Honam Mathematical J. **30** (2008), No. 3, pp. 505–512

POSITIVE IMPLICATIVE SMARANDACHE BCC-IDEALS IN SMARANDACHE BCC-ALGEBRAS

Young Bae Jun, Seok Zun Song, and Kyung Tae Kang

Abstract. The notion of positive implicative Smarandache BCC-ideals is introduced, and related properties are investigated. Relations between Smarandache BCC-ideals and positive implicative Smarandache BCC-ideals are discussed. The extension property of a positive implicative Smarandache BCC-ideals is given.

1. Introduction

Generally, in any human field, a *Smarandache Structure* on a set A means a weak structure $\mathbf{W}$ on A such that there exists a proper subset B of A which is embedded with a strong structure $\mathbf{S}$. In [11], W. B. Vasantha Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids, semi-normal subgroupoids, Smarandache Bol groupoids and strong Bol groupoids and obtained many interesting results about them. Smarandache semigroups are very important for the study of congruences, and it was studied by R. Padilla [10]. It will be very interesting to study the Smarandache structure in BCC/BCI-algebras. In [6], Y. B. Jun discussed the Smarandache structure in BCI-algebras. He introduced the notion of Smarandache (positive implicative, commutative, implicative) BCI-algebras, Smarandache subalgebras and Smarandache ideals, and investigated several properties. Also, he [7] discussed the Smarandache structure on BCC-algebras, and introduced the notion of Smarandache ideals. In this paper, we introduce the notion of positive implicative Smarandache BCC-ideals, and investigate related properties. We give relations between Smarandache BCC-ideals and positive implicative Smarandache BCC-ideals. We provide conditions for a Smarandache BCC-ideal to be a positive implicative

Received June 18, 2008. Accepted July 15, 2008.
2000 Mathematics Subject Classification: 06F35, 03G25.
Key words and phrases: Smarandache BCC-algebra, Smarandache BCC-ideal, positive implicative Smarandache BCC-ideal.

Smarandache BCC-ideal. We finally establish the extension property of a positive implicative Smarandache BCC-ideal.

2. Preliminaries

BCC-algebras were introduced by Komori [8] in a connection with some problems on BCK-algebras solved in [12], and Dudek [1, 2] redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Komori.

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCC-algebra* if it satisfies the following conditions:

(a1) $(\forall x,y,z \in X)\ (((x*y)*(z*y))*(x*z)=0)$,
(a2) $(\forall x \in X)\ (0*x=0)$,
(a3) $(\forall x \in X)\ (x*0=x)$,
(a4) $(\forall x,y \in X)\ (x*y=0, y*x=0 \Rightarrow x=y)$.

Note that every BCK-algebra is a BCC-algebra, but the converse is not true. Therefore the notion of a BCC-algebra is a generalization of that of a BCK-algebra. Also note that every BCC-algebra X satisfies the following equality:

(b1) $(\forall x \in X)\ (x*x=0)$,
(b2) $(\forall x,y \in X)\ (x*y \leq x)$,
(b3) $(\forall x,y,z \in X)\ (x \leq y \Rightarrow x*z \leq y*z,\ z*y \leq z*x)$,

where $x \leq y$ if and only if $x*y=0$. A nonempty subset I of a BCC-algebra X is called a *BCC-ideal* of X (see [4]) if it satisfies:

- $0 \in I$,
- $(\forall x,z \in X)\ (\forall y \in I)\ ((x*y)*z \in I \Rightarrow x*z \in I)$.

A BCC-algebra which is not a BCK-algebra is called a *proper BCC-algebra*. If X is a proper BCC-algebra, then $|X| \geq 4$. Note that a BCC-algebra X is a BCK-algebra if and only if it satisfies the following equality

$$(\forall x,y,z \in X)\ ((x*y)*z=(x*z)*y) \tag{2.1}$$

3. Positive implicative Smarandache BCC-ideals

We know that every proper BCC-algebra has at least four elements (see [2]), and that if X is a BCC-algebra then $\{0,a\}$, $a \in X$, is a BCK-algebra with respect to the same operation on X. Now let us consider a proper BCC-algebra $X=\{0,1,2,3,4\}$ with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	1	2
3	3	3	1	0	3
4	4	0	0	0	0

Table 3.1

Then $\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{0,4\}$, $\{0,1,2\}$, and $\{0,1,3\}$ are BCK-algebras with respect to the operation $*$ on X, and note that X does not contain BCK-algebras of order 4. Based on this result, we give the following definition.

Definition 3.1. [7] A *Smarandache BCC-algebra* is defined to be a BCC-algebra X in which there exists a proper subset Q of X such that

(i) $0 \in Q$, and $|Q| \geq 4$,
(ii) Q is a BCK-algebra with respect to the same operation on X.

Note that any proper BCC-algebra X with four elements can not be Smarandache. Hence if X is a Smarandache BCC-algebra, then $|X| \geq 5$ (see [7]). Notice that the BCC-algebra $X = \{0,1,2,3,4\}$ with Table 3.1 is not a Smarandache BCC-algebra.

Example 3.2. [7] (1) Let $X = \{0, a, b, c, d, e\}$ be a set with the following Cayley table:

$*$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	0	0	0	a
b	b	b	0	0	a	a
c	c	b	a	0	a	a
d	d	d	d	d	0	a
e	e	e	e	e	e	0

Table 3.2

Then $(X; *, 0)$ is a Smarandache BCC-algebra. Note that $Q = \{0, a, b, c\}$ is a BCK-algebra which is properly contained in X.

(2) Let $(X; *, 0)$ be a finite BCK-chain containing at least four elements and let c be its maximal element. Let $Y = X \cup \{d\}$, where $d \notin X$,

and define a binary operation $\odot$ on Y as follows:

$$x \odot y = \begin{cases} x * y & \text{if } x, y \in X, \\ 0 & \text{if } x \in Y,\ y = d, \\ d & \text{if } x = d,\ y = 0, \\ c & \text{if } x = d,\ y \in X. \end{cases}$$

Then $(Y; \odot, 0)$ is a Smarandache BCC-algebra.

(3) Let $(X; *, 0)$ be a BCK-algebra containing at least four elements in which a is the small atom. Let $Y = X \cup \{w\}$, where $w \notin X$, and define a binary operation $\odot$ on Y as follows:

$$x \odot y = \begin{cases} x * y & \text{if } x, y \in X, \\ w & \text{if } y \in X,\ x = w, \\ 0 & \text{if } x = 0,\ y = w, \\ 0 & \text{if } x = w = y, \\ a & \text{if } x \in X \setminus \{0\},\ y = w. \end{cases}$$

Then $(Y; \odot, 0)$ is a Smarandache BCC-algebra.

Definition 3.3. [7] A nonempty subset I of a Smarandache BCC-algebra X is called a *Smarandache BCC-ideal* of X related to Q if it satisfies:

(c1) $0 \in I$,
(c2) $(\forall x, z \in Q)\ (\forall y \in I)\ ((x * y) * z \in I \Rightarrow x * z \in I)$

where Q is a non-trivial BCK-algebra contained in X. If I is a Smarandache BCC-ideal of X related to every non-trivial BCK-algebra contained in X, we simply say that I is a *Smarandache BCC-ideal* of X.

Lemma 3.4. [7] *A nonempty subset I of a Smarandache BCC-algebra X is a Smarandache BCC-ideal of X related to Q if and only if it satisfies* (c1) *and*

(c3) $(\forall x \in Q)\ (\forall a \in I)\ (x * a \in I \Rightarrow x \in I)$,

where Q is a non-trivial BCK-algebra contained in X.

In what follows, let X and Q denote a Smarandache BCC-algebra and a non-trivial BCK-algebra which is properly contained in X, respectively, unless otherwise specified. We begin with the following definition.

Definition 3.5. A nonempty subset I of a Smarandache BCC-algebra X is called a *positive implicative Smarandache BCC-ideal* of X related to Q if it satisfies (c1) and

(c4) $(\forall x, y, z \in Q)\ ((x * y) * z \in I,\ y * z \in I \Rightarrow x * z \in I)$.

Example 3.6. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	2	0	1
3	3	3	3	0	0	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Table 3.3

Then $(X; *, 0)$ is a proper BCC-algebra and $(Q := \{0, 1, 2, 3, 4\}; *, 0)$ is a BCK-algebra. Hence $(X; *, 0)$ is a Smarandache BCC-algebra. It is easy to verify that subsets $I := \{0, 1, 2\}$ and $J := \{0, 1, 3\}$ are positive implicative Smarandache BCC-ideals of X related to Q. Also we know that $I := \{0, 1, 2\}$ and $J := \{0, 1, 3\}$ are positive implicative Smarandache BCC-ideals of X related to a BCK-algebra $(R := \{0, 1\}; *, 0)$.

Here we get the following question: Is every subset I of Q a (positive implicative) Smarandache BCC-ideal of X related to Q? But it is not possible. In fact, in Example 3.6, $K := \{0, 2, 3\}$ is a subset of Q which is not a Smarandache BCC-ideal, and hence not a positive implicative Smarandache BCC-ideal, of X related to Q since $1 * 3 = 0 \in K$ and $1 \notin K$.

Theorem 3.7. *Every positive implicative Smarandache BCC-ideal of X related to Q is a Smarandache BCC-ideal of X related to Q.*

Proof. Let I be a positive implicative Smarandache BCC-ideal of X related to Q and let $x \in Q$ and $a \in I$ be such that $x * a \in I$. Using (a3), we have $(x * a) * 0 = x * a \in I$ and $a * 0 = a \in I$. It follows from (a3) and (c4) that $x = x * 0 \in I$ so that I is a Smarandache BCC-ideal of X related to Q. □

The following example shows that the converse of Theorem 3.7 is not true in general.

Example 3.8. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	1	0	1	0	1
3	3	1	1	0	0	1
4	4	1	1	1	0	1
5	5	5	5	5	5	0

Table 3.4

Then $(X;*,0)$ is a proper BCC-algebra and $(Q:=\{0,1,2,3,4\};*,0)$ is a BCK-algebra. Hence $(X;*,0)$ is a Smarandache BCC-algebra. We know that $I:=\{0\}$ is a Smarandache BCC-ideal of X related to Q, but not a positive implicative Smarandache BCC-ideal of X related to Q since $(2*3)*3=1*3=0\in I$, $3*3=0\in I$, but $2*3=1\notin I$.

We provide conditions for a Smarandache BCC-ideal of X related to Q to be a positive implicative Smarandache BCC-ideal of X related to Q.

Theorem 3.9. *Let I be a Smarandache BCC-ideal of X related to Q such that*

$$(\forall x,y,z\in Q)\,((x*y)*z\in I\ \Rightarrow\ (x*z)*(y*z)\in I). \tag{3.1}$$

Then I is a positive implicative Smarandache BCC-ideal ideal of X related to Q.

Proof. Assume that $(x*y)*z\in I$ and $y*z\in I$ for all $x,y,z\in Q$. Then $(x*y)*z\in I$ implies $(x*z)*(y*z)\in I$ by (3.1), and so $x*z\in I$ by Lemma 3.4. Therefore I is a positive implicative Smarandache BCC-ideal of X related to Q. □

Corollary 3.10. *Let I be a Smarandache BCC-ideal of X related to Q such that*

$$(\forall x,y\in Q)\,((x*y)*y\in I\ \Rightarrow\ x*y\in I). \tag{3.2}$$

Then I is a positive implicative Smarandache BCC-ideal of X related to Q.

Proof. Assume that $(x*y)*z\in I$ for all $x,y,z\in Q$. Since

$$(((x*z)*(y*z))*z)*((x*y)*z)\leq((x*y)*z)*((x*y)*z)=0,$$

it follows that $(((x*z)*(y*z))*z)*((x*y)*z)=0\in I$ so from Lemma 3.4 that

$$((x*(y*z))*z)*z=((x*z)*(y*z))*z\in I.$$

Using (3.2), we get $(x * z) * (y * z) = (x * (y * z)) * z \in I$. Hence I is a positive implicative Smarandache BCC-ideal of X related to Q by Theorem 3.9. □

Proposition 3.11. *Every positive implicative Smarandache BCC-ideal I of X related to Q satisfies the following assertions.*

(i) $(\forall x, y \in Q)\ ((x * y) * y \in I \Rightarrow x * y \in I)$.

(ii) $(\forall x, y, z \in Q)\ ((x * y) * z \in I \Rightarrow (x * (y * z)) * z \in I)$.

(iii) *If I is contained in the largest BCK-algebra in X, then*

$$(\forall x, y \in Q)\,(\forall a \in I)\,(((x * y) * y) * a \in I \Rightarrow x * y \in I).$$

Proof. (i) Let $x, y \in Q$ be such that $(x*y)*y \in I$. Since $y*y = 0 \in I$, it follows from (c4) that $x * y \in I$.

(ii) Let $x, y, z \in Q$ be such that $(x * y) * z \in I$. Since

$$\begin{aligned}(((x * (y * z)) * z) * z) * ((x * y) * z) \\ = (((x * (y * z)) * ((x * y) * z)) * z) * z \\ = (0 * z) * z = 0 * z = 0 \in I,\end{aligned}$$

we have $((x * (y * z)) * z) * z \in I$ by using (c4). It follows from (i) that $(x * (y * z)) * z \in I$.

(iii) Suppose that $((x * y) * y) * a \in I$ for all $x, y \in Q$ and $a \in I$. Then $((x * a) * y) * y \in I$, and hence

$$(x * a) * y = ((x * a) * (y * y)) * y \in I$$

by (ii) and (b1). Using (c4), we conclude that $x * y \in I$. □

Theorem 3.12. (Extension Property) *Let I and J be Smarandache BCC-ideals of X related to Q and $I \subset J$. If I is a positive implicative Smarandache BCC-ideal of X related to Q, then so is J.*

Proof. Assume that $(x * y) * z \in J$ for all $x, y, z \in Q$. Then

$$((x * ((x * y) * z)) * y) * z = ((x * y) * z) * ((x * y) * z) = 0 \in I,$$

and so

$$((x * z) * (y * z)) * ((x * y) * z) = ((x * ((x * y) * z)) * (y * z)) * z \in I \subset J$$

by Proposition 3.11(ii). It follows from Lemma 3.4 that $(x*z)*(y*z) \in J$ so from Theorem 3.9 that J is a positive implicative Smarandache BCC-ideal of X related to Q. □

Acknowledgements

The authors are highly grateful to referees for their valuable comments and suggestions helpful in improving this paper.

References

[1] W. A. Dudek, *The number of subalgebras of finite BCC-algebras,* Bull. Inst. Math. Academia Sinica **20** (1992), 129–136.

[2] W. A. Dudek, *On proper BCC-algebras,* Bull. Inst. Math. Academia Sinica **20** (1992), 137–150.

[3] W. A. Dudek, *On constructions of BCC-algebras,* Selected Papers on BCK and BCI-algebras **1** (1992), 93–96. Shaanxi Scientific and Technological Press, Xian, China.

[4] W. A. Dudek and X. H. Zhang, *On ideals and congruences in BCC-algebras,* Czech. Math. J. **48(123)** (1998), 21–29.

[5] J. Hao, *Ideal theory of BCC-algebras,* Sci. Math. **1** (1998), no. 3, 373–381.

[6] Y. B. Jun, *Smarandache BCI-algebras,* Sci. Math. Japon. Online **e-2005** (2005), 271–276.

[7] Y. B. Jun, *Smarandache BCC-algebras,* Inter. J. Math. Math. Sci. **2005** (2005), no. 18, 2855–2861.

[8] Y. Komori, *The class of BCC-algebras is not a variety,* Math. Japon. **29** (1984), 391–394.

[9] J. Meng and Y. B. Jun, *BCK-algebras,* Kyungmoonsa Co. Seoul, Korea (1994).

[10] R. Padilla, *Smarandache algebraic structures,* Bull. Pure Appl. Sci., Delhi, **17E** (1998), no. 1, 119–121; http://www.gallup.unm.edu/ smarandache/alg-s-tx.txt.

[11] W. B. Vasantha Kandasamy, *Smarandache groupoids,* http://www.gallup.unm.edu/ smarandache/Groupoids.pdf.

[12] A. Wroński, *BCK-algebras do not form a variety,* Math. Japon. **28** (1983), 211–213.

Young Bae Jun
Department of Mathematics Education (and RINS)
Gyeongsang National University
Chinju 660-701, Korea
E-mail: skywine@gmail.com

Seok Zun Song
Department of Mathematics
Cheju National University
Jeju 690-756, Korea
E-mail: szsong@cheju.ac.kr

Kyung Tae Kang
Department of Mathematics
Cheju National University
Jeju 690-756, Korea
E-mail: kangkt@cheju.ac.kr