the value at $s=\frac{1}{2}$, i.e. $\left|\frac{\zeta(s)}{s^4}\right| < 16\left|\zeta\left(\frac{1}{2}\right)\right| \approx 23.3657$. The given estimation is good enough for our purpose however.

Proof. We will do the estimation in three steps, first for $t \geq 3$, then for $0.5 \leq t \leq 3$, and finally for $t \leq 0.5$.

From (3.1) we have, for $s = \frac{1}{2} + it$ and $t \geq 3$,

$$\begin{aligned}\left|\frac{s-1}{s^5}\zeta(s)\right| = \left|\frac{\zeta(s)}{s^4}\right| &\leq \frac{76.2t^{4.45(1/2)^{3/2}} log^{2/3}(t)}{t^4} \\ &\leq \frac{76.2t^{1.6} log^{2/3}(t)}{t^4} = 76.2t^{-2.4} log^{2/3}(t) \leq 5.8089.\end{aligned} \tag{4.6}$$

The function on the right hand side of the above equation is strictly decreasing with growing t, and therefore has its maximum at $t=3$ with a value ≤ 5.8089, which is well below the bound given in our theorem.

Next we will do the estimation for $0.5 \leq t \leq 3$. For $t \leq 3$, we investigate the representation (1.10) of $\zeta(s)$ as a series about 1:

$$(s-1)\zeta(s) = 1 + \sum_{k=0}^{\infty} \gamma_k (s-1)^{k+1}.$$

Using the estimation (3.7), $|\gamma_n| \leq \frac{1}{3.6^{n+1}}$, which is valid for $n \geq 7$, we get

$$|(s-1)\zeta(s)| \leq 1 + \sum_{k=0}^{\infty}\left|\frac{s-1}{3.6}\right|^{k+1} = \sum_{k=0}^{\infty}\left|\frac{s-1}{3.6}\right|^{k} = \frac{18}{18-5|s-1|} \tag{4.7}$$

for $|s-1| < 3.6$, or if $s = \frac{1}{2} + it$, for $t < \sqrt{3.6^2 - (\frac{1}{2})^2} \approx 3.5651$, i.e. for $t > 3$.

For a more detailed estimation of the series, we use the maximal modulus for points on the critical line which we are interested in currently, i.e. for $t \leq 3$, which is $|\frac{1}{2} + 3it| = \sqrt{3^2 + (\frac{1}{2})^2} \approx 3.0414$. We get

$$\begin{aligned}|(s-1)\zeta(s)| &\leq |1+\gamma_0(s-1)+\gamma_1(s-1)^2| + \sum_{k=2}^{6}|\gamma_k||s-1|^{k+1} + \sum_{k=7}^{\infty}\left|\frac{s-1}{3.6}\right|^{k+1} \\ &= |1+\gamma_0(s-1)+\gamma_1(s-1)^2| + \sum_{k=2}^{6}|\gamma_k||s-1|^{k+1} + \left|\frac{s-1}{3.6}\right|^{6}\sum_{k=0}^{\infty}\left|\frac{s-1}{3.6}\right|^{k} \\ &= |1+\gamma_0(s-1)+\gamma_1(s-1)^2| + \sum_{k=2}^{6}|\gamma_k||s-1|^{k+1} + \left|\frac{s-1}{3.6}\right|^{6}\frac{18}{18-5|s-1|}\end{aligned}$$

We put $s = \frac{1}{2} + it$:

$$\begin{aligned}\left|\left(-\frac{1}{2}+it\right)\zeta\left(\frac{1}{2}+it\right)\right| &\leq \left|1+\gamma_0\left(-\frac{1}{2}+it\right)+\gamma_1\left(-\frac{1}{2}+it\right)^2\right| \\ &\quad + \sum_{k=2}^{6}|\gamma_k||3.1|^{k+1} + \left|\frac{3.1}{3.6}\right|^6 \frac{18}{18-5\cdot 3.1} \\ &\leq \left|1+\gamma_0\left(-\frac{1}{2}+it\right)+\gamma_1\left(-\frac{1}{2}+it\right)^2\right| + 0.2105 + 2.9355 \\ &= \left|1+\gamma_0\left(-\frac{1}{2}+it\right)+\gamma_1\left(-\frac{1}{2}+it\right)^2\right| + 3.1460\end{aligned}$$

Now, for $0.5 \leq t \leq 3$,

$$\frac{\left|1+\gamma_0\left(-\frac{1}{2}+it\right)+\gamma_1\left(-\frac{1}{2}+it\right)^2\right|+3.1460}{\left|\frac{1}{2}+it\right|^5}$$

is a strictly decreasing polynomial with maximal value at $t = 0.5$, which approximately evaluates to 22.0661, just below our desired bound.

Finally, we handle the case $0 \leq t \leq \frac{1}{2}$. The remainder term of our estimation now evaluates to, using the maximal modulus, at $t = \frac{1}{2}$, $|\frac{1}{2} + it - 1| \leq |\frac{1}{2} + i\frac{1}{2}| = \sqrt{\frac{1}{2}} \approx 0.7071$,

$$\sum_{k=2}^{6}|\gamma_k||0.71|^{k+1} + \left|\frac{0.71}{3.6}\right|^6 \frac{18}{18-5\cdot 0.71} \approx 0.001838 + 0.000073 = 0.001903.$$

This yields a maximum value for

$$\frac{\left|1+\gamma_0\left(-\frac{1}{2}+it\right)+\gamma_1\left(-\frac{1}{2}+it\right)^2\right|+0.001903}{\left|\frac{1}{2}+it\right|^5}$$

at $t = 0$, of 23.4080, as stated.

Symmetry of $\zeta(s)$ about the real axis gives the same results for negative t. □

Corollary 4.3. *For $|z| <= 1$,*

$$\left|z(1-z)^4\zeta\left(\frac{1}{1-z}\right)\right| <= 23.4080. \tag{4.8}$$

Proof. From the maximum modulus principle it is enough to find the maximum on the unit circle, which is the image of the critical line under the transformation $z = 1 - 1/s$. Inverting this transformation, $(z-1)^5 f(z)$ maps back to $\frac{s-1}{s^5}\zeta(s)$, for which, on the critical line,

$$\left|\frac{s-1}{s^5}\zeta(s)\right| = \left|\frac{\zeta(s)}{s^4}\right|$$

The statement follows then from Theorem 4.1. □

Corollary 4.4. *There is only a finite number of nontrivial zeros of $\zeta(s)$ violating Riemann Hypothesis. More precisely, there are not more than 374 nontrivial zeros with imaginary part $\neq \frac{1}{2}$.*

Proof. We will apply Theorem 2.1 to $f(z) = \frac{z}{1-z}\zeta(\frac{1}{1-z})$. Regarding the preliminaries, $f(z)$ is analytic in the unit disk, and therefore meromorphic in $|z| \leq R = 1$, with $f(0) = 1 \neq 0, \infty$. $f(z)$ has no poles in the unit disk, and any zero would be an exception to Riemann Hypothesis.

Therefore we can apply (2.4), which states

$$c_0(r,f) = log|f(0)| + \sum_{|z_n|\leq r} log\frac{r}{|z_n|} - \sum_{|w_n|\leq r} log\frac{r}{|w_n|}$$

or

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} log|f(re^{i\theta})|d\theta = \sum_{|z_n|\leq r} log\frac{r}{|z_n|}, \tag{4.9}$$

where we have dropped the term regarding the poles, which is not applicable. This is of course Jensen's formula in its standard form.

Further, we have from Corollary 4.3

$$|(1-z)^5 f(z)| \leq 23.4080$$

This means, for the integral in (4.10), we have the estimation

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} log|f(re^{i\phi})|d\phi &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} log\left(\frac{23.4080}{|1-re^{i\phi}|^5}\right)d\phi \\
&\leq log(23.4080) + \frac{1}{\pi}\int_0^{\pi} log|csc(\phi)|^5 d\phi \\
&= log(23.4080) + log(32) \\
&= log(749.056) \\
&\approx 6.6188
\end{aligned}$$

(see [7] for the estimation to $csc(\phi)$). Putting this into (4.10), we get

$$\sum_{|z_n|\leq r} log\frac{r}{|z_n|} \quad\leq\quad log(749.056)$$

or

$$\prod_{|z_n|\leq r}\frac{r}{|z_n|} \quad\leq\quad 749.056 \tag{4.10}$$

Note, that we can chose r arbitrarily close to 1. If we gradually increase r, we finally will cover all zeros of $\zeta(s)$ off the critical line (otherwise there would be an infinite number of them, which would mean that $f(z) = 0$ identically, from the

Identity theorem. From numerical data (e.g. [10]) we know, that there are no exceptional zeros $\rho = \sigma + it$ up to $t = 2445999556030 \approx 2.45 \cdot 10^{12}$. This means for optional zeros z_n to have a modulus close to 1 by a magnitude $< 10^{-20}$. I.e. each of the factors on the left is a negligible amount bigger than 1. As, for symmetry reasons, any exceptional root is counted twice, this finally means that there are no more than $\lfloor \frac{749.056}{2} \rfloor = 374$ exceptional zeros.

□

By applying further parts of Theorem 2.1 to $f(z)$, we can prove the following

Theorem 4.5. *For $n > 10^{24}$ all Li coefficients λ_n are positive.*

Proof. Preconditions of Theorem 2.1 are fulfilled for $f(z) = \frac{z}{1-z}\zeta(\frac{1}{1-z})$ (see the proof of Corollary 4.3). Further, from (4.4),

$$log(f(z)) = \sum_{k=0}^{\infty} \frac{\tilde{\lambda}_k}{k} z^k$$

near $z = 0$. (2.5) now states, for $k = 1, 2, 3, ...,$

$$\begin{aligned} c_k(r, f) =& \frac{1}{2}\frac{\tilde{\lambda}_k}{k} r^k + \frac{1}{2k} \sum_{|z_n| \le r} \left[\left(\frac{r}{z_n}\right)^k - \left(\frac{\bar{z}_n}{r}\right)^k \right] \\ &- \frac{1}{2k} \sum_{|z_n| \le r} \left[\left(\frac{r}{w_n}\right)^k - \left(\frac{\overline{w}_n}{r}\right)^k \right] \end{aligned}$$

or, dropping again the expressions regarding the poles

$$\frac{1}{2}\frac{\tilde{\lambda}_k}{k} r^k = \frac{1}{2\pi}\int_{-\pi}^{\pi} log|f(re^{i\theta})|e^{-ik\theta}d\theta - \frac{1}{2k} \sum_{|z_n| \le r} \left[\left(\frac{r}{z_n}\right)^k - \left(\frac{\bar{z}_n}{r}\right)^k \right]$$

Then

$$\begin{aligned} \left| \frac{1}{2}\frac{\tilde{\lambda}_k}{k} r^k \right| =& \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} log|f(re^{i\theta})|e^{-ik\theta}d\theta - \frac{1}{2k} \sum_{|z_n| \le r} \left[\left(\frac{r}{z_n}\right)^k - \left(\frac{\bar{z}_n}{r}\right)^k \right] \right| \\ \le& \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} log|f(re^{i\theta})|e^{-ik\theta}d\theta \right| + \left| \frac{1}{2k} \sum_{|z_n| \le r} \left[\left(\frac{r}{z_n}\right)^k - \left(\frac{\bar{z}_n}{r}\right)^k \right] \right| \\ \le& \frac{1}{2\pi}\int_{-\pi}^{\pi} |log|f(re^{i\theta})||d\theta + O(10^{-50}) \end{aligned}$$

or

$$\left|\tilde{\lambda}_k\right| \le \frac{2k}{r^k} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |log|f(re^{i\theta})||d\theta + O(10^{-50}) \right) \tag{4.11}$$

As r is arbitrarily close to 1 we can ignore the factor $1/r^k$.

Now we need to compute the Nevanlinna characteristic for $f(z)$, acc. to (2.7) - (2.9). We already have, from the estimations in to proof to Corollary 4.4

$$\begin{aligned} m(r,f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} ln^{+}|f(re^{i\phi})|d\phi \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} log|f(re^{i\phi})|d\phi \\ &\leq log(23.4080) + log(32) = log(749.056) \approx 6.6188. \end{aligned}$$

Further, as $f(z)$ has no poles in the unit disk, $n(t,f) = 0$, and therefore

$$N(r,f) = \int_0^r \frac{n(t,f)}{t} dt = 0$$

From that, the Nevanlinna characteristic computes to

$$T(r,f) = m(r,f) + N(r,f) \leq log(749.056) \approx 6.6188.$$

From (2.10) we finally get the estimation

$$|c_k(r,f)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} |log|f(re^{i\theta})||d\theta \leq 2T(r,f) \approx 13.2376$$

This means, by (4.11), that

$$\left|\tilde{\lambda}_k\right| \leq 2k \cdot 13.2376 = 26.4752 \cdot k \tag{4.12}$$

With this result, we can continue to estimate the Li coefficients λ_k. We have (1.12):

$$\lambda_k = \bar{\lambda}_k + \tilde{\lambda}_k$$

With Coffey's lower bound estimation (1.14)

$$\bar{\lambda}_k \geq 0.5(1 + k(log(k) - (1-\gamma)k)) \approx 0.5 + 0.5k \cdot log(k) - 0.2114k > 0$$

we get

$$\begin{aligned} \lambda_k &\geq \bar{\lambda}_k - |\tilde{\lambda}_k| \\ &\geq 0.5 + 0.5k \cdot log(k) - 0.2114k - 26.4752k \\ &= 0.5k \cdot log(k) - 26.6866k \end{aligned}$$

which is nonnegative for $k \geq e^{53.3732}$, or $k > 10^{23.1797}$. In other words, the Li criterion is fulfilled for all $k > 10^{24}$. □

Remark 4.6. The zeros of $\zeta(s)$ are known up to a height of $Z_{max} = 2.45 \cdot 10^{12}$ on the critical line. It is known ([20], p. 8, [2], p. 441), that a violation to the positivity of λ_n could only occur at $n > Z_{max}^2 = 5.67 \cdot 10^{24}$. As we could not find a detailed account of the argument, we try elaborate this here.

Assume there would be a zero ρ of $\zeta(s)$ off the critical line, just above the known zeros, e.g. $\rho = \frac{1}{2} + \delta + i \cdot 2.4 \cdot 10^{12}$, where $|\delta| < \frac{1}{2}$ (there are no zeros outside the critical strip). Then this zero would be mapped by $z = 1 - 1/s$ to a point close to 1, with coordinates:

$$
\begin{aligned}
1-\frac{1}{\rho} &= \left(1-\frac{1/2+\delta}{(1/2+\delta)^2+2.4^2\cdot 10^{24}}\right)+i\frac{2.4\cdot 10^{12}}{(1/2+\delta)^2+2.4^2\cdot 10^{24}} \\
&\approx \left(1-\frac{1/2+\delta}{2.4^2\cdot 10^{24}}\right)+i\frac{1}{2.4\cdot 10^{12}},
\end{aligned} \tag{4.13}
$$

Now consider the Li coefficient

$$\lambda_n = \sum_\rho \left[1-\left(1-\frac{1}{\rho}\right)^n\right].$$

The contributions to λ_n from the zeros on the critical line are positive (see the proof of Li for his criterion, in [13]). Regarding the contribution of ρ, we actually have four roots to consider: $\frac{1}{2}\pm\delta\pm i\cdot 2.4\cdot 10^{12}$, due to the symmetry properties of $\zeta(s)$. The pair $\frac{1}{2}+\delta\pm i\cdot 2.4\cdot 10^{12}$ maps to the interior of the disk, with modulus < 1, and contributes a positive value to λ_n. The only chance we have to get λ_n negative comes from $\frac{1}{2}-\delta\pm i\cdot 2.4\cdot 10^{12}$, which map to just outside of the unit disk, therefore having a modulus > 1. More precisely, this modulus is

$$
\begin{aligned}
\left|1-\frac{1}{\frac{1}{2}-\delta\pm i\cdot 2.4\cdot 10^{12}}\right| &= \sqrt{\left(1-\frac{1/2-\delta}{2.4^2\cdot 10^{24}}\right)^2+\left(\frac{1}{2.4\cdot 10^{12}}\right)^2} \\
&= \sqrt{\frac{(2.4^2\cdot 10^{24}-(1/2-\delta))^2}{2.4^4\cdot 10^{48}}+\frac{2.4^2\cdot 10^{24}}{2.4^4\cdot 10^{48}}} \\
&= \sqrt{\frac{2.4^4\cdot 10^{48}+2\delta\cdot 2.4^2\cdot 10^{24}+(1/2-\delta)^2}{2.4^4\cdot 10^{48}}} \\
&\approx \sqrt{1+\frac{2\delta}{2.4^2\cdot 10^{24}}} \\
&\geq 1+\frac{\delta}{2.4^2\cdot 10^{24}}
\end{aligned} \tag{4.14}
$$

Applying again Bernoulli's estimation, and ignoring $Arg(\rho)$ for a moment, i.e. assuming $1-(1-1/\rho)$ being real, we get

$$1-\left(1-\frac{1}{\rho}\right)^n \leq 1-\left(1+\frac{\delta}{2.4^2\cdot 10^{24}}\right)^n \leq 1-exp\left(\frac{n/2}{2.4^2\cdot 10^{24}}\right)$$

or generally,

$$1-\left(1-\frac{1}{\rho}\right)^n \leq 1-exp\left(\frac{n/2}{\mathrm{Im}(\rho)^2}\right) \tag{4.15}$$

We see already, that n has to be at least of order $\mathrm{Im}(\rho)^2$ or $5.7\cdot 10^{24}$ to achieve a negative contribution to λ_n.

But counting to the sum of zeros is not the modulus, but (the real part of) the pair $(1-1/\rho)^n+(1-1/\bar{\rho})^n$, which is approximately the modulus times $2cos(nArg(1-1/\rho))\approx 2cos(n\cdot arcsin(\mathrm{Im}(1-1/\rho)))\approx 2cos(n\cdot\mathrm{Im}(1-1/\rho))\approx 2-2n\cdot\mathrm{Im}(1-1/\rho)$. So we get

$$2-\left(1-\frac{1}{\rho}\right)^n-\left(1-\frac{1}{\bar{\rho}}\right)^n \leq 2-2\left(1-\frac{n}{2.4\cdot 10^{12}}\right)\left(1-exp\left(\frac{n/2}{2.4^2\cdot 10^{24}}\right)\right)$$
$$=\frac{n^2-2n\cdot 2.4^2\cdot 10^{24}}{2.4^3\cdot 10^{36}}-exp\left(\frac{n/2}{2.4^2\cdot 10^{24}}\right)$$

Finally, we have to consider the size of the trend $\bar{\lambda}_n$, which is $\approx \frac{1}{2}k\cdot log(k)$ and which has to be compensated in order to get a negative λ_n. This means we have to resolve

$$\frac{1}{2}n\cdot log(n)+\frac{n^2-2n\cdot 2.4^2\cdot 10^{24}}{2.4^3\cdot 10^{36}}-exp\left(\frac{n/2}{2.4^2\cdot 10^{24}}\right)<0$$

for n, which yields the condition $n>7.5\cdot 10^{26}$ (the evaluation also shows that we can ignore the contribution of $\mathrm{Arg}(\rho)$, as is to be expected by the properties of the exponential function).

For our purpose it is safe to say that, given an exceptional zero with imaginary part $>2.4\cdot 10^{12}$, n has to be $>10^{25}$ for λ_n to become negative.

We now can state

Theorem 4.7. *Riemann hypothesis is true, i.e. all nontrivial zeros of $\zeta(s)$ lie on the critical line* $\mathrm{Re}(s)=\frac{1}{2}$.

Proof. By Theorem 4.5 and Remark 4.6, Li's criterion is proved, which is equivalent to Riemann Hypothesis [13]. □

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