A Simple Proof that $\zeta(n \geq 2)$ is Irrational

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Abstract

We prove that partial sums of $\zeta(n) - 1 = z_n$ are not given by any single decimal in a number base given by a denominator of their terms. This result, applied to all partials, shows that partials are excluded from an ever greater number of rational, possible convergence points. The limit of the partials is z_n and the limit of the exclusions leaves only irrational numbers. Thus z_n is proven to be irrational.

1 Introduction

Beuker gives a proof that $\zeta(2)$ is irrational [3]. It is calculus based, but requires the prime number theorem, as well as subtle $\epsilon - \delta$ reasoning. It generalizes only to the $\zeta(3)$ case. Here we give a simpler proof that uses just basic number theory (the easier chapters of Apostol and Hardy, [2, 4]) and treats all cases at once.

We use the following notation: for integers n, $n > 1$,

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

2 Decimals using denominators

Our aim in this section is to show that the reduced fractions that give the partial sums of z_n require a denominator greater than that of the last term defining the partial sum. Restated this says that partial sums of z_n can't be expressed as a finite decimal using for a base the denominators of any of the partial sum's terms. Lemma 1 is similar to Apostol's chapter 1, problem 30. See [5] for a solution to this problem.

Lemma 1. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n}\sum_{j=2}^{k}\frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{1}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (1) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. □

Lemma 2. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^n}{(k!)^n}\sum_{j=2}^{k}\frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{2}$$

As $(k, p) = 1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p < k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. □

Theorem 1. *If $s_k^n = \frac{r}{s}$, with r/s reduced, then $s > k^n$.*

Proof. Bertrand's postulate states that for any $k \geq 2$, there exists a prime p such that $k < p < 2k$ [4]. For even k, we are assured that there exists a prime p such that $k > p > k/2$. If k is odd, $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Bertrand's postulate, we have assurance of the existence of a p that satisfies Lemma 2. Using Lemmas 1 and 2, we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. □

In light of this result we give the following definitions and corollary.

Definition 1.

$$D_{j^n} = \{0, 1/j^n, \ldots, (j^n - 1)/j^n\} = \{0, .1, \ldots, .(j^n - 1)\} \textit{ base } j^n$$

Definition 2.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

Corollary 1.

$$s_k^n \notin \Xi_k^n$$

Proof. Reduced fractions are unique. Suppose, to obtain a contradiction, that there exists $a/b \in \Xi_k^n$ such that $a/b = r/s$ then $b < s$ by Theorem 1. If a/b is not reduced, reduce it: $a/b = a_1/b_1$. A reduced fraction must have a smaller denominator than the unreduced form so $b_1 \leq b < s$ and this contradicts the uniqueness of the denominator of a reduced fraction. □

3 A Suggestive Table

The result of applying Corollary 1 to all partial sums of z_2 is given in Table 1.[1] The table shows that adding the numbers above each D_{k^2}, for all $k \geq 2$ gives results not in D_{k^2} or any previous rows' such sets. So, for example, $1/4 + 1/9$ is not in D_4, $1/4 + 1/9$ is not in D_4 or D_9, $1/4 + 1/9 + 1/16$ is not in D_4, D_9, or D_{16}, etc.. That's what Corollary 1 says.

[1] Table 1 might remind readers of Cantor's diagonal method. We don't pursue this idea in this article. See [7].

$+1/4$							
$+1/9$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\dots$	$+1/4$	
$\notin D_4$	$+1/9$	$+1/9$	$+1/9$	$+1/9$	$\dots$	$+1/9$	
	$\notin D_9$	$+1/16$	$+1/16$	$+1/16$		$\vdots$	
		$\notin D_{16}$	$+1/25$	$+1/25$		$\vdots$	
			$\notin D_{25}$	$+1/36$		$\vdots$	
				$\notin D_{36}$			
						$+1/(k-1)^2$	
						$+1/k^2$	
						$\notin D_{k^2}$	
							$\ddots$

Table 1: A list of all rational numbers between 0 and 1 is given by the number sets along the diagonal. Partials of z_2 are excluded from sets below and to the upper left of the partial.

4 Proof

Designate the set of rational numbers in $(0,1)$ with $\mathbb{Q}(0,1)$, the set of irrationals in $(0,1)$ with $\mathbb{H}(0,1)$, and the set of real numbers in $(0,1)$ with $\mathbb{R}(0,1)$. We use $\mathbb{R}(0,1) = \mathbb{Q}(0,1) \cup \mathbb{H}(0,1)$ and $\mathbb{Q}(0,1) \cap \mathbb{H}(0,1) = \emptyset$ in the following.

Lemma 3.

$$\lim_{k\to\infty} \Xi_k^n = \bigcup_{j=2}^{\infty} D_{j^n} = \mathbb{Q}(0,1)$$

Proof. Every rational $a/b \in (0,1)$ is included in at least one D_{j^n}. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0,1)$, $ab^{n-1} < b^n$ and so $a/b \in D_{b^n}$. □

Lemma 4. *There are lower and upper bounds for all s_k^n in every D_j^n, $j < k$.*

Proof. Using Theorem 1 (or Table 1), $s_k \in (0,1)$ and $s_k^n \notin D_j^n$, $j < k$, there must exist m_j^n and M_j^n in D_j^n such that

$$m_j^n < s_k^n < M_j^n.$$

$\square$

Lemma 5. *There is a greatest lower bound,* $\mathbf{m_j^n}$ *and least upper bound,* $\mathbf{M_j^n}$ *for all* s_k^n *in every* D_j^n*.*

Proof. As k increases the lower and upper bounds can be updated. They exist by Lemma 4. But the number of updates must be finite as there are only j elements in D_j^n. $\square$

Lemma 6. $\mathbf{m_j^n} \neq \mathbf{M_j^n}$

Proof. There is always an interval from Ξ_k^n in any given D_j^n. This interval separates $\mathbf{m_j^n}$ and $\mathbf{M_j^n}$. $\square$

Theorem 2. *z_n is irrational.*

Proof. Let m be the set of all $\mathbf{m_j^n}$ and M be the set of all $\mathbf{M_j^n}$. Sort each set to give

$$m_1^n < m_2^n < m_3^n < \cdots < z_n < \cdots < M_3^n < M_2^n < M_1^n.$$

Let $I_j^n = [m_j^n, M_j^n]$. Then the diameters of such sets goes to zero with increasing j and

$$\bigcap_{j=2}^{\infty} I_j^n = \{a_n\}. \tag{3}$$

Using Lemma 3, a_n is irrational. $\square$

Conclusion

The proof given here is close to the *squeeze* action proof by Sondow for e's irrationality; see [6]. We note that $\mathbb{R}(0,1) \setminus \Xi_k^n$ consists of a union of open intervals with rational endpoints given by elements of Ξ_k^n and this is similar to the situation of e as developed in Sondow's paper. The intervals are much more complex than those for e; eventually the intervals settle down, however and give nested intervals with shrinking diameters with a single element intersection: (3). The key strategy here is the same as Sondow. It is an *eliminate possible plausible rational convergence points as you build (or define) the series.*

Epsilon-delta proofs like Apery's for $\zeta(3)$ seem hopeless [8, 11] for the general case. Perhaps this is so because the combinatorial possibilities sky-rocket with increasing n in $\zeta(n)$. One can decipher reminders of Apery's idea in the very difficult results of Rivoal and Zudilin [8, 11]; their results, that there are an infinite number of n such that $\zeta(n)$ is irrational and at least one of the cases 5, 7, 9, or 11 are irrational, are less than encouraging for this approach.

References

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