A proof of the Riemann Hypothesis

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Abstract: This paper investigates the characteristics of the Riemann xi function. A detailed investigation of two integration approaches to Riemann's original equation (firstly as a series including incomplete gamma functions and secondly as a power series), focussing firstly on the characteristics of the expressions including incomplete gamma functions (highlighting the properties of the continued fraction components and the characteristics of the series terms as they reduce rapidly in magnitude) and secondly on the implications of the power series representation in the case where the imaginary component is zero leads to the conclusion that the Riemann xi function only has real zeros.

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Introduction

This paper addresses one of the key unresolved questions arising from Riemann's original 1859 paper regarding the distribution of prime numbers ('Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse'[1] - translation in Edwards [2]) - the truth or otherwise of the Riemann Hypothesis ('One finds in fact about this many real roots within these bounds and it is very likely that all of the roots are real' - referring to the roots of the Riemann Xi function).

This paper starts from Riemann's original definition of the Xi function and develops the function both as a series of modified incomplete gamma functions and as a power series by using two different integration approaches.

Section 1 develops the function into a series of modified incomplete gamma functions and investigates in detail the behaviour of each component of the series and the addition of those components, particularly for the case where the Xi function has no imaginary component, showing that the function has all real zeros or real zeros up to a certain value.

Section 2 develops the function into a power series, investigates the detailed characteristics of the series (including the existence of real zeros up to the limit) and uses a variable substitution to help show that if the function has only real zeros when there is no imaginary component, there will be no additional entire function zeros generated when there is an imaginary component.

Section 3 develops the implications of the earlier investigations and the change of coordinates, leading to the conclusion that the Xi function has only real zeros.

1 Original Equation and Integration resulting in a series of modified incomplete Gamma Functions.

Riemann's original equation in his paper (Riemann)[1]:

$\xi(t)=4\int_1^\infty (d/dx(x^{3/2}\psi'(x)))x^{-1/4}cos(\frac{t}{2}logx)dx$, where $\psi(x)=\sum_{m=1}^{\infty} e^{-m^2\pi x}$

$d/dx(x^{3/2}\psi'(x)) = d/dx(-\sum_{m=1}^{\infty} x^{3/2}m^2\pi e^{-m^2\pi x}) = \sum_{m=1}^{\infty}(m^4\pi^2 x-(3/2)m^2\pi)x^{1/2}e^{-m^2\pi x}$

Leading to: $4\int_1^\infty \sum_{m=1}^{\infty}(m^4\pi^2x^{5/4}-(3/2)m^2\pi x^{1/4})e^{-m^2\pi x}cos(\frac{t}{2}logx)dx$

Looking at the terms of the expression under the integral sign and setting t/2=a+bi:

$\int_1^\infty \sum_{m=1}^{\infty} m^4\pi^2x^{5/4}e^{-m^2\pi x}cos((a+bi)logx)dx$-$\int_1^\infty \sum_{m=1}^{\infty}(3/2)m^2\pi x^{1/4}e^{-m^2\pi x}cos((a+bi)logx)dx$

Taking the first of these expressions and remembering that $cos(x)=\frac{1}{2}(e^{ix}+e^{-ix})$:

= $\int_1^\infty \sum_{m=1}^{\infty} \frac{1}{2}m^4\pi^2x^{5/4}e^{-m^2\pi x}(x^{i(a+bi)}+x^{-i(a+bi)})dx$ and now substituting y=$m^2\pi$x, so that dy=$m^2\pi$dx:

=$\int_{m^2\pi}^\infty \sum_{m=1}^{\infty} \frac{1}{2}m^2\pi y^{5/4}e^{-y}(y^{i(a+bi)}+y^{-i(a+bi)})dy$

= $\int_{m^2\pi}^\infty \sum_{m=1}^{\infty} \frac{1}{2}m^2\pi e^{-y}((\frac{1}{m^2\pi}^{(\frac{5}{4}-b+ai)}y^{(\frac{5}{4}-b+ai)})+(\frac{1}{m^2\pi}^{(\frac{5}{4}+b-ai)}y^{(\frac{5}{4}+b-ai)}))dy$

= $\int_{m^2\pi}^\infty \sum_{m=1}^{\infty} \frac{1}{2}(m^2\pi)^{-\frac{1}{4}}e^{-y}(((m^2\pi)^{(b-ai)}y^{(\frac{5}{4}-b+ai)})+((m^2\pi)^{(-b+ai)}y^{(\frac{5}{4}+b-ai)}))dy$

= $\sum_{m=1}^{\infty} \frac{1}{2}(m^2\pi)^{-\frac{1}{4}}((m^2\pi)^{(b-ai)}\Gamma(\frac{9}{4}-b+ai,m^2\pi)+(m^2\pi)^{(-b+ai)}\Gamma(\frac{9}{4}+b-ai,m^2\pi))$ - expression 1.

By a similar process, the second of the expressions results in:

$-\int_1^\infty \sum_{m=1}^{\infty}(3/2)m^2\pi x^{1/4}e^{-m^2\pi x}cos((a+bi)logx)dx$

= $\sum_{m=1}^{\infty} -\frac{3}{4}(m^2\pi)^{-\frac{1}{4}}((m^2\pi)^{(b-ai)}\Gamma(\frac{5}{4}-b+ai,m^2\pi)+(m^2\pi)^{(-b+ai)}\Gamma(\frac{5}{4}+b-ai,m^2\pi))$ - expression 2.

$\Gamma(s,x)$ is the upper incomplete gamma function. Looking at the properties of these incomplete gamma functions in more detail:

For all that follows, we shall restrict the value of b from +1/2 to -1/2. Riemann proved in his original paper that there are no zeros of the Riemann xi function with t having imaginary parts outside the region of +$\frac{1}{2}$i to -$\frac{1}{2}$i, which is equivalent to restricting b in the above expressions.

$\Gamma(\frac{9}{4}-/+b+/-ai,m^2\pi)$ and $\Gamma(\frac{5}{4}-/+b+/-ai,m^2\pi))$ are all upper incomplete gamma functions with the real part of s positive and smaller than x (if we consider the function in the form $\Gamma(s,x)$). Nielsen[7] proved that all zeros of $\Gamma(s,x)$ for x > 0, lie in the half-plane Re (s) > x. In these cases, $m^2\pi$ is always greater than (5/4+/-b) and (5/4+/-b) is always positive - this means that each incomplete gamma function has no zeros.

With fixed b and varying a, as a increases then the magnitude of the function decreases. This can be seen if we consider the product representation of the factorial function for the complex number s as known to Euler [11] and remembering that s$\Gamma(s)=\Pi(s)$:

$\Pi(s)=\lim_{N\to\infty}\frac{N!}{(s+1)(s+2)...(s+N)}(N+1)^s$, resulting in: $\Gamma(s)=\lim_{N\to\infty}\frac{N!}{s(s+1)(s+2)...(s+N)}(N+1)^s$

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For s=(b+ai), where b is fixed, then, for any N, as a increases in magnitude both real and imaginary components of the $(N+1)^s$ term are oscillatory and the term has a constant magnitude, while the denominator increases in magnitude. In the limit, this leads to a function of monotonically decreasing magnitude and tending to zero.

At the same time, referring to the series expansion of $\Gamma(s,x)$[8] from DLMF 8.7.3:

$\Gamma(s,x) = \Gamma(s) - \sum_{k=0}^{\infty} \frac{(-1)^k x^{s+k}}{k!(s+k)})$

It can be seen that for fixed real part of s (with real part of s>0) and increasing size of imaginary part (positive or negative) and $x = m^2\pi$, the sum component monotonically decreases in magnitude and tending to zero in the limit. Since $\Gamma(s)$ also decreases and tends to zero at the same limit, then we know that $\Gamma(s,x)$ also monotonically decreases and tends to zero with these parameters.

In addition:

$\frac{1}{2}(m^2\pi)^{-\frac{1}{4}}$ results in a positive real result, reducing in magnitude as m increases.

$(m^2\pi)^{(b-ai)}$ and $(m^2\pi)^{(-b+ai)}$ both result in constant magnitude results for fixed b and varying a (real results for a=0, complex results otherwise). When b=0, they are of identical magnitude with identical real parts and complex parts reflected across the vertical axis. They are non-zero for all a,b.

At this point, we can conclude that each of the expressions below (individual components of expressions 1 and 2) has no zeros, except in the limit as a tends to infinity. In addition, we note that the sum of expressions 1 and 2 is the sum of each of these functions multiplied by constant magnitude terms, so we can conclude that the complete function tends to zero in the limit:

$\sum_{m=1}^{\infty} \frac{1}{2}(m^2\pi)^{-\frac{1}{4}}((m^2\pi)^{(b-ai)}\Gamma(\frac{9}{4} - b + ai, m^2\pi))$

$\sum_{m=1}^{\infty} \frac{1}{2}(m^2\pi)^{-\frac{1}{4}}(+(m^2\pi)^{(-b+ai)}\Gamma(\frac{9}{4} + b - ai, m^2\pi))$

$\sum_{m=1}^{\infty} -\frac{3}{4}(m^2\pi)^{-\frac{1}{4}}((m^2\pi)^{(b-ai)}\Gamma(\frac{5}{4} - b + ai, m^2\pi))$

$\sum_{m=1}^{\infty} -\frac{3}{4}(m^2\pi)^{-\frac{1}{4}}((m^2\pi)^{(-b+ai)}\Gamma(\frac{5}{4} + b - ai, m^2\pi))$

It is also useful to investigate the relative magnitudes of the incomplete gamma functions as b is varied.

Starting from $\Gamma(\frac{5}{4} + b +/- ai, m^2\pi) = \int_{m^2\pi}^{\infty} t^{(\frac{5}{4} - 1 + b +/- ai)} e^{-t} dt =$

$\int_{m^2\pi}^{\infty} e^{(\frac{5}{4} - 1 + b +/- ai)log(t)} e^{-t} dt =$

$\int_{m^2\pi}^{\infty} e^{((\frac{5}{4} - 1 + b)log(t) - t)} e^{(+/- ai)log(t)} dt$

We can see from the structure of this integral that for any fixed value of a and t being greater than or equal to $m^2\pi$ (and therefore log(t) will always be greater than 1), the magnitude of the integral will increase as (positive) b increases since the magnitude of the real part of the integrand increases by $e^{blog(t)}$.

This behaviour is also seen as a consequence of this known property of the upper incomplete gamma function: $\Gamma(s+1,x) = s\Gamma(s,x) + x^s e^{-x}$

The magnitude of $(m^2\pi)^{(\frac{5}{4} +/- b +/- ai)} e^{-m^2\pi}$ is always greater than $(\frac{5}{4} +/- b +/- ai)\Gamma(\frac{5}{4} +/- b +/- ai, m^2\pi)$, since we have established that the relevant gamma functions have no zeros except in the limit as a increases - leading to the conclusion that the magnitude of $\Gamma(s+1, m^2\pi)$ is always greater than $\Gamma(s, m^2\pi)$.

See figures 1 and 2 for illustrations of the incomplete gamma functions in expressions 1 and 2. Note the constant period oscillations for both real and imaginary components of each function.

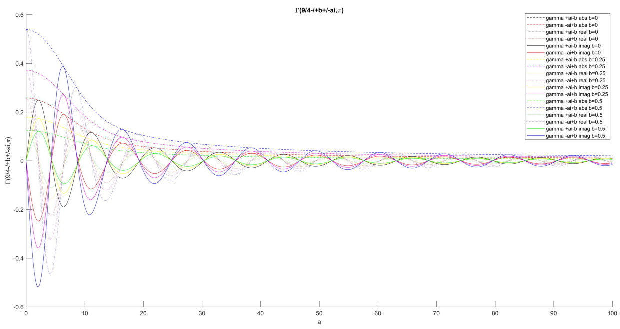

Fig. 1: Incomplete Gamma Functions 1.

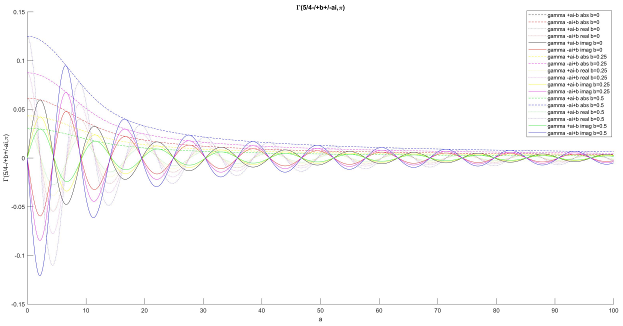

Fig. 2: Incomplete Gamma Functions 2.

Remembering this property of the incomplete gamma function:

$$\Gamma(s+1,x)=s\Gamma(s,x)+x^se^{-x}$$

Combining expression 1 and expression 2 and making use of the above property:

$$\sum_{m=1}^{\infty}\frac{1}{2}(m^2\pi)^{-\frac{1}{4}}((m^2\pi)^{(b-ai)}\Gamma(\tfrac{9}{4}-b+ai,m^2\pi)+(m^2\pi)^{(-b+ai)}\Gamma(\tfrac{9}{4}+b-ai,m^2\pi))-\frac{3}{4}(m^2\pi)^{-\frac{1}{4}}((m^2\pi)^{(b-ai)}\Gamma(\tfrac{5}{4}-b+ai,m^2\pi)+(m^2\pi)^{(-b+ai)}\Gamma(\tfrac{5}{4}+b-ai,m^2\pi))=$$

$$\sum_{m=1}^{\infty}\frac{1}{2}(m^2\pi)^{-\frac{1}{4}}((m^2\pi)^{(b-ai)}(\Gamma(\tfrac{5}{4}-b+ai,m^2\pi)(\tfrac{5}{4}-b+ai)+(m^2\pi)^{(\frac{5}{4}-b+ai)}e^{-m^2\pi})+(m^2\pi)^{(-b+ai)}(\Gamma(\tfrac{5}{4}+b-ai,m^2\pi)(\tfrac{5}{4}+b-ai)+(m^2\pi)^{(\frac{5}{4}+b-ai)}e^{-m^2\pi}))-\frac{3}{4}(m^2\pi)^{-\frac{1}{4}}((m^2\pi)^{(b-ai)}\Gamma(\tfrac{5}{4}-b+ai,m^2\pi)+(m^2\pi)^{(-b+ai)}\Gamma(\tfrac{5}{4}+b-ai,m^2\pi))=$$

$$\sum_{m=1}^{\infty}(m^2\pi)^{-\frac{1}{4}}((m^2\pi)^{(b-ai)}(\Gamma(\tfrac{5}{4}-b+ai,m^2\pi)(\tfrac{5}{8}-\tfrac{b}{2}+\tfrac{a}{2}ai-\tfrac{3}{4})+\tfrac{1}{2}(m^2\pi)^{(\frac{5}{4}-b+ai)}e^{-m^2\pi})+(m^2\pi)^{(-b+ai)}(\Gamma(\tfrac{5}{4}+b-ai,m^2\pi)(\tfrac{5}{8}+\tfrac{b}{2}-\tfrac{a}{2}i-\tfrac{3}{4})+\tfrac{1}{2}(m^2\pi)^{(\frac{5}{4}+b-ai)}e^{-m^2\pi}))=$$

$$\sum_{m=1}^{\infty}(m^2\pi)^{-\frac{1}{4}}((m^2\pi)^{(b-ai)}\Gamma(\tfrac{5}{4}-b+ai,m^2\pi)(\tfrac{5}{8}-\tfrac{b}{2}+\tfrac{a}{2}ai-\tfrac{3}{4})+\tfrac{1}{2}(m^2\pi)^{\frac{5}{4}}e^{-m^2\pi}+(m^2\pi)^{(-b+ai)}\Gamma(\tfrac{5}{4}+b-ai,m^2\pi)(\tfrac{5}{8}+\tfrac{b}{2}-\tfrac{a}{2}i-\tfrac{3}{4})+\tfrac{1}{2}(m^2\pi)^{\frac{5}{4}}e^{-m^2\pi})$$

Separating this into two expressions:

$\sum_{m=1}^{\infty}(m^2\pi)^{-\frac{1}{4}}((m^2\pi)^{(b-ai)}\Gamma(\frac{5}{4}-b+ai,m^2\pi)(\frac{5}{8}-\frac{b}{2}+\frac{a}{2}i-\frac{3}{4})+\frac{1}{2}(m^2\pi)^{\frac{5}{4}}e^{-m^2\pi})$ - Expression 3

$\sum_{m=1}^{\infty}(m^2\pi)^{-\frac{1}{4}}((m^2\pi)^{(-b+ai)}\Gamma(\frac{5}{4}+b-ai,m^2\pi)(\frac{5}{8}+\frac{b}{2}-\frac{a}{2}i-\frac{3}{4})+\frac{1}{2}(m^2\pi)^{\frac{5}{4}}e^{-m^2\pi})$ - Expression 4

To investigate these more closely, it is also useful to use the continued fraction representation of the incomplete gamma function from Abramowitz and Stegun 6.5.31[6]

$$\Gamma(s,x)=e^{-x}x^s\left[\frac{1}{x+}\frac{1-s}{1+}\frac{1}{x+}\frac{2-s}{1+}\frac{2}{x+}\cdots\right]$$

Substituting into expression 3:

$$\sum_{m=1}^{\infty}(m^2\pi)^{-\frac{1}{4}}((m^2\pi)^{(b-ai)}e^{-m^2\pi}(m^2\pi)^{(\frac{5}{4}-b+ai)}\left[\frac{1}{m^2\pi+}\frac{1-(\frac{5}{4}-b+ai)}{1+}\frac{1}{m^2\pi+}\frac{2-(\frac{5}{4}-b+ai)}{1+}\frac{2}{m^2\pi+}\cdots\right](-\tfrac{1}{8}-\tfrac{b}{2}+\tfrac{a}{2}i)+\tfrac{1}{2}(m^2\pi)^{\frac{5}{4}}e^{-m^2\pi})=$$

$\sum_{m=1}^{\infty}(m^2\pi)e^{-m^2\pi}(\left[\frac{1}{m^2\pi+}\frac{1-(\frac{5}{4}-b+ai)}{1+}\frac{1}{m^2\pi+}\frac{2-(\frac{5}{4}-b+ai)}{1+}\frac{2}{m^2\pi+}\cdots\right](-\frac{1}{8}-\frac{b}{2}+\frac{a}{2}i)+\frac{1}{2})$ - Expression 5

Similarly for expression 4:

$$\sum_{m=1}^{\infty}(m^2\pi)^{-\frac{1}{4}}((m^2\pi)^{(-b+ai)}\Gamma(\tfrac{5}{4}+b-ai,m^2\pi)(\tfrac{5}{8}+\tfrac{b}{2}-\tfrac{a}{2}i-\tfrac{3}{4})+\tfrac{1}{2}(m^2\pi)^{\frac{5}{4}}e^{-m^2\pi})=$$

$\sum_{m=1}^{\infty}(m^2\pi)e^{-m^2\pi}(\left[\frac{1}{m^2\pi+}\frac{1-(\frac{5}{4}+b-ai)}{1+}\frac{1}{m^2\pi+}\frac{2-(\frac{5}{4}+b-ai)}{1+}\frac{2}{m^2\pi+}\cdots\right](-\frac{1}{8}+\frac{b}{2}-\frac{a}{2}i)+\frac{1}{2})$ - Expression 6

Investigating the properties of the function represented by the Continued Fraction (CF) components shown above in expressions 5 and 6:

Firstly, rearranging the definition:

$\Gamma(s,x)=e^{-x}x^s\left[\frac{1}{x+}\frac{1-s}{1+}\frac{1}{x+}\frac{2-s}{1+}\frac{2}{x+}\cdots\right]$ to $\left[\frac{1}{x+}\frac{1-s}{1+}\frac{1}{x+}\frac{2-s}{1+}\frac{2}{x+}\cdots\right]=\Gamma(s,x)/e^{-x}x^s$

Remembering that (for complex s and fixed real part of s and real x), the incomplete gamma function decreases in magnitude for increasing imaginary part of s and has oscillatory real and imaginary parts generated by real terms to the power (s+integers) in the series expansion from DLMF, we would expect the Continued Fraction function to have non-oscillatory real and imaginary components and decreasing magnitude (since the definition above shows the Continued Fraction is equal to the incomplete gamma function divided by a term of constant magnitude and an oscillatory component in x^s).

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Note that this means that individually the real and imaginary components will have no zeros. Also note that for fixed real part, the continued fraction values for $\Gamma(5/4+b-ai,m^2\pi)$ and $\Gamma(5/4-b+ai,m^2\pi)$ will both have positive real components but the CF values for $\Gamma(5/4-b+ai,m^2\pi)$ will have positive imaginary and the CF values for $\Gamma(5/4+b-ai,m^2\pi)$ will have negative imaginary components. When b=0, the real components of each function will be identical and the imaginary components of the functions will identically sum to zero.

At the same time, referring to the alternative series expansion of $\Gamma(s,x)$[8] from DLMF 8.7.3:

$\Gamma(s,x) = \Gamma(s)(1 - x^s e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(s+k+1)})$ (series expansion).

Comparing this with $\Gamma(s,x)$ = $e^{-x}x^s\left[\frac{1}{x+}\,\frac{1-s}{1+}\,\frac{1}{x+}\,\frac{2-s}{1+}\,\frac{2}{x+}\cdots\right]$ (continued fraction expansion).

We can see that for fixed s and increasing x (with re(s)>1 and $x = m^2\pi$), the magnitude of the continued fraction term reduces as x increases (since the x^k terms in the series expansion increase in magnitude).

A final consideration is the derivative of the incomplete gamma function:

$\frac{\delta\Gamma(s,x)}{\delta x} = -x^{s-1}e^{-x}$.

This means that for constant s (and x=$m^2\pi$), as x increases, then the derivative $\Gamma(s,x)$ decreases in magnitude.

This is in fact what we observe in the following figures for m=1 - figure 3, m=2 - figure 4 and m=3 - figure 5.

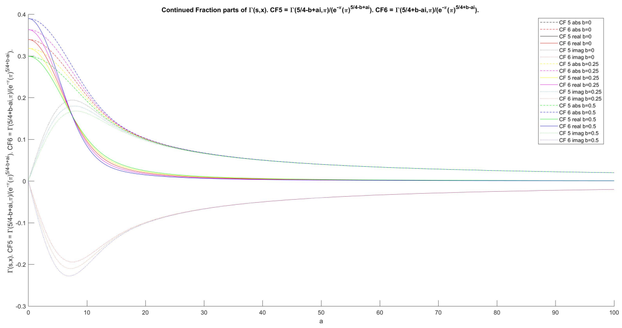

Fig. 3: Continued Fraction Components m=1.

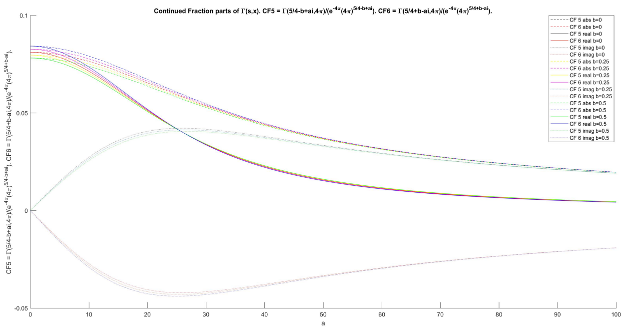

Fig. 4: Continued Fraction Components m=2.

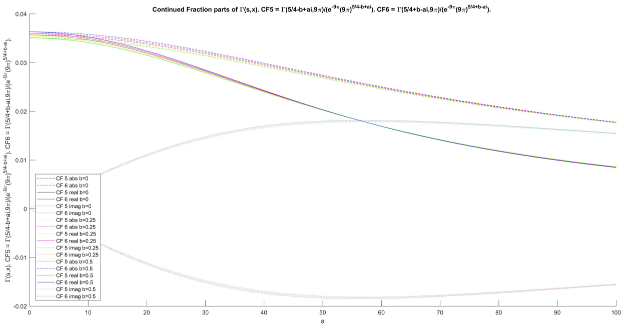

Fig. 5: Continued Fraction Components m=3.

It is now useful to consider the other variable components of expressions 5 and 6.

Firstly the $(m^2\pi)e^{-m^2\pi}$ components. As m increases, these components are real and reduce in magnitude very rapidly and consistently.

Secondly the $(-\frac{1}{8} - \frac{b}{2} + \frac{a}{2}i)$ and $(-\frac{1}{8} + \frac{b}{2} - \frac{a}{2}i) + \frac{1}{2})$ components as seen in figure 6 below (increasing in magnitude almost linearly):

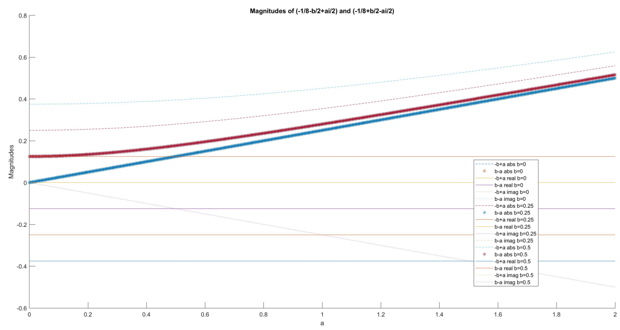

Fig. 6: Plots of $(-\frac{1}{8} - \frac{b}{2} + \frac{a}{2}i)$ and $(-\frac{1}{8} + \frac{b}{2} - \frac{a}{2}i)$

Considering the implications of this in expressions 5 and 6 (where the continued fraction expressions are multiplied by these components):

Considering $(\Big[\frac{1}{m^2\pi+}\frac{1-(\frac{5}{4}-b+ai)}{1+}\frac{1}{m^2\pi+}\frac{2-(\frac{5}{4}-b+ai)}{1+}\frac{2}{m^2\pi+}...\Big](-\frac{1}{8} - \frac{b}{2} + \frac{a}{2}i) + \frac{1}{2})$ from expression 5:

and $(\Big[\frac{1}{m^2\pi+}\frac{1-(\frac{5}{4}+b-ai)}{1+}\frac{1}{m^2\pi+}\frac{2-(\frac{5}{4}+b-ai)}{1+}\frac{2}{m^2\pi+}...\Big](-\frac{1}{8} + \frac{b}{2} - \frac{a}{2}i) + \frac{1}{2})$ from expression 6:

If the components of the continued fraction component (especially the imaginary component) are large enough, then it is possible for the complete expression to have a negative real part for large enough imaginary part of s. The real part would start positive for no imaginary part, reduce in value to negative for some value of the imaginary part of s, then changing direction and tending to zero in the limit from below (the monotonically decreasing nature of the incomplete gamma function above does not allow for an additional change of sign when multiplied by a linear term as in this case). If the components are not large enough, then the real part of the expression stays positive. NB, as $m^2\pi$ increases, the continued fraction components reduce in size for constant s.

In addition, the characteristics of similar sign real parts and opposite sign imaginary parts will continue, with the sum of the imaginary parts of expression 5 and expression 6 vanishing when b=0.

This is what we observe in the figures below, showing the magnitude of, real part of and imaginary part of (expression 5 + expression 6), summing m=1,2 and 3 and in addition the separate real parts of m=1, m=2 and m=3, all for varying b (nb note the negative portion of the real curve for m=1, compared with the much smaller, always positive curves for m=2 and m=3). See figure 7 for an overview, with more detail in figure 8.
Focussing now on the case when b=0 and looking in detail at the characteristics of the (expression 5 + expression 6) values for increasing values of m. In this case, there is only a real component.

Firstly, we have observed that for m=1, the expression is negative above a certain value of a, tending to zero in the limit from below. Thus for m=1, the expression has one zero.

For m=2, we observe that the expression is always positive (and tending to zero at the limit). This means that for each m>2, the expression will be positive and tending to zero. As m increases, the magnitude of each expression reduces rapidly (dependent on $(m^2\pi)e^{-m^2\pi}$).

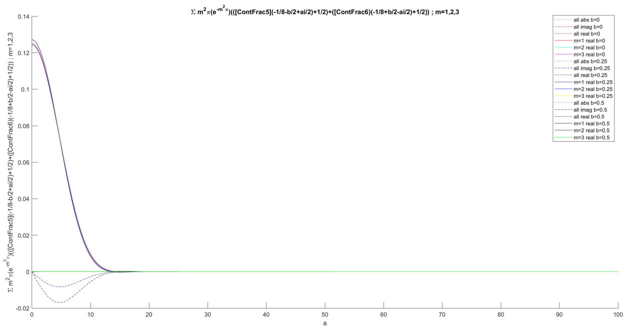

Fig. 7: Combined Expression 5 and Expression 6.

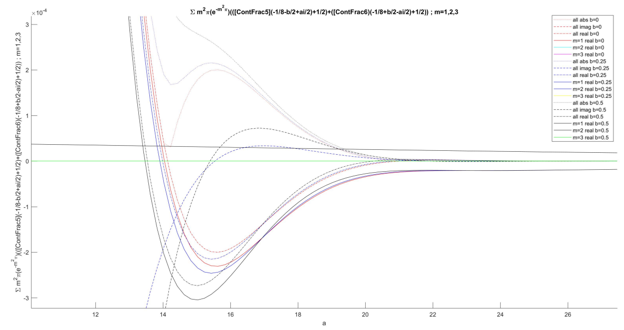

Fig. 8: Combined Expression 5 and Expression 6 Detail.

Considering now the possibilities when adding the expressions when m=1,2,3, etc. Firstly, adding the expressions for m=1 and m=2. The results will depend on the relative magnitudes of the two expressions. Given the m=1 expression as fixed, if the magnitude of the m=2 expression is relatively large, the combined expression will have an additional zero (as the expression turns positive) and will then remain positive. This means that the expression will have no more zeros (and no additional zeros as the m=3,4, etc expressions are added since they are all positive). If the m=2 expression is relatively small, then the combined expression will not have any more zeros as it will remain negative (remembering that the magnitudes of the m=2,3,4 etc expressions decrease very rapidly). If the magnitude of the expression is within a certain range, then there will be an even number of zeros as the expression turns positive (and then potentially negative and positive again, depending on the exact shape of the curves) and finally negative again - meaning that it is possible for more zeros to be generated when the m=3 expression is added. The argument holds as the expression for each value of m is added. The key implication is that if adding any of the expressions either generates an odd number of new zeros or no new zeros, then there will be no new zeros generated for expressions with greater values of m - ie the real zeros will stop above a certain value of a.

This is what we observe in the following figures, showing the ((m=1)), ((m=1)+(m=2)), ((m=1)+(m=2)+(m=3)) and ((m=1)+(m=2)+(m=3)+(m=4)) curves, as well as the -(m=2), -(m=3) and -(m=4) curves for illustration of the relative magnitudes and interaction with the main curve. Figure 9 for the overview, figure 10 for m=1,2 and figure 11 for m=1,2,3.

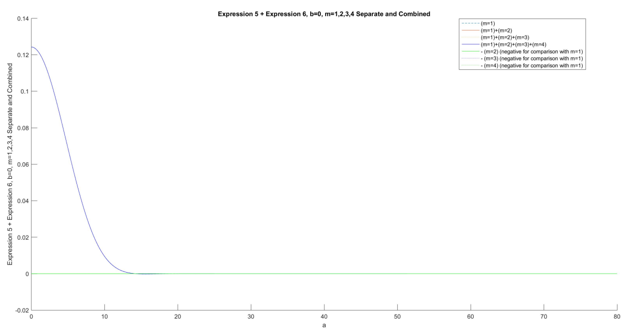

Fig. 9: Combined Expression 5 and Expression 6, b=0, m=1,2,3,4.

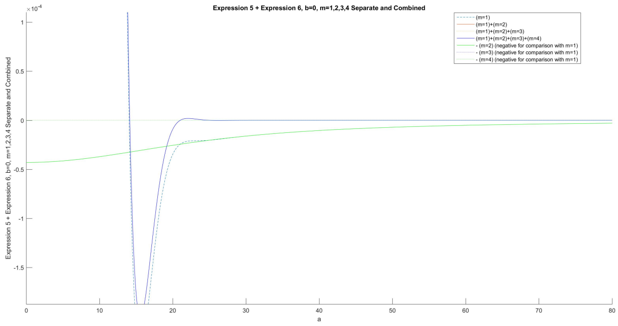

Fig. 10: Combined Expression 5 and Expression 6, b=0, m=1,2,3,4 - Highlighting m=1,2.

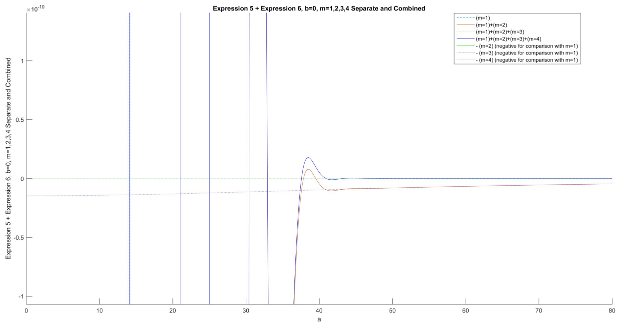

Fig. 11: Combined Expression 5 and Expression 6, b=0, m=1,2,3,4 - Highlighting m=1,2,3.

2 Original Equation and Integration resulting in a power series.

2.1 Original Equation

Riemann's original equation in his paper (Riemann)[1]:

ξ(t)=4$\int_1^\infty (d/dx(x^{3/2}\psi'(x)))x^{-1/4} cos(\frac{t}{2} logx)dx$ where $\psi(x) = \sum_{m=1}^\infty e^{-m^2\pi x}$

To avoid ξ-Ξ confusion, the equation from Edwards[3] is used:

ξ(s)=4$\int_1^\infty (d/dx(x^{3/2}\psi'(x)))x^{-1/4} cosh(\frac{1}{2}(s-\frac{1}{2})logx)dx$

This leads to (Edwards)[3]:

$\xi(s) = \sum_{n=0}^\infty a_{2n}(s-\frac{1}{2})^{2n}$ where a_{2n}=4$\int_1^\infty (d/dx(x^{3/2}\psi'(x))x^{-1/4}\frac{logx^{2n}}{2^{2n}(2n)!})dx$

Now, if t=(x+yi), then (s-$\frac{1}{2}$) = it = (xi-y), and:

ξ(s)=$\sum_{n=0}^\infty a_{2n}(xi-y)^{2n}$

Due to the fact that all a_{2n} are positive (Edwards p41)[4], it immediately follows from the above that if x=0, there are no real zeros of the function and if y=0 then there are potentially many real zeros (depending on the actual values of a_{2n}).

It is important to note at this point that it has been proven that

ξ(s)=$\sum_{n=0}^\infty a_{2n}(xi-y)^{2n}$

(a polynomial in tt) has been proven to converge as the coefficients decrease rapidly; this result is necessary in the convergence of the product representation (Hadamard) [5].

2.2 Real Curve Shape Components

Investigating some of the properties of the terms of the Riemann definition of ξ(s) (Edwards P16)[9] for $s = (\frac{1}{2} + ri)$; this is the equivalent of varying x and setting y=0 in the power series (we will see below in the polar coordinates section why r is used as the variable):

ξ(s) = $\Pi(\frac{s}{2})(s-1)\pi^{\frac{-s}{2}}\zeta(s)$

Since we know that this expression has the same zeros as $\zeta(s)$ and no other zeros, we can consider the magnitudes of the various components.

a) The $\Pi(\frac{s}{2})$ term, where Π is the factorial function. The magnitude of the factorial function for real numbers of increasing size increases rapidly.

However, the behaviour for complex numbers with a fixed real part and an imaginary part of increasing size is very different - the magnitude of the function decreases very rapidly with increasing imaginary number size.

In addition, it is oscillatory for both real and complex components.

This behaviour can be seen by investigating the product representation of the factorial function for the complex number s as known to Euler [11]:

$$\Pi(s) = \lim_{N\to\infty} \frac{N!}{(s+1)(s+2)...(s+N)}(N+1)^s$$

For s=(a+bi), where |a|<1, then, for any N, as b increases in magnitude both real and imaginary components of the $(N+1)^s$ term are oscillatory with a constant magnitude of oscillation, while the denominator increases rapidly in magnitude.

In the limit, this leads to a function of rapidly decreasing magnitude.

See Figure 12 and Figure 13 for illustrations.

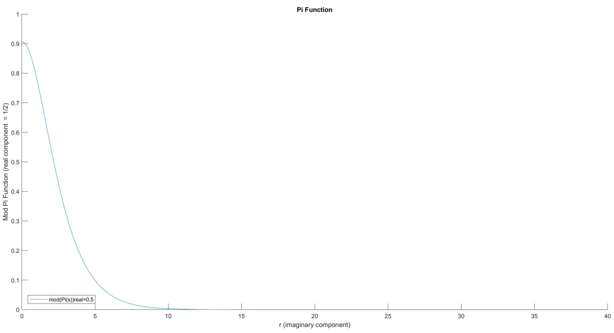

Fig. 12: Mod $\Pi(\frac{1}{2}+ri)$, r$<$ 18

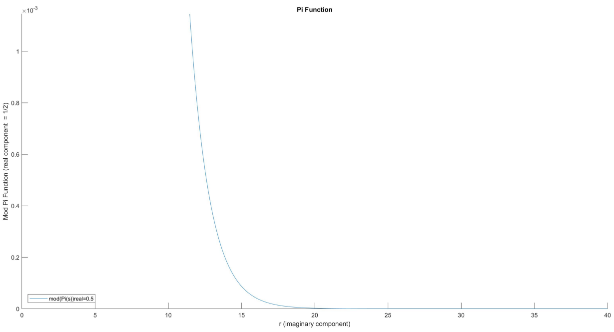

Fig. 13: Mod $\Pi(\frac{1}{2}+ri)$, r$<$ 30 Note rapid decrease in magnitude

b) (s-1) The magnitude of this (non oscillatory) term increases slowly with the magnitude of s.

c) $\pi^{\frac{-s}{2}}$. The real and imaginary components of this term oscillate with a fixed magnitude of oscillation. The magnitude of the term is constant for fixed real part value and varying imaginary part.

d) $\zeta(s)$. The real and imaginary components of this term oscillate with a very slowly increasing magnitude (see Figure 14 for illustration).

Looking at the product of the individual terms, this means that the curve is oscillatory with decreasing magnitude of oscillation as r increases.

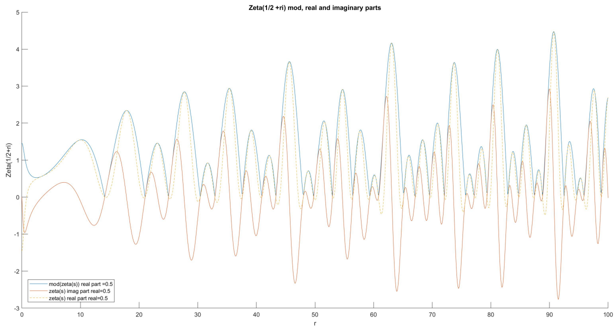

Fig. 14: $\zeta(\frac{1}{2} + ri)$

In addition, since we know that for b=0 then there is no imaginary element of the function, the function tends to zero in the limit (as expected).

The above two conclusions mean that for any value of r there will be real zeros greater than r.

This reduction in magnitude can be seen in the curves with the actual a_{2n} below. Figure 15 and Figure 16 show 2 sections of ξ(s) with b=0.

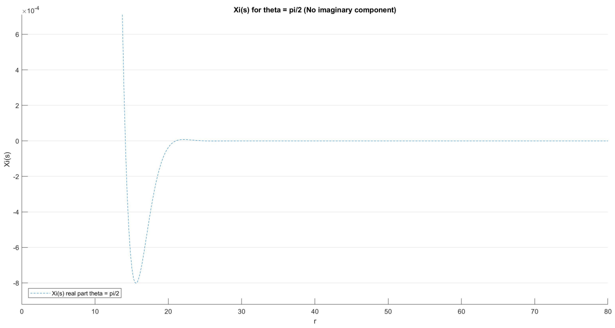

Fig. 15: $\xi(s)$ No imaginary component, r< 25

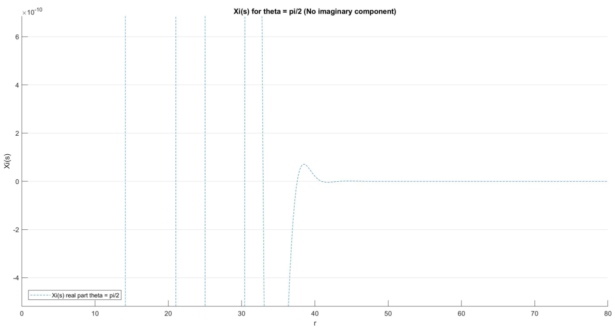

Fig. 16: $\xi(s)$ No imaginary component, r< 40 Note rapid decrease in magnitude

2.2 Polar Coordinates

2.2.1 Substitution

Using de Moivre's Theorem (Heading p115 [10]) $(ai - b)^{2n}$ can be rewritten as $r^{2n}(cos\theta + isin\theta)^{2n}$ and expanded as $r^{2n}cos2n\theta + ir^{2n}sin2n\theta$, taking r to range from 0 to ∞ and θ to range from $\frac{\pi}{2}$ to π for the most relevant quadrant. The structure of the expression (see below) means that the π to 2π half is a reflection of the 0 to π half. The behaviour of the expression is markedly different for $r \leq 1$. From this point, I will consider only $r > 1$ (since we know that there are no relevant zeros for $r \leq 1$).

2.2.2 Complete Expression

The above results in: ξ(s)=$\sum_{n=0}^{\infty} a_{2n} r^{2n} (cos\theta + isin\theta)^{2n}$

=$a_0 + a_2 r^2 cos2\theta + a_4 r^4 cos4\theta + a_6 r^6 cos6\theta + a_8 r^8 cos8\theta...$

$+i(a_2 r^2 sin2\theta + a_4 r^4 sin4\theta + a_6 r^6 sin6\theta + a_8 r^8 sin8\theta...)$

Both real and imaginary parts of the expression are single valued for each r, θ combination. Both real and imaginary parts have a period of π. For $\theta = \frac{\pi}{2}$ the expression is equal to ξ(s)=$\sum_{n=0}^{\infty} a_{2n} (ai)^{2n}$. For the actual values of a_{2n} the expression does have multiple real zeros.

Figure 17 shows the variation of the function with variation in θ.

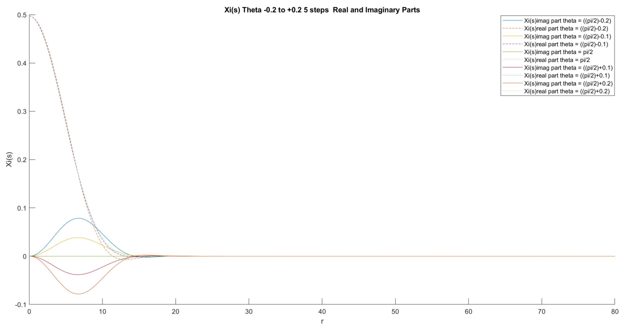

Fig. 17: $\xi(s)$ with varying θ

2.3 Paths of Zeros

2.3.1 Real Part Zeros

Using $\theta = (\frac{\pi}{2} + \epsilon)$:

cos2$\theta = cos(2(\frac{\pi}{2} + \epsilon))$ = cosπcos2ϵ - sinπsin2ϵ = - cos2ϵ and $cos(2(\frac{\pi}{2} - \epsilon))$ = cosπcos2ϵ + sinπsin2ϵ = - cos2ϵ

Similar expressions can be generated for cos2nθ for all values of n with similar results (except alternating signs).

This means that the path of the function $a_0 + a_2 r^2 cos2\theta + a_4 r^4 cos4\theta + a_6 r^6 cos6\theta + a_8 r^8 cos8\theta...$=0 is reflected across $\theta = \frac{\pi}{2}$ for varying r.

This equation describes a family of curves.

No function in the family intersects any other function in the family (the function is single valued for each r,θ combination).

For the actual values of a_{2n}, it appears that each function in the family will pass through $\theta = \frac{\pi}{2}$.

When $\theta \neq \frac{\pi}{2}$, then we know that the function extends from the zeros on the $\theta = \frac{\pi}{2}$ line.

If a function does not pass through $\theta = \frac{\pi}{2}$, then it will have 2 reflected branches on either side of $\theta = \frac{\pi}{2}$ (non-intersecting with any other of the family of functions).

2.3.2 Imaginary Part Zeros

Using $\theta = (\frac{\pi}{2} + \epsilon)$:

sin2$\theta = sin(2(\frac{\pi}{2} + \epsilon))$ = sinπcos2ϵ + cosπsin2ϵ = - sin2ϵ and $sin(2(\frac{\pi}{2} - \epsilon))$ = sinπcos2ϵ - cosπsin2ϵ = + sin2ϵ

Similar expressions can be generated for sin2nθ for all values of n with similar results (except alternating signs).

This means that the path of the function $a_2 r^2 sin2\theta + a_4 r^4 sin4\theta + a_6 r^6 sin6\theta + a_8 r^8 sin8\theta...$=0 is reflected across $\theta = \frac{\pi}{2}$ for varying r.

This equation describes a family of curves.

No function in the family intersects any other function in the family (the function is single valued for each r,θ combination).

For the actual values of a_{2n} it appears that each function in the family will pass through $\theta = \frac{\pi}{2}$.

When $\theta = \frac{\pi}{2}$, then we know that the function is identically zero.

If a function does not pass through $\theta = \frac{\pi}{2}$, then it will have 2 reflected branches on either side of $\theta = \frac{\pi}{2}$ (non-intersecting with any other of the family of functions).

The real part and imaginary part have the same number of pairs of zeros (the imaginary part has an additional zero at r=0).

See Figure 18 for an illustration of the paths of real and imaginary zeros for the actual values of a_{2n} in the same graph.

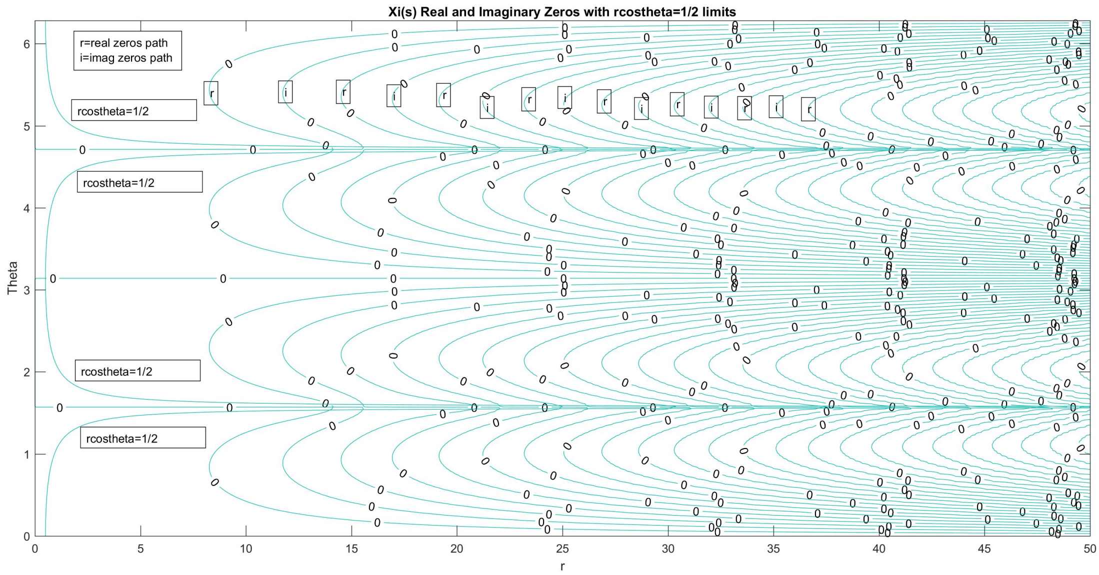

Fig. 18: $\xi(s)$ real and imaginary part zeros.

Note that the paths of the zeros do not intersect in these samples (as shown above), except where the imaginary function is identically zero.

2.3.3 (Real + Imaginary) and (Real - Imaginary) Part Zeros

The complete function will be zero when both real and imaginary expressions are equal to each other and both zero. Thus we are looking for common zeros of these two expressions:

$a_0 + a_2r^2cos2\theta + a_4r^4cos4\theta + a_6r^6cos6\theta + a_8r^8cos8\theta$...=0 (1)
and $a_2r^2sin2\theta + a_4r^4sin4\theta + a_6r^6sin6\theta + a_8r^8sin8\theta$...=0 (2)

It is useful to investigate the combined expressions ((1)+(2)) and ((1)-(2)). If and only if they are simultaneously zero then the complete function is zero.

Reusing: cos2$\theta = cos(2(\frac{\pi}{2} + \epsilon))$ = cosπcos2ϵ - sinπsin2ϵ = - cos2ϵ and $cos(2(\frac{\pi}{2} - \epsilon))$ = cosπcos2ϵ + sinπsin2ϵ = - cos2ϵ
sin2$\theta = sin(2(\frac{\pi}{2} + \epsilon))$ = sinπcos2ϵ + cosπsin2ϵ = - sin2ϵ and $sin(2(\frac{\pi}{2} - \epsilon))$ = sinπcos2ϵ - cosπsin2ϵ = + sin2ϵ

Similar expressions can be generated for sin2nθ and cos2nθ for all values of n with similar results (except alternating signs).

(1)+(2) for ϵ is the same as (1)-(2) for - ϵ and (1)-(2) for ϵ is the same as (1)+(2) for - ϵ - that is, the expressions (1)+(2) and (1)-(2) are reflected through $\theta = \frac{\pi}{2}$ and if they cross $\theta = \frac{\pi}{2}$ then there will be a coincident pair of real zeros on $\theta = \frac{\pi}{2}$.

If they do not cross $\theta = \frac{\pi}{2}$, then there will be a reflected pair of imaginary zeros tracing reflected paths.

In addition, as the functions are single valued for each r, θ combination then there are no intersections with any other of the same family of functions.

This means that there will be no additional complete function zeros generated - each pair of imaginary part zeros will only coincide with one pair of real part zeros.

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One can also see from the above expressions that as ϵ tends to $\frac{\pi}{2}$ and individual components tend to zero, the functions both tend to horizontal (i.e one would expect to see an increasing number of almost parallel, almost horizontal functions as r increases).

See Figure 19 for an illustration of the paths of the ((1)+(2)) and ((1)-(2)) zeros for the actual values of a_{2n}.

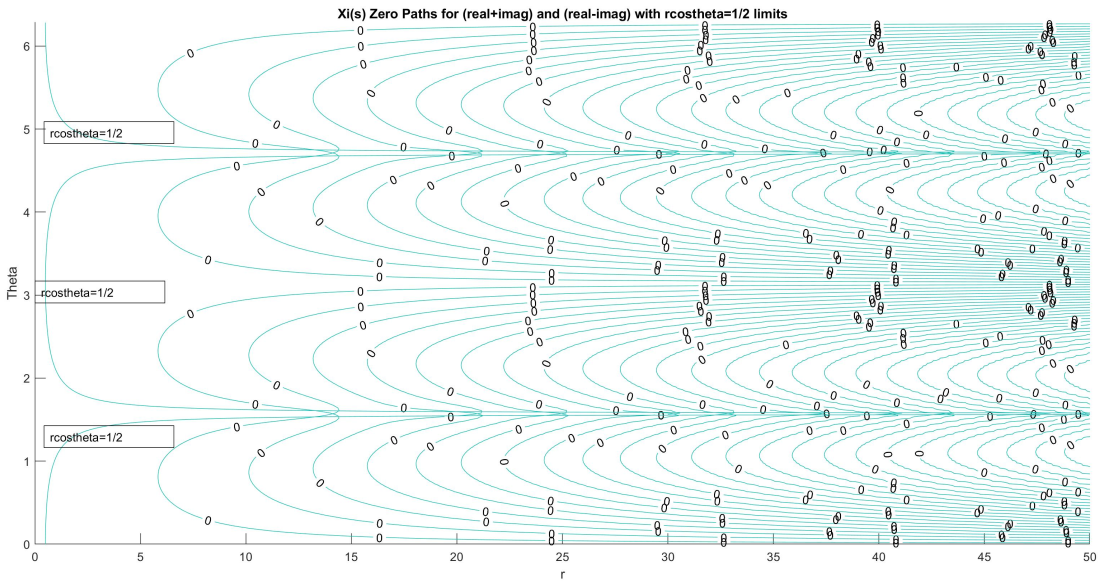

Fig. 19: $\xi(s)$ (real + imaginary) and (real - imaginary) zero paths.

3 Conclusions

Known previously - ξ(s) (with no imaginary component) does not have zeros outside the critical strip.

In Section 1 it was shown that $\xi(s)$ with no imaginary component (ie b=0) will either have all real zeros or all real zeros for a less than a certain value.

In Section 2.1 it was shown that $\xi(s)$ no imaginary component (b=0) has real zeros greater than any fixed value of a.

This means that $\xi(s)$ with no imaginary component has only real zeros.

In Section 2.2 it was shown that there are no additional zeros of the complete function due to the coincidence of imaginary zeros from the real and imaginary parts of ξ(s).

Combining these conclusions, all of the roots of the Riemann Xi function (where s=($\frac{1}{2} + ti$) - no imaginary component) are real - QED.

1 References

1 Riemann, B.: ‘Gesammelte Werke.’(Teubner, Leipzig, 1892; reprinted by Dover Books, New York, 1953.) p145.
2 Edwards, H.M.: ‘Riemann’s Zeta Function’(Dover Publications, 2001) p299
3 Edwards, H.M.: ‘Riemann’s Zeta Function’(Dover Publications, 2001) p17
4 Edwards, H.M.: ‘Riemann’s Zeta Function’(Dover Publications, 2001) p41
5 Hadamard, J.: ‘Étude sur les Propriétés des Fonctions Entières et en Particulier d’une Fonction Considérée par Riemann. *J. Math. Pures Appl.* [4] **9**, pp. 171-215 (1893)
6 Abramowitz, M.,Stegun I.A. eds.:’Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables’(10th Printing with corrections Dec 1972 of original June 1964. Applied Mathematics Series 55 United States Department of Commerce) p263, 6.5.31
7 Neilsen, Niels:’Handbuch der Theorie der Gammafunktion’1906 Teubner Leipzig p212, ChapXV
8 https://dlmf.nist.gov/8.7E3
9 Edwards, H.M.: ‘Riemann’s Zeta Function’(Dover Publications, 2001) p16
10 Heading, J.: ‘Mathematical Methods in Science and Engineering’(2nd Edition, Edward Arnold 1970) p115
11 Euler, L.: ‘De progressionibus transcendentibus....’*Comm. Acad. Sci. Petropolitanae* **5**, pp. 36-57 (1737).(Also "Opera" (1), Vol.14, pp. 1-24).

All figures created in MATLAB.