Abel's Lemma and Dirichlet's Test Incorrectly Determine that a Trigonometric Version of the Dirichlet Series $\zeta(s) = \sum n^{-s}$ is Convergent Throughout the Critical Strip at $t \neq 0$

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Abstract

Euler's formula is used to derive a trigonometric version of the Dirichlet series $\zeta(s) = \sum n^{-s}$, which is divergent in the half-plane $\sigma \leq 1$, wherein $s \in \mathbb{C}$ and $s = \sigma + it$. Abel's lemma and Dirichlet's test incorrectly hold that trigonometric $\zeta(s)$ is convergent in the critical strip $0 < \sigma \leq 1$ at $t \neq 0$, because they fail to consider a divergent monotonically decreasing series (e.g. the harmonic series) in combination with a bounded oscillating function having an increasing period duration (e.g. $f(t, n) = \sin(t \cdot \ln(n))$).

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Contents

1 Introduction

Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{1.1}$$

is used to derive a trigonometric version of the Dirichlet series $\zeta(s)$,

$$\zeta(s) = \sum n^{-s} \tag{1.2}$$

the result being:

$$Re\Big[\zeta(s)\Big] = \sum_{n=1}^{\infty}\Big[n^{-\sigma}\cdot\cos\Big(t\cdot\ln(n)\Big)\Big] \tag{1.3}$$

$$Im\Big[\zeta(s)\Big] = -i\cdot\sum_{n=1}^{\infty}\Big[n^{-\sigma}\cdot\sin\Big(t\cdot\ln(n)\Big)\Big] \tag{1.4}$$

wherein $s \in \mathbb{C}$ and $s = \sigma + it$.

The Dirichlet series $\zeta(s) = \sum n^{-s}$ is known to be divergent in the half-plane $\sigma \leq 1$, wherein $s \in \mathbb{C}$ and $s = \sigma + it$. Trigonometric $\zeta(s)$ is shown in this paper to be divergent at the value of s corresponding to the first zero of the Riemann-Siegel formula on the critical line $\sigma = 0.5$ in the critical strip $0 < \sigma \leq 1$. Abel's lemma and Dirichlet's test incorrectly determine that trigonometric $\zeta(s)$ is convergent in the critical strip, where $t \neq 0$ (in other words, except on the σ-axis).

The errors in Abel's lemma and Dirichlet's test are identified. The sequence $f(n,\sigma) = n^{-\sigma}$ is monotonically decreasing and has a limit of zero, when $\sigma > 0$ and n is the set of non-zero natural numbers. But the series $f(n,\sigma) = \sum n^{-\sigma}$ is divergent if σ is in the critical strip $0 < \sigma \leq 1$. Moreover, the functions $f_1(t,n) = \cos\Big(t\cdot\ln(n)\Big)$ and $f_2(t,n) = \sin\Big(t\cdot\ln(n)\Big)$ are both bounded oscillating functions with ever-increasing half-period duration (due to the nested $\ln(n)$ expression). Both Abel's lemma and Dirichlet's test fail to consider such functions.

2 A Trigonometric Version of the Dirichlet Series, Derived by Using Euler's Formula

We define the zeta function as the following Dirichlet series:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{2.1}$$

The trigonometric version of the Dirichlet series is derived as follows. First, separate the real and imaginary portions of the complex exponent $s = \sigma + it$,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-(\sigma+it)} \tag{2.2}$$

and then separate the terms, based on the laws of exponents:

$$\zeta(s) = \sum_{n=1}^{\infty} \left[n^{-\sigma} \cdot n^{-it}\right] \tag{2.3}$$

The term n^{-it} can be rewritten into trigonometric form, first by rewriting it in exponential form

$$n^{-it} = \exp(\ln(n^{-it})) = e^{-it\cdot\ln(n)} \tag{2.4}$$

and then by using the Euler formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{2.5}$$

the term n^{-it} can be rewritten as:

$$n^{-it} = e^{-it\cdot\ln(n)} = \cos\Big(-t\cdot\ln(n)\Big) + i\sin\Big(-t\cdot\ln(n)\Big) \tag{2.6}$$

According to the laws of trigonometry, $\cos(-\theta) = \cos(\theta)$, and $\sin(-\theta) = (-\sin(\theta))$, so the above equation can be rewritten as follows:

$$n^{-it} = \cos\Big(t\cdot\ln(n)\Big) - i\sin\Big(t\cdot\ln(n)\Big) \tag{2.7}$$

Replacing n^{-it} in the Riemann zeta function with its trigonometric equivalent produces the following:

$$\zeta(s) = \sum_{n=1}^{\infty} \left[n^{-\sigma} \cdot \Big(\cos\big(t\cdot\ln(n)\big) - i\sin\big(t\cdot\ln(n)\big)\Big)\right] \tag{2.8}$$

This trigonometric version of the Riemann zeta function can be separated into real and imaginary components:

$$Re\Big[\zeta(s)\Big] = \sum_{n=1}^{\infty} \left[n^{-\sigma} \cdot \cos\Big(t\cdot\ln(n)\Big)\right] \tag{2.9}$$

$$Im\Big[\zeta(s)\Big] = -i \cdot \sum_{n=1}^{\infty} \Big[n^{-\sigma} \cdot \sin\Big(t \cdot \ln(n)\Big)\Big] \tag{2.10}$$

3 The Trigonometric Dirichlet Series is Divergent Along the Sigma Axis in the Critical Strip

The σ-axis (the line $t = 0$) in the complex plane s corresponds to the set of real values ($s \in \mathbb{R}$). Along the σ-axis the trigonometric Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \Big[n^{-\sigma} \cdot \Big(\cos\big(t \cdot \ln(n)\big) - i \sin\big(t \cdot \ln(n)\big)\Big)\Big] \tag{3.1}$$

is reduced to

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-\sigma} \tag{3.2}$$

because $t = 0$. The values of s on the σ-axis do not have any imaginary component. So,

$$Re\Big[\zeta(s)\Big] = \sum_{n=1}^{\infty} n^{-\sigma} \tag{3.3}$$

$$Im\Big[\zeta(s)\Big] = 0 \tag{3.4}$$

The divergence of $Re\Big[\zeta(s)\Big]$ in the critical strip ($0 < \sigma \leq 1$, $t = 0$) is easily proven, by the integral test for convergence (a.k.a. the Maclaurin–Cauchy test). For example, according to the integral test at $\sigma = 0.5$,

$$Re\Big[\zeta(0.5)\Big] = \int_1^M \frac{1}{n^{0.5}} = \int_1^M n^{-0.5} = \Big(2 \cdot n^{0.5} + C\Big)\Big|_1^M = \Big(2 \cdot M^{0.5} - 2\Big)$$

and $(2 \cdot M^{0.5} - 2) \to \infty$ as $M \to \infty$. Conceptually, at $0 < \sigma \leq 1$ the series is an infinite sum of items that are successively smaller in size as $M \to \infty$, but whose sum continues to grow.

4 The Trigonometric Dirichlet Series is Divergent at a Zero of the Riemann-Siegel Formula in the Critical Strip

As discussed in a preceding section, the Trigonometric Dirichlet Series along the line segment $(0 < \sigma \leq 1, t = 0)$, is defined as:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-\sigma} \tag{4.1}$$

and is divergent throughout the line segment $(0 < \sigma \leq 1, t = 0)$.

But for values of s in the critical strip $(0 < \sigma < 1)$ and along the line $\sigma = 1$ where $t \neq 0$, the trigonometric version of the Dirichlet series of the Riemann zeta function is:

$$\zeta(s) = \sum_{n=1}^{\infty} \left[n^{-\sigma} \cdot \Big(\cos\big(t \cdot \ln(n)\big) - i \sin\big(t \cdot \ln(n)\big) \Big) \right] \tag{4.2}$$

This formula can be separated into real and imaginary portions, as in Equations (2.8) and (2.9):

$$Re\Big[\zeta(s)\Big] = \sum_{n=1}^{\infty} \left[n^{-\sigma} \cdot \cos\Big(t \cdot \ln(n)\Big) \right] \tag{4.3}$$

$$Im\Big[\zeta(s)\Big] = -i \cdot \sum_{n=1}^{\infty} \left[n^{-\sigma} \cdot \sin\Big(t \cdot \ln(n)\Big) \right] \tag{4.4}$$

At $t = 0$, as indicated by the p-test, the Dirichlet series $f(n) = \sum n^{-\sigma}$ is divergent for values of s located on the line segment $(0 < \sigma \leq 1, t = 0)$. At $t \neq 0$, multiplying the divergent Dirichlet series by $\cos(t \cdot \ln(n))$ in the real component, and by $\sin(t \cdot \ln(n))$ in the imaginary component (both of which have a nested logarithmic function of n), does not produce a conditionally convergent function.

Instead, as shown in the Figures below, the nested $\ln(n)$ function forces the partial sums of $\zeta(s)$ to oscillate between diverging to $+\infty$ and diverging to $-\infty$, as $n \to \infty$. Both the real and imaginary portions of the Dirichlet series behave in this manner.

These results can be proven as follows, beginning with a discussion of the imaginary portion of the Zeta function,

$$Im\Big[\zeta(s)\Big] = \sum_{n=1}^{\infty} \left[n^{-\sigma} \cdot \sin\Big(t \cdot \ln(n)\Big) \right] \tag{4.5}$$

The function $f(n) = \sin\Big(t \cdot \ln(n)\Big)$ is a bounded oscillating function with ever-increasing half-period duration (due to the nested $\ln(n)$ expression). This is shown by comparing the zeros of

$$f_1(n) = \sin(t \cdot n) \tag{4.6}$$

to the zeros of

$$f_2(n) = \sin(t \cdot \ln(n)) \tag{4.7}$$

For the sake of the following discussion only, assume that $n \in \mathbb{R}$.

The function $f_1(n) = \sin(t \cdot n)$ has zeros at

$$n = 0, \frac{\pi}{t}, \frac{2\pi}{t}, \frac{3\pi}{t}, ...\frac{n\pi}{t} \tag{4.8}$$

so each half-period is of equal duration. If $t = 1$, the zeros of $f_1(n)$ are at

$$n = 0, \pi, 2\pi, 3\pi, ...n\pi \tag{4.9}$$

and thus each half-period has a duration equal to π.

In contrast, the function $f_2(n) = \sin(t \cdot \ln(n))$ has zeros at

$$n = \frac{e^{\pi}}{t}, \frac{e^{2\pi}}{t}, \frac{e^{3\pi}}{t}, \frac{e^{4\pi}}{t}, ...\frac{e^{n\pi}}{t} \tag{4.10}$$

so if $t = 1$, the zeros of $f_2(n)$ are at

$$n = e^{\pi}, e^{2\pi}, e^{3\pi}, e^{4\pi}, ...e^{n\pi} \tag{4.11}$$

So the durations of the first three half-periods p_1, p_2, and p_3 are

$$p_1 = e^{2\pi} - e^{\pi} = e^{\pi}(e^{\pi} - 1) \tag{4.12}$$

$$p_2 = e^{3\pi} - e^{2\pi} = e^{2\pi}(e^{\pi} - 1) = e^{\pi} p_1 \tag{4.13}$$

$$p_3 = e^{4\pi} - e^{3\pi} = e^{3\pi}(e^{\pi} - 1) = e^{\pi} p_2 \tag{4.14}$$

so by induction, each subsequent half-period has a duration that is larger than its immediate predecessor by a factor of e^{π} (which is approximately 23.14).

Since it is assumed here that $n \in \mathbb{R}$, replace the summation symbol $\sum$ with the integral symbol $\int$, as follows

$$Im\Big[\zeta(s)\Big] = \int_{n=+0}^{\infty} \Big[n^{-\sigma} \cdot \sin\Big(t \cdot \ln(n)\Big)\Big] \tag{4.15}$$

Now, break up the infinite series of the imaginary portion of the zeta function, into finite portions. This is done at the zeros of $f_2(n) = \sin(t \cdot \ln(n))$. (Ignore the very first portion from $n = +0$ to the first zero at $n = \exp(\pi)/t$. In an infinite series, it is irrelevant). The function $f_3(n)$ used in the equations below is defined as $f_3(n) = n^{-\sigma} \cdot \sin(t \cdot \ln(n))$

$$Im\Big[\zeta(s)\Big] = \int_{\exp(\pi)/t}^{2\exp(\pi)/t} f_3(n) + \int_{2\exp(\pi)/t}^{3\exp(\pi)/t} f_3(n) + \int_{3\exp(\pi)/t}^{4\exp(\pi)/t} f_3(n) + ... \tag{4.16}$$

The absolute value of each finite portion is larger than the absolute value of the finite portion that precedes it. (This is easily derived from the results of the computer program, as shown in the next section).

$$\Big|\int_{\exp(\pi)/t}^{2\exp(\pi)/t} f_3(n)\Big| < \Big|\int_{2\exp(\pi)/t}^{3\exp(\pi)/t} f_3(n)\Big| < \Big|\int_{3\exp(\pi)/t}^{4\exp(\pi)/t} f_3(n)\Big| < ... \tag{4.17}$$

Since $f_3(n)$ comprises a sine function, the finite portions also alternate in terms of +/- sign. The increasing magnitudes, combined with the alternating signs, results in the sum of the finite portions of $Im\Big[\zeta(s)\Big]$ oscillating between diverging to $+\infty$ to diverging to $-\infty$, as $n \to \infty$.

The above discussion is also applicable to $Re\Big[\zeta(s)\Big]$, because the only difference between $Im\Big[\zeta(s)\Big]$ and $Re\Big[\zeta(s)\Big]$ is that $Im\Big[\zeta(s)\Big]$ has a cosine function instead of a sine function. This means that $\sin(t \cdot \ln(n))$ and $\cos(t \cdot \ln(n))$ differ by a phase shift of $\pi/2$, because $\sin(\theta + \pi/2) = \cos(\theta)$, and $\theta = t \cdot \ln(n)$.

5 Computer Calculations Confirm that the Trigonometric Dirichlet Series is Divergent at a Zero of the Riemann-Siegel Formula in the Critical Strip

The Figures below were created using data output from the computer program listed at the end of this paper. The graphs confirm the oscillating divergence of the real and imaginary portions of the Dirichlet series at ($\sigma = 0.5$, $t = 14.134725$). These coordinates approximately correspond to the first zero of the analytic continuation of

$\zeta(s)$, according to Odlyzko's[Odl] calculations of the Riemann-Siegel formula. (The value of t has been truncated to $t = 14.134725$). The simulation was run for $n = 1$ to 8.0 x 10^8.

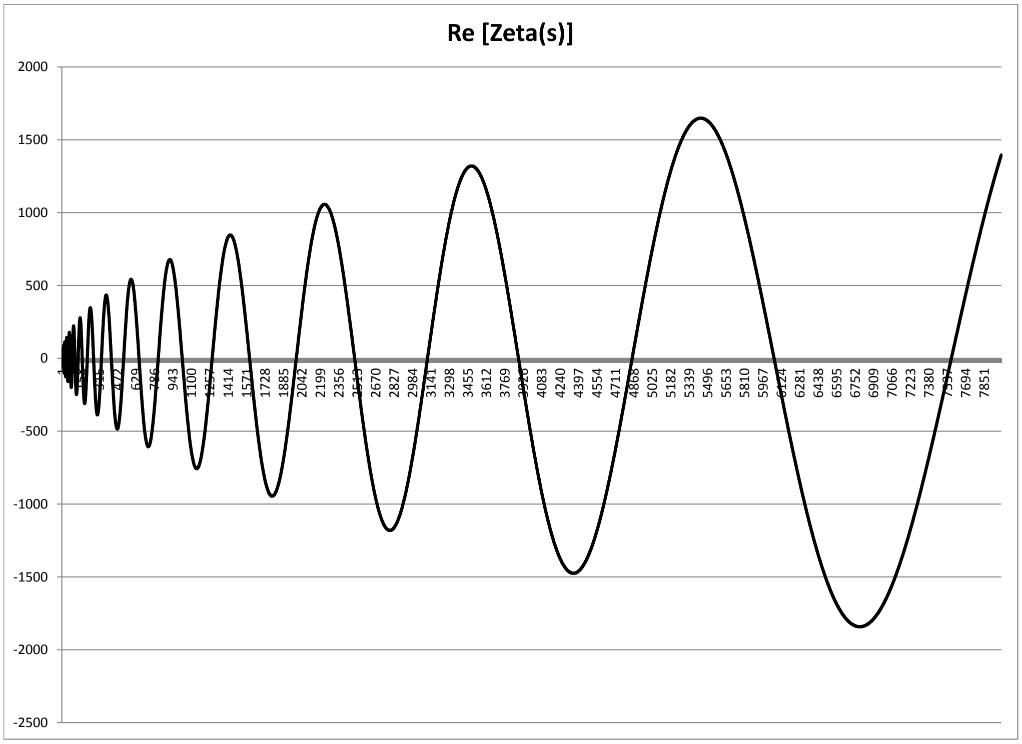

Figure 2: $Re[\zeta(s)]$ **at** ($\sigma = 0.5$**,** $t = 14.134725$**), and** $n = 1$ **to** 8.0 **x** 10^8**.**

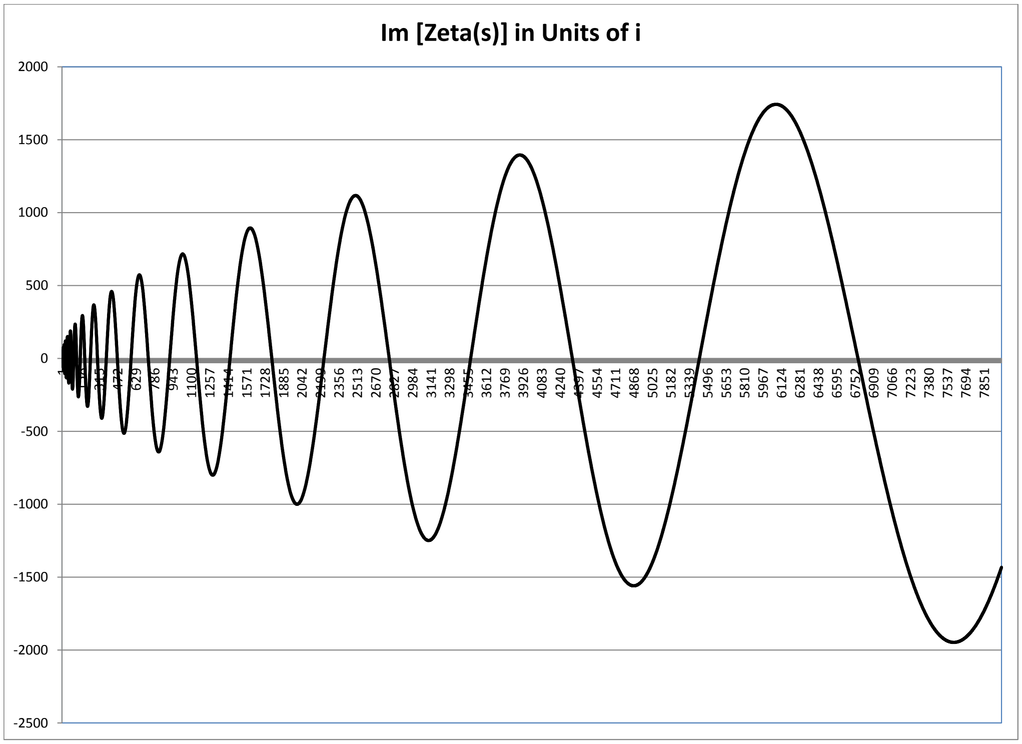

Figure 3: $Im[\zeta(s)]$**, at (**$\sigma = 0.5$**,** $t = 14.134725$**), and** $n = 1$ **to** 8.0 **x** 10^8**.**

6 Abel's Lemma, and Its Proof

It has been shown that trigonometric $\zeta(s)$ is divergent at the first zero of the Riemann-Siegel formula, which is located in the region $(0 < \sigma \leq 1,\ t \neq 0)$. Abel's lemma incorrectly determines that trigonometric $\zeta(s)$ is convergent at this point. Tolstov's [Tol16] version of Abel's Lemma[1], and the proof thereof, are provided below. (Abbott's [Abb10] version of Abel's lemma[2] is similar and discussed later in this paper).

Abel's Lemma. *Let*

$$u_0 + u_1 + u_2 + \cdots + u_n + \cdots$$

be a numerical series (with real or complex terms), whose partial sums σ_n *satisfy the*

[1]pages 97-98
[2]page 171

condition

$$|\sigma_n| \leq M,$$

where M is a constant. Then, if the positive numbers $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ approach zero monotonically, the series

$$\alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n + \cdots \tag{6.1}$$

converges, and its sum s satisfies the inequality

$$|s| \leq M\alpha_0 \tag{6.2}$$

Proof of Abel's Lemma. Let

$$s_n = \alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n;$$

then, since $u_0 = \sigma_0$ and $u_n = \sigma_n - \sigma_{n-1}$ for $(n = 1, 2, \cdots)$,

$$s_n = \alpha_0(\sigma_0) + \alpha_1(\sigma_1 - \sigma_0) + \alpha_2(\sigma_2 - \sigma_1) + \cdots + \alpha_n(\sigma_n - \sigma_{n-1});$$

or

$$s_n = \sigma_0(\alpha_0 - \alpha_1) + \sigma_1(\alpha_1 - \alpha_2) + \cdots + \sigma_{n-1}(\alpha_{n-1} - \alpha_n) + \sigma_n\alpha_n.$$

It follows that

$$s_n - \sigma_n\alpha_n = \sigma_0(\alpha_0 - \alpha_1) + \sigma_1(\alpha_1 - \alpha_2) + \cdots + \sigma_{n-1}(\alpha_{n-1} - \alpha_n). \tag{6.3}$$

Now consider the series

$$\sigma_0(\alpha_0 - \alpha_1) + \sigma_1(\alpha_1 - \alpha_2) + \cdots + \sigma_{n-1}(\alpha_{n-1} - \alpha_n) + \cdots . \tag{6.4}$$

This series converges, since the absolute values of its terms do not exceed the corresponding terms of the following convergent series with nonnegative terms:

$$M(\alpha_0 - \alpha_1) + M(\alpha_1 - \alpha_2) + \cdots + M(\alpha_{n-1} - \alpha_n) + \cdots$$
$$= M(\alpha_0 - \alpha_1 + \alpha_1 - \alpha_2 + \cdots + \alpha_{n-1} - \alpha_n + \cdots)$$
$$= M\alpha_0. \tag{6.5}$$

The right-hand side of Equation (6.3) is the nth partial sum of the series (6.4), and therefore, as $n \to \infty$, it approaches a definite limit, whose absolute value does not exceed the number $M\alpha_0$. But then the left-hand side of Equation (6.3) also approaches a limit as $n \to \infty$, and

$$|\lim_{n\to\infty}(s_n - \sigma_n\alpha_n)| \leq M\alpha_0$$

Since

$$|\sigma_n\alpha_n| \leq M\alpha_n$$

thus

$$\lim_{n\to\infty} \sigma_n\alpha_n = 0.$$

Therefore,

$$\lim_{n\to\infty} s_n = s$$

exists, i.e., the series (6.1) converges, and s satisfies the inequality (6.2).

7 Abel's Lemma is Incorrect when Applied to the Trigonometric Dirichlet Series in the Critical Strip

Tolstov's [Tol16] version of Abel's Lemma[3] and Abbott's [Abb10] version of Abel's lemma[4] incorrectly determine that the Riemann Zeta Function $\zeta(s)$ is convergent at the first zero of the Riemannn-Siegel formula, because the trigonometric $\zeta(s)$ at that those coordinates satisfies all of the explicit conditions of Abel's lemma.

The expression $f_1(t,n) = \cos\Big(t \cdot \ln(n)\Big)$ forms a numerical series of real terms, and the expression $f_2(t,n) = -i\sin\Big(t\cdot\ln(n)\Big)$ forms a numerical series of imaginary terms. Each of these series can be assigned to Abel's numerical series $u_0 + u_1 + u_2 + \cdots + u_n$, whose partial sums σ_n satisfy the condition $|\sigma_n| \leq M$, wherein M is a constant. This

[3] pages 97-98

[4] page 171

is because both sine and cosine functions are bounded functions, with both maximum and minimum partial sums. (The nested logarithm function does not change this fact).

Moreover, the positive numbers $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n$ of the Dirichlet series $\sum n^{-\sigma}$ monotonically approach a limit of zero, as n increases to ∞. However, at any point in the region $(0 < \sigma \leq 1,\ t \neq 0)$, contrary to the determination according to Abel's lemma, the series $\alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$ does not converge, and its sum s does not satisfy the inequality $|s| \leq M\alpha_0$.

This is because for any point in the region $(0 < \sigma \leq 1, t \neq 0)$, neither the right-hand side of Equation (6.3) (which is the nth partial sum of the series (6.4)) approaches a definite limit as $n \to \infty$, nor does the left-hand side of Equation (6.3) approach a limit as $n \to \infty$,

Abel's lemma overlooks the possibility that there exists a sequence of positive numbers $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n$ that monotonically decreases to zero and also forms a divergent series when summed. Abel's lemma also overlooks the possibility that sequence

$$u_0, u_1, u_2, \cdots, u_n, \cdots$$

is bounded, but with a period that increases in duration with each half-cycle. The trigonometric Dirichlet Series in the critical strip $(0 < \sigma \leq 1,\ t \neq 0)$ satisfies both of these criteria.

Abbott's [Abb10] version of Abel's Lemma (page 171) does not expressly claim that the series

$$\alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n + \cdots$$

converges. It only claims that the sum s of the above series satisfies the inequality

$$|s| \leq 2M\alpha_0$$

which is less strict than Tolstov's version that claims that said sum s satisfies the inequality

$$|s| \leq M\alpha_0$$

However, since Abbott's version claims that this inequality holds true for all $n \in \mathbb{N}$, Abbott's version inherently (and erroneously) claims that the series

$$\alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n + \cdots$$

is bounded by the constant $2M\alpha_0$ (and therefore does not diverge to $+\infty$ or $-\infty$, or oscillate between diverging to one and then to the other). Abbott's version of Abel's lemma is wrong for the same reason that Tolstov's version is wrong.

8 Dirichlet's Test is Incorrect when Applied to the Trigonometric Dirichlet Series in the Critical Strip

Dirichlet's test for convergence incorrectly determines that the Dirichlet Series of the Riemann Zeta function $\zeta(s)$ is convergent in the strip $(0 < \sigma \leq 1, t \neq 0)$. As defined by Abbott (page 69) [Abb10]:

"Dirichlet's Test for convergence states that if the partial sums of

$$\sum_{n=1}^{\infty} x_n \tag{8.1}$$

are bounded (but not necessarily convergent), and if (y_n) is a sequence satisfying

$$y_1 \geq y_2 \geq y_3 \geq \ldots \geq 0 \tag{8.2}$$

with $\lim y_n = 0$ then the series $\sum_{n=1}^{\infty} x_n y_n$ converges."

Additional versions can be found at Weisstein[Wei], Haggstrom[Hag12], Feldman[Fel08], and Wikipedia[Ano].

The trigonometric Riemann zeta function $\zeta(s)$ in the critical strip $(0 < \sigma \leq 1, t \neq 0)$, defines the function x_n as $x_n = \sin\big(t \cdot \ln(n)\big)$, and y_n as $y_n = 1/n^{\sigma}$. These functions x_n and y_n satisfy all of the requirements of Dirichlet's test. Regarding the partial sums of x_n:

$$\Big|\sum_{n=1}^{N} \sin\big(t \cdot \ln(n)\big)\Big| \leq M \tag{8.3}$$

for every positive integer N, wherein M is a constant greater than or equal to the integral of sine over the interval $[0, \pi]$:

$$M \geq \int_0^{\pi} \sin(x) \cdot dx \tag{8.4}$$

Moreover, regarding the function y_n, the trigonometric $\zeta(s)$ satisfies these requirements as well:

$$\frac{1}{n^{\sigma}} \geq \frac{1}{(n+1)^{\sigma}} \geq \frac{1}{(n+2)^{\sigma}} \ldots \geq 0 \tag{8.5}$$

and as $n \to \infty$,

$$\lim_{n\to\infty} \frac{1}{n^\sigma} = 0 \tag{8.6}$$

Therefore, if $0 < \sigma \leq 1$ and $t \neq 0$, then Dirichlet's Test incorrectly determines that the imaginary portion of the trigonometric Dirichlet series converges:

$$Im\Big[\sum_{n=1}^{\infty} x_n y_n\Big] = -i \cdot \sum_{n=1}^{\infty} \frac{\sin\big(t \cdot \ln(n)\big)}{n^\sigma} \tag{8.7}$$

However, in reality, $\sum 1/n^\sigma$ is divergent when $(0 < \sigma \leq 1)$, and $f(t, n) = \sin\big(t \cdot \ln(n)\big)$ has a period that increases with each half-cycle. So the series is divergent.

In regards to the real portion of the trigonometric Dirichlet series:

$$Re\Big[\sum_{n=1}^{\infty} x_n y_n\Big] = \sum_{n=1}^{\infty} \frac{\cos\big(t \cdot \ln(n)\big)}{n^\sigma} \tag{8.8}$$

the results of Dirichlet's test differ from the results of Dirichlet's test for the imaginary portion only at $t = 0$. Unlike the series produced by the sine function (which equals zero when $t = 0$), the series produced by the cosine function becomes the following at $t = 0$:

$$Re\Big[\sum_{n=1}^{\infty} x_n y_n\Big] = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \tag{8.9}$$

which according to the P-Test is divergent throughout the line segment $(0 < \sigma \leq 1, t = 0)$. Therefore, except in regards to values of s that are directly on the σ-axis in the critical strip, Dirichlet's test incorrectly determines that the trigonometric Dirichlet series is convergent in the critical strip $(0 < \sigma \leq 1)$.

9 Conclusion

Abel's lemma and Dirichlet's test incorrectly determine that the trigonometric Dirichlet series is convergent in the critical strip $(0 < \sigma \leq 1)$.

10 C Language Source Code Used to Calculate the Trigonometric Riemann Zeta Function

The following is the C language source code used to generate the data for Figures 2 and 3. Please note: The constant "sigma" is assigned a negative value, in order to avoid the step of multiplying by -1 during the computation of $n^{-\sigma}$.

```
#include <stdio.h>
#include <math.h>

int main()
{
    double sigma = -0.500000;
    double t = 14.134725;
    double sum_real = 0;
    double sum_imaginary = 0;

     FILE *file_ptr ;
     file_ptr = fopen( "Zeta_data.txt", "w" ) ;

     if ( file_ptr != NULL )
        {
          printf( "File created\n" );

          printf("N                    1/(n^sig)    nth_Real
             nth_Imag     Sum_Real        Sum_Imag    \n");

          printf("--------------- ------------    ------------
             ------------     ----------------    -----------------
             \n");

          fprintf( file_ptr, "N                    1/(n^sig)
             nth_Real     nth_Imag     Sum_Real        Sum_Imag \n");

          fprintf( file_ptr, "--------------- ------------
             ------------    ------------     ----------------
             ----------------- \n");

          for (register int n=1; n<800000001; n++)
          {
            double ln_n = log((double) n);
            double theta = t * ln_n;

            double coefficient = pow((double) n, sigma);   // This
               is 1 / (n ^ sigma)

            double real = coefficient * cos(theta);
            double imaginary = coefficient * sin(theta);
```

```
            double sum_real = sum_real + real;
            double sum_imaginary = sum_imaginary - imaginary;

            if (n == 1 || n % 100000 == 0)
              {
               printf("%d  %f  %f   %f   %f  %f \n", n, coefficient
                   , real, imaginary, sum_real, sum_imaginary);
               fprintf( file_ptr, "%d  %f  %f   %f   %f  %f  \n", n
                   , coefficient, real, imaginary, sum_real,
                   sum_imaginary);
              }
          }
      printf("\n");
      fclose( file_ptr ) ;
      return 0;
    }
    else
      { printf( "Unable to create file\n" ) ; return 1; }
}
```

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