Solution of a High-School Algebra Problem to Illustrate the Use of Elementary Geometric (Clifford) Algebra

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Acknowledgment

I thank Dennis Richards, a teacher at Rio Lindo Academy, Healdsburg, California, for suggesting that the LinkedIn group "Pre-University Geometric Algebra" (https://www.linkedin.com/groups/8278281) prepare documents such as this for the benefit of high school teachers.

1 Introduction

This document is the first in what is intended to be a collection of solutions of high-school-level problems via Geometric Algebra (GA). GA is very much "overpowered" for such problems, but students at that level who plan to go into more-advanced math and science courses will benefit from seeing how to "translate" basic problems into GA terms, and to then solve them using GA identities and common techniques.

I invite readers to send their comments, criticisms, and suggestions. Your participation in the LinkedIn group "Pre-University Geometric Algebra" (https://www.linkedin.com/groups/8278281) will be welcome and appreciated.

2 Statement of the Problem

The following problem was contributed by Dennis Richards, a teacher at Rio Lindo Academy, Healdsburg, California, on 16 February 2018:

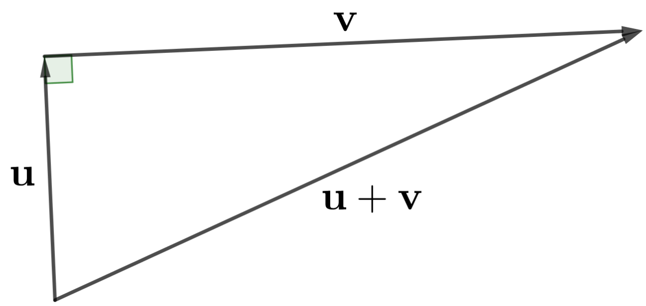

Figure 1: The vector representation of the right triangle discussed in Solution 1 to Problem 1.

> A piece of wire 400mm long is bent to form a right-angled triangle whose hypotenuse is 170mm long. What are the lengths of the adjacent sides?

2.1 Solution 1

(Contributed by Nicholas White.)

Please note that this problem does not require the power of GA, since the problem does not involve vectors where both the length and direction are necessary for the problem's solution. Define $\mathbf{u}$ and $\mathbf{v}$ as shown in Fig. 1, such that the lengths of the adjacent sides are the two vectors' norms ($\|\mathbf{u}\|$ and $\|\mathbf{v}\|$), and the hypotenuse is the vector $\mathbf{u}+\mathbf{v}$. Thus, using [1] (p. 58) as reference,

$$\|\mathbf{u}+\mathbf{v}\| = 170\text{mm} \qquad \text{(length of hypotenuse)} \tag{2.1.1}$$

The length of the perimeter of the triangle is equal to the length of the wire:

$$\|\mathbf{u}\| + \|\mathbf{v}\| + \|\mathbf{u}+\mathbf{v}\| = 400\text{mm}. \tag{2.1.2}$$

Using the Pythagorean Theorem (a special case of a theorem in Ref. [1] p. 58) gives

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u}+\mathbf{v}\|^2 = 170^2. \tag{2.1.3}$$

Next, we recognize that the sum of the lengths of the adjacent sides is equal to the perimeter minus the length of the hypotenuse: $\|\mathbf{u}\| + \|\mathbf{v}\| = 400 - 170 = 230\text{mm}$. Therefore, $\|\mathbf{u}\| = 230\text{mm} - \|\mathbf{v}\|$. Substituting into Eq. (2.1.3), we obtain the quadratic equation

$$230^2 - 460\|\mathbf{v}\| + 2\|\mathbf{v}\|^2 = 170^2,$$
$$\therefore \quad 2\|\mathbf{v}\|^2 - 460\|\mathbf{v}\| + 230^2 - 170^2 = 0, \tag{2.1.4}$$

which we solve via the quadratic formula to find that there are two solutions: $\|\mathbf{v}\| = 150\text{mm}$, in which case $\|\mathbf{u}\| = 80\text{mm}$, and $\|\mathbf{v}\| = 80\text{mm}$, in which case $\|\mathbf{u}\| = 150\text{mm}$

2.2 Solution 2

(Contributed by Jim Smith.)

The vector representation that we will use in this solution is shown in Fig. 2 .

2.2.1 Review of possibly-useful facts, definitions, observations, and methods

See also Refs. [2] and [3].

- Because of the closure property of vector addition, the sum of two or more vectors evaluates to a single vector. Thus, we may, as suits us, think of $\mathbf{c} + \mathbf{b}$ as the sum of those two vectors, or as a single vector.
- The length of a segment (for example, of a side of a polygon) is equal to the norm of the vector that joins that segment's endpoints. Thus, the lengths of the two adjacent sides are $\|\mathbf{b}\|$ and $\|\mathbf{c} + \mathbf{b}\|$, while that of the hypotenuse is $\|\mathbf{c}\|$.
- The square of a vector is equal to the dot product of the vector with itself.
- The square of the norm of a vector is the square of the vector itself. Thus, the norm of a vector is the square root of the square of the vector itself. Therefore,

$$\begin{aligned}\|\mathbf{c}+\mathbf{b}\| &= \sqrt{(\mathbf{c}+\mathbf{b})(\mathbf{c}+\mathbf{b})}\\ &= \sqrt{\mathbf{cc}+\mathbf{cb}+\mathbf{bc}+\mathbf{bb}}\\ &= \sqrt{\|\mathbf{c}\|^2+\mathbf{c}\cdot\mathbf{b}+\mathbf{c}\wedge\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\wedge\mathbf{c}+\|\mathbf{b}\|^2}\\ &= \sqrt{\|\mathbf{c}\|^2+2\mathbf{c}\cdot\mathbf{b}+\|\mathbf{b}\|^2},\end{aligned}$$

  because $\mathbf{c}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{c}$. We can write that result more conveniently as

$$\|\mathbf{c}+\mathbf{b}\| = \sqrt{c^2+2\mathbf{c}\cdot\mathbf{b}+b^2}, \qquad (2.2.1)$$

  where b $(= \|\mathbf{b}\|)$ and c $(= \|\mathbf{c}\|)$ are the lengths of the corresponding sides. The term $\mathbf{c}\cdot\mathbf{b}$ is unknown, but see below.
- The vectors $\mathbf{b}$ and $\mathbf{c}+\mathbf{b}$ are perpendicular; therefore, $(\mathbf{c}+\mathbf{b})\cdot\mathbf{b} = 0$. Expanding the left-hand side of that equation, and rearranging, we obtain an expression for $\mathbf{c}\cdot\mathbf{b}$:

$$\mathbf{c}\cdot\mathbf{b} = -b^2. \qquad (2.2.2)$$

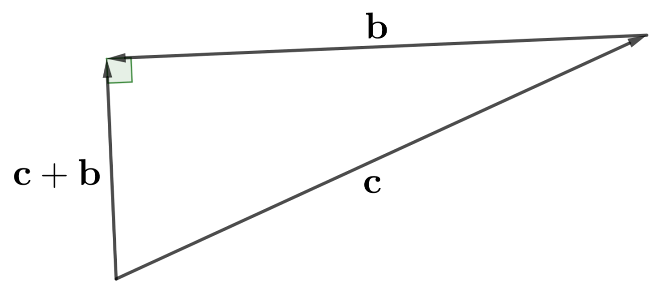

Figure 2: The vector representation of the right triangle discussed in Solution 2 to Problem 1.

Thus, in Eq. (2.2.1), we can express the unknown term $\mathbf{c} \cdot \mathbf{b}$ (which doesn't interest to us) in terms of b, which we wish to identify.

- From the statement of the problem, we can deduce that $\|\mathbf{c} + \mathbf{b}\|$ (i.e., the length of the side represented by that vector) is $400\text{mm} - 170\text{mm} - b$; therefore
$$\|\mathbf{c} + \mathbf{b}\| = 230\text{mm} - b. \tag{2.2.3}$$

2.2.2 Formulation of a strategy, and the solution

Our review of facts, observations, and methods, etc. indicates that by using Eqs. (2.2.2) and (2.2.3) to make appropriate substitutions in Eq. (2.2.1), we will obtain a quadratic equation that we can solve for b:

$$\begin{aligned}
\|\mathbf{c} + \mathbf{b}\| &= \sqrt{c^2 + 2\mathbf{c} \cdot \mathbf{b} + b^2} \\
230 - b &= \sqrt{c^2 + 2\left(-b^2\right) + b^2} \\
230^2 - 460b + b^2 &= c^2 - b^2 \\
\therefore 2b^2 - 460b + 230^2 - 170^2 &= 0, \quad \text{because } c = 170\text{mm}.
\end{aligned} \tag{2.2.4}$$

Equation (2.2.4) is identical to Eq. (2.1.4), and yields two solutions: (1) $b = 150\text{mm}$, $\|\mathbf{c} + \mathbf{b}\| = 80\text{mm}$; and (2) $b = 80\text{mm}$, $\|\mathbf{c} + \mathbf{b}\| = 150\text{mm}$ (Fig. 3). The two solutions are actually identical.

References

[1] A. Macdonald, *Linear and Geometric Algebra* (First Edition) p. 126, CreateSpace Independent Publishing Platform (Lexington, 2012).

[2] D. Hestenes, 1999, *New Foundations for Classical Mechanics*, (Second Edition), Kluwer Academic Publishers (Dordrecht/Boston/London).

[3] J. A. Smith, 2017, "Some Solution Strategies for Equations that Arise in Geometric (Clifford) Algebra", http://vixra.org/abs/1610.0054 .

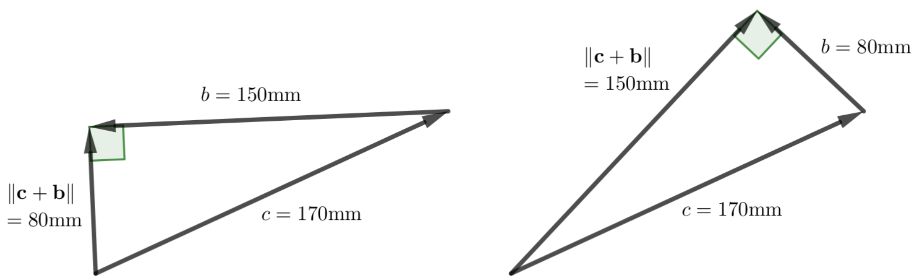

Figure 3: The two solutions (which are actually identical) obtained via the second method.