

SMARANDACHE DISJOINT IN BCK/D-ALGEBRAS

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ABSTRACT. In this paper we include several new families of Smarandache-type  $P$ -algebras and we study some of their properties in relation to the properties of previously defined Smarandache-types.

Let  $(X, *)$  be a binary system/algebra. Then  $(X, *)$  is a *Smarandache-type  $P$ -algebra* if it contains a subalgebra  $(Y, *)$ , where  $Y$  is non-trivial, i.e.,  $|Y| \geq 2$ , or  $Y$  contains at least two distinct elements, and  $(Y, *)$  is itself of type  $P$ . Thus, we have *Smarandache-type semigroups* (the type  $P$ -algebra is a semigroup), *Smarandache-type groups* (the type  $P$ -algebra is a group), *Smarandache-type abelian groups* (the type  $P$ -algebra is an abelian group). Smarandache semigroup in the sense of Kandasamy is in fact a Smarandache-type group (see [2]). Smarandache-type groups are of course a larger class than Kandasamy’s Smarandache semigroups since they may include non-associative algebras as well.

In this paper we include several new families of Smarandache-type  $P$ -algebras and we study some of their properties in relation to the properties of previously defined Smarandache-types.

A *d-algebra* ([4])  $(X, *, 0)$  is an algebra satisfying the following axioms: (i)  $x * x = 0$  for all  $x \in X$ ; (ii)  $0 * x = 0$  for all  $x \in X$ ; (iii)  $x * y = y * x = 0$  if and only if  $x = y$ .

If  $X = [0, \infty) = \{x \in R \mid x \geq 0\}$ , where  $R$  is the collection of real numbers, and if  $x * y = \max\{0, x - y\}$ , then  $(X, *, 0)$  is a  $d$ -algebra.  $d$ -algebras are quite common and occur in many situations as the example above indicates.

Instead of asking whether a  $d$ -algebra can be a semigroup using the same operation, we can instead ask the much wider question: Can a  $d$ -algebra be a Smarandache-type semigroup?

**Theorem 1.** *If  $(X, *, 0)$  is a  $d$ -algebra, then it cannot be a Smarandache-type semigroup.*

*Proof.* Suppose that it is in fact a Smarandache-type semigroup. Let  $|Y| \geq 2$ , where  $(Y, *)$  is a subalgebra of  $(X, *)$ , which is also a semigroup. Thus, if  $y \in Y$ , then  $y * y = 0 \in Y$  as well. Therefore  $(y * y) * y = 0 * y = 0 = y * (y * y) = y * 0$ . But  $y * 0 = 0 * y = 0$  by condition (iii) for  $d$ -algebras implies  $y = 0$ , so that in fact  $Y = \{0\}$  and  $|Y| = 1$ , a contradiction.  $\square$

**Corollary 2.** *If  $(X, *, 0)$  is a  $d$ -algebra, then it cannot be a Smarandache-type group.*

We can say the following however:

**Theorem 3.** *If  $(X, *)$  is a semigroup, then it cannot be a Smarandache-type  $d$ -algebra.*

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*Proof.* Suppose that  $(Y, *, 0)$  is a non-trivial sub- $d$ -algebra of  $(X, *)$ . Then if  $y \in Y$ , we have  $y * y = 0$  and  $0 \in Y$  and  $(y * y) * y = 0 * y = 0 = y * (y * y) = y * 0$ , so that  $y = 0$  and  $|Y| = 1$ ,  $Y = \{0\}$ . Hence, the condition  $|Y| \geq 2$  is impossible, i.e.,  $(X, *)$  is not a Smarandache-type  $d$ -algebra.  $\square$

Given a  $d$ -algebra and abelian group, we can construct an algebra which is both a Smarandache-type  $d$ -algebra and a Smarandache-type abelian group. Let  $(Y, *, 0)$  be a  $d$ -algebra and  $(Z, +)$  be the additive abelian group of natural numbers. Let  $(X, \otimes)$  be defined for  $X = Y \times Z = \{(a, m) \mid a \in Y, m \in Z\}$  as follows:  $(a, m) \otimes (b, n) := (a * b, m + n)$ . Then  $(a, 0) \otimes (b, 0) = (a * b, 0)$ , and thus  $(Y \times \{0\}, \otimes, (0, 0))$  is a subalgebra of  $(X, \otimes)$  which is isomorphic to the  $d$ -algebra  $(Y, *, 0)$ . On the other hand,  $(\{0\} \times Z, \otimes)$  has a product  $(0, m) \otimes (0, n) = (0 * 0, m + n) = (0, m + n)$  so that  $(\{0\} \times Z, \otimes)$  is an example of an algebra which is a Smarandache-type abelian group. Hence  $(X, *)$  is both a Smarandache-type  $d$ -algebra and Smarandache-type abelian group.

Given algebra types  $(X, *)$  (type- $P_1$ ) and  $(X, \circ)$  (type- $P_2$ ), we shall consider them to be *Smarandache disjoint* if the following two conditions hold:

- (A) If  $(X, *)$  is a type- $P_1$ -algebra with  $|X| > 1$  then it cannot be a Smarandache-type- $P_2$ -algebra  $(X, \circ)$ ;
- (B) If  $(X, \circ)$  is a type- $P_2$ -algebra with  $|X| > 1$  then it cannot be a Smarandache-type- $P_1$ -algebra  $(X, *)$ .

This condition does not exclude the existence of algebras  $(X, \diamond)$  which are both Smarandache-type- $P_1$ -algebras and Smarandache-type- $P_2$ -algebras. In fact we have already produced an example of such an algebra where  $P_1 \equiv$  'semigroup' and  $P_2 \equiv$  ' $d$ -algebra'.

If  $(X, *, 0)$  is a  $d$ -algebra which also satisfy the following conditions:

- (iv)  $(x * (x * y)) * y = 0$  for all  $x, y \in X$ ;
- (v)  $((x * y) * (x * z)) * (z * y) = 0$  for all  $x, y, z \in X$ ,

then it is a *BCK*-algebra (see [1, 3]). Since *BCK*-algebras are  $d$ -algebras it follows that:

**Theorem 4.** *Semigroups and  $d$ -algebras are Smarandache disjoint.*

**Corollary 5.** *Semigroups and *BCK*-algebras are Smarandache disjoint.*

**Corollary 6.** *Groups and  $d$ -algebras are Smarandache disjoint.*

Of course, since groups are semigroups we have:

**Corollary 7.** *Groups and semigroups are not Smarandache disjoint.*

Consider the collection of left semigroups, i.e., semigroups  $(X, *)$  with an associative product  $x * y = x$ . If  $(Y, *)$  is a subgroup of  $(X, *)$ , then  $x * y = x$  means that  $y$  is the multiplicative identity of  $(Y, *)$  and since  $y \in Y$  is arbitrary, it follows that  $|Y| = 1$  as well. Hence we show that:

**Theorem 8.** *If  $(X, *)$  is a left semigroup, then  $(X, *)$  cannot be a Smarandache-type group.*

If  $(X, *, e)$  is a group and if  $(Y, *)$  is a left semigroup, then being closed, we have for  $x, y \in Y$ ,  $x * y = x$ , whence from the group structure of  $(X, *)$  we find  $y = e$ , and since  $y \in Y$

is arbitrary,  $|Y| = 1$  as well. Thus  $(X, *)$  cannot be a Smarandache-type left semigroup and hence we have shown that:

**Theorem 9.** *Left semigroups and groups are Smarandache disjoint.*

**Corollary 10.** *If  $(X, *)$  with  $x * y = y$  is a right semigroup, then it follows that right semigroups and groups are Smarandache disjoint.*

The notion of Smarandache disjointness illustrated here appears to be novel and of interest as well.

**Question.** *Give an examples of a special class of  $d$ -algebras which is Smarandache disjoint from the class of BCK-algebras.*

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