Natural Squarefree Numbers: Statistical Properties II.

Helmut Preininger
1200 Vienna
Austria
January 12, 2018
mailto: helmut.preininger@chello.at
hosted at: www.vixra.org

Abstract

This paper is an appendix of *Natural Squarefree Numbers: Statistical Properties.* [PR04]. In this appendix we calculate the probability of c is squarefree, where c=a*b, a is an element of the set X and b is an element of the set Y.

1 Notation and Results

Notation 1. ***[Subsets of*** $\mathbb{N}$***]***

N *... natural (i.e.* $\mathbb{N}$*),*
NE *... natural and even,*
NO *... natural and odd*

S *... squarefree (i.e.* $a \in \mathbb{N}$ *and a is squarefree),*
SE *... squarefree and even,*
SO *... squarefree and odd*

F *... none squarefree (i.e.* $a \in \mathbb{N}$ *and a is none squarefree),*
FE *... none squarefree and even,*
FO *... none squarefree and odd*

Notation 2. ***[Probabilities]***
P_X*: Probability:* $a \in X$ *and is squarefree*
$M_{X,Y}$*: Let* $a \in X$*,* $b \in Y$ *and* $c = a * b$*. Probability: c is squarefree.*

1.1 Results

Note, for our numerical calculations we use the programming language PureBasic (see http://www.purebasic.com).

1.1.1 Probability of $a \in X$ and a is squarefree

(see [PRE04] section 3.1,3.2,3.3)

choosen set X	N	NE	NO
ProbabilityP_X	P_N	$\frac{2P_N}{3}$	$\frac{4P_N}{3}$

where $P_N := \prod_i^\infty \left(1 - \frac{1}{p_i^2}\right) = \left(1 - \sigma\left(\frac{1}{p_i^2}\right)\right) = \frac{6}{\pi^2}$, p_i prim.

1.1.2 Probability of $a \in X$, $b \in Y$ and $a * b$ is squarefree

Obviously hold
a is even and b is even then $a * b$ is none squarefree
a is none squarefree or b is none squarefree then $a * b$ is none squarefree.

$+$	N	NE	NO	S	SE	SO	F	FE	FO
N	$P_N^2 M_{S,S}$	$\frac{P_N^2 M_{S,S}}{2}$	$\frac{3P_N^2 M_{S,S}}{2}$	$P_N M_{S,S}$	$\frac{3P_N M_{S,S}}{4}$	$\frac{9P_N M_{S,S}}{8}$	0	0	0
NE		0	$P_N^2 M_{S,S}$	$\frac{P_N M_{S,S}}{2}$	0	$\frac{3P_N M_{S,S}}{4}$	0	0	0
NO			$2P_N^2 M_{S,S}$	$\frac{3P_N MS,S}{2}$	$\frac{3P_N M_{S,S}}{2}$	$\frac{3P_N M_{S,S}}{2}$	0	0	0
S				$M_{S,S}$	$\frac{3M_{S,S}}{4}$	$\frac{9M_{S,S}}{8}$	0	0	0
SE					0	$\frac{9M_{S,S}}{8}$	0	0	0
SO						$\frac{9M_{S,S}}{8}$	0	0	0
F							0	0	0
FE								0	0
FO									0

where $M_{S,S} = \left(1 - \sigma\left(\frac{1}{(p_i+1)^2}\right)\right) = \prod_{i=1}^\infty (1 - \frac{1}{(p_i+1)^2}) \approx 0.775...$, p_i prim.

2 Multiplication a*b: Experimental Data, Numerical and Analytical Calculation

Let M the probability of $a * b$ is squarefree, for various sets of a and b.

Experimental Data: *Note, this data gives only a hint, we do not invest much work.* We choose two intervals of $\sqrt{w}$ consecutive number of the appropriate form, both with random (between $1..N$) starting value. Then we test w pairs of numbers, given by the intervals. We repeat this procedure 36 times.

Numerical Calculation: *Note, this data gives only a hint, we use only the 8 byte Double* *datatype (see: IEEE 754 standard)* . We estimate a lower and upper bound for the probability (we use $\sigma_1^n(\gamma)$, $\sigma_2^n(\gamma)$, $\sigma_3^n(\gamma)$, $\sigma_4^n(\gamma)$ with the summation over the first n primes), (see [PRE04] section 2) . Let p prime and $f(a)$ the probability of $a \equiv m \mod p^2$ where $0 \leq m < p^2$. To estimate γ we count all possible pairs m_a, m_b of $f(a)$ and $f(b)$ and all sufficient pairs m_a, m_b of $f(a)$ and $f(b)$ with $(m_a + m_b) \equiv 0 \mod p^2$. We get

$$\gamma_i = \frac{\text{number of all sufficient pairs}}{\text{number of all possible pairs}}$$

2.1 a is natural, b is natural

Experimental Data:

Table: a is natural times natural

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.289186	0.000023	0.004825	+/- 0.000804	+/- 0.002412
40000	1e+4	0.289001	0.000007	0.002574	+/- 0.000429	+/- 0.001287
10000	8e+4	0.288186	0.000016	0.003967	+/- 0.000661	+/- 0.001984
90000	8e+4	0.286467	0.000005	0.002160	+/- 0.000360	+/- 0.001080

Analytical Calculation: We have (see section 2.14, 1.1.2)

$$M_{N,N} = P_N^2 M_{S,S}$$

2.2 a is natural, b is natural and even

Experimental Data:

Table: a is natural times natural and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.143639	0.000010	0.003172	+/- 0.000529	+/- 0.001586
40000	1e+4	0.143693	0.000002	0.001380	+/- 0.000230	+/- 0.000690
10000	8e+4	0.140450	0.000015	0.003907	+/- 0.000651	+/- 0.001954
90000	8e+4	0.143159	0.000002	0.001431	+/- 0.000238	+/- 0.000715

Analytical Calculation: We have (see section 2.15, 1.1.2)

$$M_{N,NE} = P_N \cdot \frac{2P_N}{3} \cdot M_{S,SE} = \frac{P_N^2 M_{S,S}}{2}$$

2.3 a is natural, b is natural and odd

Experimental Data:

Table: a is natural times natural and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.425078	0.000140	0.011839	+/- 0.001973	+/- 0.005919
40000	1e+4	0.428042	0.000012	0.003426	+/- 0.000571	+/- 0.001713
10000	8e+4	0.423497	0.000161	0.012682	+/- 0.002114	+/- 0.006341
90000	8e+4	0.431666	0.000012	0.003428	+/- 0.000571	+/- 0.001714

Analytical Calculation: We have (see section 2.16, 1.1.2)

$$M_{N,NO} = P_N \cdot \frac{4P_N}{3} \cdot M_{S,SO} = \frac{3P_N^2 M_{S,S}}{2}$$

2.4 a is natural, b is squarefree

Experimental Data:

Table: a is natural times squarefree

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.478669	0.000041	0.006419	+/- 0.001070	+/- 0.003210
40000	1e+4	0.473551	0.000062	0.007865	+/- 0.001311	+/- 0.003933
10000	8e+4	0.474381	0.000093	0.009632	+/- 0.001605	+/- 0.004816
90000	8e+4	0.472101	0.000030	0.005512	+/- 0.000919	+/- 0.002756

Analytical Calculation: We get (see section 2.14)

$$M_{N,S} = P_N M_{S,S}$$

2.5 a is natural, b is squarefree and even

Experimental Data:

Table: a is natural times squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.351508	0.000036	0.005968	+/- 0.000995	+/- 0.002984
40000	1e+4	0.354625	0.000021	0.004584	+/- 0.000764	+/- 0.002292

10000	8e+4	0.356961	0.000086	0.009250	+/- 0.001542	+/- 0.004625
90000	8e+4	0.352621	0.000016	0.003947	+/- 0.000658	+/- 0.001974

Analytical Calculation: We have (see section 2.15)

$$M_{N,SE} = P_N \cdot M_{S,SE} = \frac{3P_N M_{S,S}}{4}$$

2.6 a is natural, b is squarefree and odd

Experimental Data:

Table: a is natural times squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.528844	0.000195	0.013978	+/- 0.002330	+/- 0.006989
40000	1e+4	0.537424	0.000052	0.007209	+/- 0.001201	+/- 0.003604
10000	8e+4	0.533633	0.000243	0.015592	+/- 0.002599	+/- 0.007796
90000	8e+4	0.529081	0.000014	0.003760	+/- 0.000627	+/- 0.001880

Analytical Calculation: We have (see section 2.16)

$$M_{N,SO} = P_N \cdot M_{S,SO} = \frac{9P_N M_{S,S}}{8}$$

2.7 a is natural and even, b is natural and odd

Experimental Data:

Table: a is natural and even times natural and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.279114	0.000140	0.011823	+/- 0.001971	+/- 0.005912
40000	1e+4	0.284399	0.000006	0.002488	+/- 0.000415	+/- 0.001244
10000	8e+4	0.281661	0.000074	0.008628	+/- 0.001438	+/- 0.004314
90000	8e+4	0.284733	0.000018	0.004296	+/- 0.000716	+/- 0.002148

Analytical Calculation: We have (see section 2.17, 1.1.2)

$$M_{NO,NE} = M_{NE,NO} = \frac{2P_N}{3} \cdot \frac{4P_N}{3} \cdot M_{SE,SO} = P_N^2 M_{S,S}$$

Remark 3. .
$M_{N,N} = (1/4)M_{NO,NO} + (1/4)M_{NE,NE} + (1/4)M_{NE,NO} + (1/4)M_{NO,NE}$

2.8 a is natural and even, b is squarefree

Experimental Data:

Table: a is natural and even times squarefree

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.236064	0.000008	0.002758	+/- 0.000460	+/- 0.001379
40000	1e+4	0.233300	0.000032	0.005692	+/- 0.000949	+/- 0.002846
10000	8e+4	0.237289	0.000024	0.004871	+/- 0.000812	+/- 0.002436
90000	8e+4	0.236981	0.000002	0.001301	+/- 0.000217	+/- 0.000650

Analytical Calculation: We have (see section 2.15, 1.1.2)

$$M_{NE,S} = \frac{2P_N}{3} \cdot M_{S,SE} = \frac{P_N M_{S,S}}{2}$$

2.9 a is natural and even, b is squarefree and odd

Experimental Data:

Table: a is natural and even times squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.356386	0.000080	0.008935	+/- 0.001489	+/- 0.004468
40000	1e+4	0.353802	0.000030	0.005463	+/- 0.000910	+/- 0.002731
10000	8e+4	0.352817	0.000394	0.019855	+/- 0.003309	+/- 0.009927
90000	8e+4	0.357616	0.000010	0.003105	+/- 0.000517	+/- 0.001552

Analytical Calculation: We have (see section 2.17, 1.1.2)

$$M_{NE,SO} = \frac{2P_N}{3} \cdot M_{SE,SO} = \frac{3P_N M_{S,S}}{4}$$

2.10 a is natural and odd, b is natural and odd

Experimental Data:

Table: a is natural and odd times natural and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.567639	0.000098	0.009923	+/- 0.001654	+/- 0.004962
40000	1e+4	0.579964	0.000028	0.005270	+/- 0.000878	+/- 0.002635

```
10000 |    8e+4 | 0.572047 | 0.000150 | 0.012236 | +/- 0.002039 | +/- 0.006118
90000 |    8e+4 | 0.576509 | 0.000006 | 0.002498 | +/- 0.000416 | +/- 0.001249
-------------------------------------------------------------------------------
```

Analytical Calculation: We have (see section 2.18, 1.1.2)

$$M_{NO,NO} = \frac{4P_N}{3} \cdot \frac{4P_N}{3} \cdot M_{SO,SO} = 2P_N^2 M_{S,S}$$

2.11 a is natural and odd, b is squarefree

Experimental Data:

```
Table: a is natural and odd times squarefree
    w |       N |     mean |      Var |  Std.Var |       68.3 % |       99.7 %
-------------------------------------------------------------------------------
10000 |    1e+4 | 0.711942 | 0.000101 | 0.010047 | +/- 0.001674 | +/- 0.005023
40000 |    1e+4 | 0.704685 | 0.000075 | 0.008645 | +/- 0.001441 | +/- 0.004323
10000 |    8e+4 | 0.712006 | 0.000110 | 0.010504 | +/- 0.001751 | +/- 0.005252
90000 |    8e+4 | 0.707492 | 0.000028 | 0.005310 | +/- 0.000885 | +/- 0.002655
-------------------------------------------------------------------------------
```

Analytical Calculation: We have (see section 2.16, 1.1.2)

$$M_{NO,SO} = \frac{4P_N}{3} \cdot M_{S,SO} = \frac{3P_N MS, S}{2}$$

2.12 a is natural and odd, b is squarefree and even

Experimental Data:

```
Table: a is natural and odd times squarefree and even
    w |       N |     mean |      Var |  Std.Var |       68.3 % |       99.7 %
-------------------------------------------------------------------------------
10000 |    1e+4 | 0.704339 | 0.000083 | 0.009096 | +/- 0.001516 | +/- 0.004548
40000 |    1e+4 | 0.708424 | 0.000094 | 0.009715 | +/- 0.001619 | +/- 0.004857
10000 |    8e+4 | 0.707011 | 0.000150 | 0.012266 | +/- 0.002044 | +/- 0.006133
90000 |    8e+4 | 0.704445 | 0.000037 | 0.006111 | +/- 0.001018 | +/- 0.003055
-------------------------------------------------------------------------------
```

Analytical Calculation: We have (see section 2.17, 1.1.2)

$$M_{NO,SE} = \frac{4P_N}{3} \cdot M_{SO,SE} = \frac{3P_N M_{S,S}}{2}$$

2.13 a is natural and odd, b is squarefree and odd

Experimental Data:

Table: a is natural and odd times squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.711336	0.000124	0.011117	+/- 0.001853	+/- 0.005559
40000	1e+4	0.703386	0.000049	0.006988	+/- 0.001165	+/- 0.003494
10000	8e+4	0.707878	0.000193	0.013876	+/- 0.002313	+/- 0.006938
90000	8e+4	0.711050	0.000023	0.004752	+/- 0.000792	+/- 0.002376

Analytical Calculation: We have (see section 2.18, 1.1.2)

$$M_{NO,SO} = \frac{4P_N}{3} \cdot M_{SO,SO} = \frac{3P_N M_{S,S}}{2}$$

2.14 a is squarefree, b is squarefree

Experimental Data:

Table: a is squarefree times squarefree

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.774867	0.000006	0.002482	+/- 0.000414	+/- 0.001241
40000	1e+4	0.775826	0.000001	0.000805	+/- 0.000134	+/- 0.000402
10000	8e+4	0.774342	0.000004	0.001877	+/- 0.000313	+/- 0.000939
90000	8e+4	0.775916	0.000002	0.001422	+/- 0.000237	+/- 0.000711

Numerical Calculation: Since a is squarefree, $a \equiv 0 \mod p^2$ does not exist and we have $f(a) = \{0, 1, \ldots, 1\}$ (see [PRE04] section 4.25). We have $(p-1)^2$ sufficient pairs and $(p^2-1)^2$ possible pairs and therefore

$$M_{S,S} = 1 - \sigma\left(\frac{1}{(p+1)^2}\right)$$

Calculation of the Error-Term R_n: Let $\gamma = (1/(p+1)^2)$

$$\sigma^n(\gamma) \leq \sigma(\gamma) \leq \sigma^n(\gamma) + \sum_{i=\frac{p_n+1}{2}}^{\infty} \frac{1}{((2i+1)+1)^2} = \sigma^n(\gamma) + \frac{1}{4}\left(\frac{\pi^2}{6} - \sum_{i=1}^{\frac{p_n+1}{2}} \frac{1}{i^2}\right)$$

and we get

$$R_n = \frac{1}{4}\left(\frac{\pi^2}{6} - \sum_{i=1}^{\frac{p_n+1}{2}} \frac{1}{i^2}\right)$$

Summation over the first 360 primes gives

$$0.7756918339 \leq M_{S,S} = 1 - \sigma(\gamma) \leq 0.7759702486$$

2.15 a is squarefree, b is squarefree and even

Experimental Data:

Table: a is squarefree times squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.582106	0.000052	0.007192	+/- 0.001199	+/- 0.003596
40000	1e+4	0.581292	0.000029	0.005344	+/- 0.000891	+/- 0.002672
10000	8e+4	0.585214	0.000072	0.008497	+/- 0.001416	+/- 0.004248
90000	8e+4	0.585136	0.000014	0.003753	+/- 0.000625	+/- 0.001876

Analytical Calculation: We have $f(a) = \{0, 1, \ldots, 1\}$ and

$$f(b) = \begin{cases} \{0, 0, 1, 0\}, & p = 2 \\ \{0, 1, \ldots, 1\}, & \text{otherwise} \end{cases}$$

Case $p = 2$: We have 3 possible pairs and 1 sufficient pair therefore $1/3$.
Case $p > 2$: We have $(p^2 - 1)^2$ possible pairs and $(p - 1)^2$ sufficient pairs.

We get $\gamma = \{1/(p_1 + 1), 1/(p_2 + 1)^2, \ldots\}$ and (see [PRE04] section 4.26)

$$M_{S,SE} = \frac{3M_{S,S}}{4}$$

2.16 a is squarefree, b is squarefree and odd

Experimental Data:

Table: a is squarefree times squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.869822	0.000004	0.002104	+/- 0.000351	+/- 0.001052
40000	1e+4	0.872924	0.000004	0.002090	+/- 0.000348	+/- 0.001045
10000	8e+4	0.872511	0.000005	0.002172	+/- 0.000362	+/- 0.001086
90000	8e+4	0.872411	0.000001	0.001143	+/- 0.000190	+/- 0.000571

Analytical Calculation: We have $f(a) = \{0, 1, \ldots, 1\}$ and

$$f(b) = \begin{cases} \{0,1,0,1\}, & p = 2 \\ \{0,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

Case $p = 2$: We have 6 possible pairs and 0 sufficient pair therefore 0.
Case $p > 2$: We have $(p^2 - 1)^2$ possible pairs and $(p-1)^2$ sufficient pairs.

We get $\gamma = \{0, 1/(p_2+1)^2, \ldots\}$ and (see [PRE04] section 4.27)

$$M_{S,SO} = \frac{9M_{S,S}}{8}$$

2.17 a is squarefree and even, b is squarefree and odd

Experimental Data:

Table: a is squarefree and even times squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.874839	0.000014	0.003759	+/- 0.000627	+/- 0.001880
40000	1e+4	0.872698	0.000003	0.001714	+/- 0.000286	+/- 0.000857
10000	8e+4	0.871972	0.000008	0.002787	+/- 0.000465	+/- 0.001394
90000	8e+4	0.872926	0.000000	0.000639	+/- 0.000106	+/- 0.000319

Analytical Calculation: Since

$$f(a) = \begin{cases} \{0,0,1,0\}, & p = 2 \\ \{0,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and since

$$f(b) = \begin{cases} \{0,1,0,1\}, & p = 2 \\ \{0,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

we have $\gamma = \{0, 1/(p_2+1)^2, 1/(p_3+1)^2, \ldots\}$ (see [PRE04] section 4.32).

$$M_{SE,SO} = M_{SO,SE} = (1 - \sigma(\gamma)) = \frac{9M_{S,S}}{8}$$

Remark 4. .
$M_{S,S} = (1/9)M_{SE,SE} + (4/9)M_{SO,SO} + (4/9)M_{SE,SO}$

2.18 a is squarefree and odd, b is squarefree and odd

Experimental Data:

Table: a is squarefree and odd times squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	1e+4	0.871164	0.000009	0.003072	+/- 0.000512	+/- 0.001536
40000	1e+4	0.872149	0.000004	0.002083	+/- 0.000347	+/- 0.001041
10000	8e+4	0.875775	0.000010	0.003169	+/- 0.000528	+/- 0.001585
90000	8e+4	0.872846	0.000003	0.001640	+/- 0.000273	+/- 0.000820

Analytical Calculation: Since

$$f(a) = f(b) = \begin{cases} \{0,1,0,1\}, & p = 2 \\ \{0,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

, we have $\gamma = \{0, 1/(p_2+1)^2, 1/(p_3+1)^2, \ldots\}$ (see [PRE04] section 4.36).

$$M_{SO,SO} = 1 - \sigma(\gamma') = \frac{9M_{S,S}}{8}$$

References

[PRE04] H. Preininger, *Natural Squarefree Numbers: Statistical Properties*, 2017, http://vixra.org/pdf/1712.0441v1.pdf