A note on a problem invoving a square in a curvilinear triangle

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Abstract. A problem involving a square in the curvilinear triangle made by two touching congruent circles and their common tangent is generalized.

Keywords. square in a curvilinear triangle

Mathematics Subject Classification (2010). 01A27, 51M04

Let α_1 and α_2 be touching circles of radius a with external common tangent t. In this note we consider the following problem [1, 4, 5] (see Figure 1).

Problem 1. $ABCD$ is a square such that the side DA lies on t and the points C and B lie on α_1 and α_2, respectively. Show that $2a = 5|AB|$.

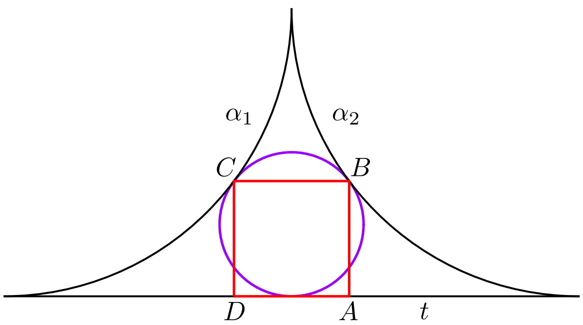

Figure 1.

If γ_1, γ_2, $\cdots$, γ_n are congruent circles touching a line s from the same side such that γ_1 and γ_2 touch and γ_i $(i = 3, 4, \cdots, n)$ touches γ_{i-1} from the side opposite to γ_1, then γ_1, γ_2, $\cdots$, γ_n are called congruent circles on s. The curvilinear triangle made by α_1, α_2 and t is denoted by Δ. The incircle of Δ touches α_1 and α_2 at C and B, respectively as in Figure 1. Indeed the problem is generalized as follows (see Figure 2):

Theorem 1. *If β_1, β_2, $\cdots$, β_n are congruent circles on t lying in Δ such that β_1 touches α_1 at a point C and β_n touches α_2 at a point B and A is the foot of perpendicular from B to t, then the following relations hold.*
(i) $n|AB| = |BC|$.
(ii) $2a = \left((\sqrt{n}+1)^2 + 1\right)|AB|$.

Proof. Let b be the radius of β_1. By Theorem 5.1 in [2] we have

$$a = (\sqrt{n}+1)^2 b. \tag{1}$$

Let $d = |AB|$. Since C divides the segment joining the centers of α_1 and β_1 in the ratio $a : b$ internally, we have

$$\frac{d-b}{b} = \frac{a-b}{a+b}. \tag{2}$$

Eliminating b from (1) and (2), and solving the resulting equation for d, we get $d = 2a/(1 + (1 \pm \sqrt{n})^2)$. But in the minus sign case we get $2b - d = 2a(1 - 4\sqrt{n})/(n^2 - n + 2\sqrt{n} + 2) < 0$ by (1). Hence $d = 2a/(1 + (1 + \sqrt{n})^2)$. This proves (ii). Let $|BC| = 2h$. Then from the right triangle formed by the line BC, the segment joining the centers of α_1 and β_1, and the perpendicular from the center of α_1 to BC, we get $(a - h)^2 + (a - d)^2 = a^2$. Solving the equation for h, we have $h = a - \sqrt{(2a - d)d} = an/(1 + (1 + \sqrt{n})^2)$. This proves (i). □

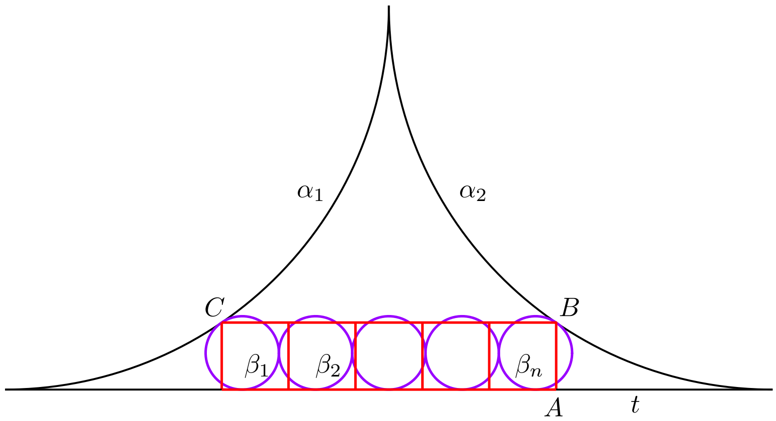

Figure 2: $n = 5$

The figure consisting of α_1, α_2, β_1, β_2, $\cdots$, β_n and t is denoted by $\mathcal{B}(n)$ and considered in [2]. Since the inradius of Δ equals $a/4$, the next theorem is also a generalization of Problem 1 (see Figure 3).

Theorem 2. *Let β_1, β_2, $\cdots$, β_n be congruent circles on a line s. If a circle α of radius a touches s and β_1 and β_n externally at points C and B, respectively, A is the foot of perpendicular from B to s, then the following relations hold.*
(i) $(n - 1)|AB| = |BC|$.
(ii) $2a = ((n - 1)^2 + 4)|AB|/4$.

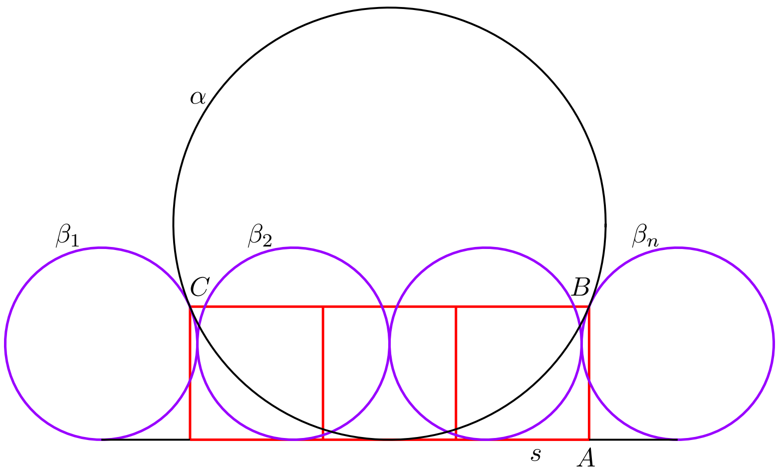

Figure 3: $n = 4$

Theorem 2 is proved in a similar way as Theorem 1 using the fact that the ratio of the radii of α and β_1 equals $(n - 1)^2 : 4$ [3]. The figure consisting of α, β_1, β_2, $\cdots$, β_n and s is denoted by $\mathcal{A}(n)$ and considered in [2].

References

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Tohoku Univ. WDB is short for Tohoku University Wasan Material Database.