A NEW SPECIAL FUNCTION AND IT'S APPLICATION

ZERAOULIA RAFIK

This paper is dedicated to my students.

ABSTRACT. In this note we present a new special function such that behaves more like error function and since we arn't able to express it as elementary function using previous standard functions ,we only give it's simple expression in some range values using numerical approximation . and we will show how it helps to get values of complicated integral which they arn't available at wolfram alpha and in the same time we will show it's relationship with error function and cumulative distribution function .

1. INTRODUCTION

Integrals of the error function occur in a great variety of applications usually in problems involving multiple integration where the integrand contains exponentials of the squares of the argument, example of applications can be cited from atomic physics [1] , astrophysics [2] and statistical analysis [3] and [4] , This note gives and offer a new special function which is related to the error function and it has a relationship with Normal CDF [8] , This function is not montioned here [6] which it is a general reference for all standard error functions with them integration .Let us now starting to define that function with it's numerical approximation.

The function is defined over all real numbers and it is convergent only for positive line , for every $a \in (0, +\infty)$ as:

$$I(a) = \int_0^a \left(e^{-x^2}\right)^{\operatorname{erf}(x)} dx \tag{1.1}$$

with $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. is the error function .

1.1. Numerical approximation of $\int_0^a \left(e^{-x^2}\right)^{\operatorname{erf}(x)} dx$ in some ranges values.

Now, if we really need a simple expression for $I(a)$ in some range of values, there are ways to get various approximations.

The function is very nice. It goes to its limit at ∞ very very fast. Here's below in Figure 1. the plot of $I(a)$ for $a \in [0, 10]$:

So (depending on the accuracy we need) we can easily take $I(a) = I(\infty)$ for $a > a_0$ with a_0 around 3 or 4.

Mathematica gives for the first 100 digits:
$I(\infty) = 0.97210699276917859315107787544239117555427 2$
$18338556990097229104084418887599582200334106782184012587 34$

Date: 13 /02/2018.

2000 *Mathematics Subject Classification.* Primary 3317; Secondary 6525.

Key words and phrases. error function, special function,new special function.

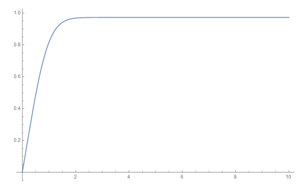

FIGURE 1. the plot of $I(a)$ for $a \in [0, 10]$

Now, what we can do for small a?

The function is so nice, we can just use the Taylor expansion around $a = 0$. The first term is:

$$I(a) \approx a$$

Here's the plot for $a \in [0, 1]$ The proof is simple. The Taylor series look like this:

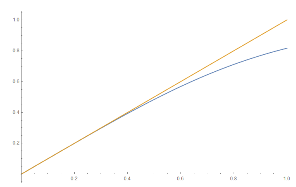

FIGURE 2. the plot of $I(a)$ for $a \in [0, 1]$

$$I(a) = I(0) + I'(0)a + \frac{I''(0)}{2!}a^2 + \frac{I'''(0)}{3!}a^3 + \dots$$

We can see that:

$$I(0) = 0$$

$$I'(0) = e^{-a^2(a)}\Big|_{a=0} = 1$$

Now let's find a better approximation by computing the higher derivatives:

$$I''(a) = \left(e^{-a^2(a)}\right)' = -\frac{2}{\sqrt{\pi}}ae^{-a^2(\mathrm{erf}(a)+1)}\left(\sqrt{\pi}e^{a^2}\mathrm{erf}(a) + a\right)$$

$$I''(0) = 0$$

—

we use Mathematica as a shortcut, but it's easy to do it by hand, if we remember that :

$$\mathrm{erf}'(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}$$

$$I'''(0) = 0$$

$$I^{IV}(0) = -\frac{12}{\sqrt{\pi}}$$

So our next approximation is:

$$I(a) \approx a - \frac{1}{2\sqrt{\pi}}a^4$$

The plot with both approximations (orange, green) and the function itself (blue) is given below: we can continue in the same way for higher derivatives. Now I admit

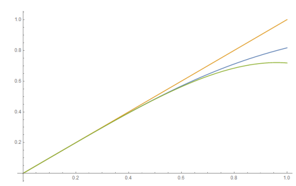

FIGURE 3

that it's possible you need the values of $I(a)$ for all the possible a and with high precision, so the approximations won't do. Then you need to turn to numerical integration (as Mathematica did for me to plot the function). Another way to approximate the function is using its derivative:

$$\frac{dI}{da} = e^{-a^2(a)}$$

But this is an ordinary differential equation, which can be solved numerically.

As an illustration, here's a simple explicit Euler scheme for the step size h:

$$\frac{I(a+h) - I(a)}{h} = e^{-a^2(a)}$$

$$I(a+h) = I(a) + he^{-a^2(a)}$$

We can use an initial value $I(0) = 0$.

For $h = \frac{1}{10}$ we have the following result (red dots) compared to the exact function (blue line):

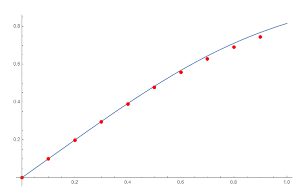

FIGURE 4

For $h = \frac{1}{50}$ see figure 5 :

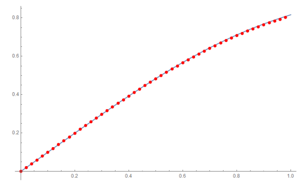

FIGURE 5

This way can serve as a good alternative to numerical integration (depending on the context and the application of course).

let us now to show the relationship between this function and other standard special functions as Error function and cumulative distribution function for normale distribution in the context of it uses , The function (1.1) could be used to find values of complicated integral which are not available in any references of standard special functions and also not available to get them values in wolfram alpha for example :

$$\int_0^{+\infty} e^{x^2(1-2\Phi(x\sqrt{2}))}dx, \tag{1.2}$$

with $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}\mathrm{d}z.$ is CDF (cumulative distribution function for normal distribution) if someone were asked to find the value of this integral he w'd be confused how he can do it's evaluation because it is very complicated and probably he can't show that is convergent or not and for wolfram alpha as a best means of computation can't recognize at a least that ϕ is a cumulative normal distribution then no result w'd be obtained about the value of this integral. let us compute (1.2) using (1.1) and we will conclude that they have the same value and both are identical function and identical integral.

The well known formula which express the relationship between Error function and Cumulative density function is defined as:

$\mathrm{Erf}(x) = 2(\Phi(x\sqrt{2}) - \Phi(0)) = 2\left(\Phi(x\sqrt{2}) - \frac{1}{2}\right) = 2\Phi(x\sqrt{2}) - 1.$ and it is easy to check that is always hold for every real numbers by the following short proof

Proof. By definition, the Error Function [7]:

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t. \tag{1.3}$$

Writing $t^2 = z^2/2$ implies $t = z/\sqrt{2}$ (because t is not negative), whence $\mathrm{d}t = \mathrm{d}z/\sqrt{2}$. The endpoints $t = 0$ and $t = x$ become $z = 0$ and $z = x\sqrt{2}$. To convert the resulting integral into something that looks like a cumulative distribution function (CDF), it must be expressed in terms of integrals that have lower limits of $-\infty$, thus:

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{2\pi}}\int_0^{x\sqrt{2}} e^{-z^2/2}\mathrm{d}z = 2\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x\sqrt{2}} e^{-z^2/2}\mathrm{d}z - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-z^2/2}\mathrm{d}z\right).$$

Those integrals on the right hand size are both values of the CDF of the standard Normal distribution,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} \mathrm{d}z.$$

Specifically,

$$\mathrm{Erf}(x) = 2(\Phi(x\sqrt{2}) - \Phi(0)) = 2\left(\Phi(x\sqrt{2}) - \frac{1}{2}\right) = 2\Phi(x\sqrt{2}) - 1. \tag{1.4}$$

□

Now from (1.4) we have :$\mathrm{Erf}(x) = 2\Phi(x\sqrt{2}) - 1.$,
then Multiplying both side by $-x^2$ we get this :
$-x^2\mathrm{Erf}(x) = -x^2(2\Phi(x\sqrt{2}) - 1.)$ which it is :$-x^2\mathrm{Erf}(x) = x^2(1 - 2(\Phi(x\sqrt{2}))$,Now just to raise exp for both side and integrating both side over positive real line $(0, \infty)$ we w'd get the following identity :

$$\int_0^{+\infty} \left(e^{-x^2}\right)^{\mathrm{erf}(x)} dx = \int_0^{+\infty} e^{x^2(1-2\Phi(x\sqrt{2}))} \tag{1.5}$$

Now since the LHS of (1.5) has a known value which it is $0.97210699\cdots$, then the right hand side also equal's :$0.97210699\cdots$ hence we came up with the following identity :

$$\int_0^{a} \left(e^{-x^2}\right)^{\mathrm{erf}(x)} dx = \int_0^{a} e^{x^2(1-2\Phi(x\sqrt{2}))} \tag{1.6}$$

and we are done it for every positive real number a

conclusion: The titled function could be used in wide area in the side of computation for complicated integral which montioned in probability and excuse me if i say : $zeraoulia(a) = \int_0^a \left(e^{-x^2}\right)^{\mathrm{erf}(x)} dx$ then $zeraoulia(+\infty) = 0.97210699\cdots$ since it's not refer to anyone .

Acknowledgements

I w'd like to thank Yuriy S for and j. jackline for their helping me to build this paper .

References

1. Zimering, *S., 1. Mathematical Phys. (1965)* I, Ann. Inst. Fourier (Grenoble) **6, 336** (1965)
2. Ng, E. W., *Astrophysical 1. ISO, 787 (1967).*, Amer. Math. Soc., Providence, RI, **1993**.
3. Zelen, M, . *and Severo, N. c.,* Chapter 26 of [5],
4. Amer , *Proc. Sympos. Appl. Math., vol. 36, . Math. Soc., Providence, RI,* 1986, pp. **3456**
5. Abramowitz, M. and Stegun, I. A. *(Editors), Handbook of Mathematical Functions, NBS AMS **55**, (Government Print ing Office', Washington , D. C. 1965)*
6. Edward W. Ng and Murray Geller *A Table of Integrals of the Error Functions,JOURNAL OF RESEARCH of the Natianal Bureau of Standards - B. Mathematical Sciences Vol. **73**B, No. 1, January-March 1969* (October 23, 1968)
7. *https://en.wikipedia.org/wiki/Error$_f$unction*
8. *http://mathworld.wolfram.com/NormalDistribution.html*

Department of Mathematics, university Batna, Algeria
Current address: Department of Mathematics ,Algeria
E-mail address: zeraouliarafik@gmail.com