# Spinors, Twistors, Quaternions and Complex Space

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Utilizing the spinor approach, electromagnetic and gravitational metrics are mapable to the twistor algebra, which corresponds to the complexified Minkowski space. Quaternion transformations relate to spin and rotation corresponding to the twistor analysis.

## 1. Introduction

In this chapter we will present a formalism that uniquely relates electromagnetic and gravitational fields. Through this formalism and the relationship of the spinor calculus and the twistor algebra we can demonstrate the fundamental conditions of such a system which accommodates macroscopic astrophysical phenomena as well as microscopic quantum phenomena. The generalized Minkowski formalism has large scale astrophysical as well as quantum level consequences.

The generalized hyperdimensional Minkowski manifold has nonlocal as well as anticipatory properties. Also, briefly we discuss the $720^\circ$ symmetry of the so-termed Dirac string trick within the context of the relativistic form of the Dirac formalism. Twistors and spinors are examined and are applicable to the quaternion formalism. The quaternion formalism can be related to the hyperdimensional complexified Minkowski space, Lie groups, $SU_n$, as well as Riemannian topologies and the Dirac equation.

In Sec. 2 we present the formalism for the role of the spinor calculus which is utilized to relate the expression for the metric tensor to gravitational and electromagnetic field components through the relationship of the twistor algebra and spinor calculus. The Minkowski space formalism consistent with this approach uniquely relates to the twistors in Sec. 3. In this section, we demonstrate the manner in which the approaches presented in this paper relate to the current supersymmetry and GUT models as well as string theory. We further elaborate on the symmetry principles of the complexification of Minkowski space, twistors and their properties. A fundamental relationship between complex Minkowski space, twistor algebra and quaternions are developed in Sec. 4. Of interest are the non-Abelian nature of quaternions, the $SU_n$ groups, quantum theory and Penrose topology.

## 2. The Spinor Formalism and their Relationship to Twistors

The approach to unification of the electromagnetic and gravitational fields was developed by Kaluza [1] and Klein [2] in the 1920s and their work was seriously considered by Einstein in the 1930's. This

5D geometry utilizes the spinor calculus to account for the coupling of the electromagnetic field to the gravitational field, in which the spinor is treated as a rolled-up dimension rather than as the four extended dimensions of the gravitational field. The concept of small Planck scale rotational "extra dimensions" (XD) is postulated in current 10D and 11D supersymmetry models. They are considered to be ultramicroscopic because they are not seen. However, following Randall our cosmology utilizes large scale XD in our model. Kaluza-Klein Planck-scale XD is not the only interpretation; even infinite size XD can be hidden behind the barrier of uncertainty by a topological switching process of subtractive interferometry [36,37]. Theory is treated as a subset of this supersymmetry, including the grand unification theory (GUT) and theory of everything (TOE).

The Kaluza-Klein Theory requires a periodicity of the 5D spinor fields to unify electromagnetism and gravity based on the homeomorphism between the Lorentz group and the unimodular transformation of Maxwell's equations and the weak Weyl limit of the gravitational field. A discussion of the Kaluza-Klein model, Rauscher [3,4], Newman [5] and Hansen and Newman complex 8-space is given in [6]. In the approach of the latter three references, the spinor calculus is demonstrated as mapable one-to-one with the twistor algebra of the complex 8-space and, hence, the Penrose twistor [3].

The coupling of the electromagnetic field with the gravitational field in the Kaluza-Klein may also yield a connection through the photon description of the twistor algebra. The photon is the quanta of the electromagnetic field and the interaction mediation between leptons, such as the electron. The 5D spinor calculus has been developed within a 5D relativistic formalism [1-3]. The spinor calculus developed in the 5D spinor formalism accounts for the coupling of the electromagnetic field to the gravitational metric.

This approach is manifestly 5-covariant in a special 5D space. The specific spin frames of reference of the 5D Kaluza-Klein geometry reduces to the spinor formalism of curved spacetime. The theory of spinors in 4D space is based upon the transformation $L'$ and the group of unimodular transformation $U_1$ in $SL(2,C)$. This formalism is related to 2-toroidal space $U_1 \times U_1$. Unimodular action of the symplectic automorphism group $SL(2,R)$ of the Heisenberg 2-step nilpotent Lie group, $N$ has the discrete subgroups $SL(2,Z)$ of $SL(2,R)$. The 2D compact unit sphere = $S_2$ (Riemannian sphere) and the 3D spherical component unit sphere can map as $S_3 \to R^4$.

It has been established that the 5D 4-component spinor calculus is related to the 4D spinor formalism in order to account for the coupling of the electromagnetic field as a periodic 5D spinor field to the curved space of the gravitational Riemannian metric [7]. We can utilize projective geometry to relate 5D spinor calculus to the 4D twistor space.

An isomorphism between vectors $v^{\beta}$ and spinors $v^{AA'}$ satisfies the condition

$$\bar{\pi}^{AA'} = \pi^{AA'} \tag{1}$$

so that the spinor equivalent to a vector $v^{\beta}$ is

$$\pi^{AA'} = \tau_{\beta}^{AA'} v^{\beta} \tag{2}$$

where $\tau_{\beta}^{AA'}$ is a tensor.

Therefore,

$$v^{\beta} = \tau_{AA'}^{\beta}\, \pi^{AA'} \tag{3}$$

where $v^{\beta}$ is real for $\bar{\pi}^{AA'} = \pi^{AA'}$. The covering map $SL(2,C)$ goes to $O(1,3)$ by using the vector-spinor correspondence.

We present some of the properties and structure of this significant advancement in developing a unified force theory for the electromagnetic and gravitational fields which can be related to the twistor algebra. In addition to the general coordinate transformations of the four coordinates $x^{\mu}$, the preferred coordinate system permits the group relation,

$$x'^5 = x^5 + f\left(x^1, x^2, x^3, x^4\right). \qquad (4)$$

Using this condition and the 4D cylindrical metric or $ds^2 = \gamma_{ik} dx^i dx^k$ yields the form

$$ds^2 = \left(dx^5 + \gamma_{\mu 5} dx^{\mu}\right)^2 + g_{\mu\nu} dx^{\mu} dx^{\nu} \qquad (5)$$

where the second term is the usual 4-space metric. The quantity $\gamma_{\mu 5}$ in the above equation, transforms like a gauge [7]

$$\gamma'_{\mu 5} = \gamma_{\mu 5} - \frac{\partial f}{\partial x^{\mu}} \qquad (6)$$

where the function $f$ is introduced as an arbitrary function. Returning to our 4D metrical form in its 5-compact form and 4D and 5D form gives,

$$\gamma_{\mu\nu} = g_{\mu\nu} + \gamma_{\mu 5}\gamma_{\nu 5}. \qquad (7)$$

Proceeding from the metrical form in a "cylindrical" space, $ds^2 = \gamma_{ik} dx^i dx^k$ where indices $i, k$ run 1 to 5, we introduce the condition of cylindricity which can be described in a coordinate system in which the $\gamma_{ik}$ are independent of $x^5$ or

$$\frac{\partial \gamma_{ik}}{\partial x^5} = 0. \qquad (8)$$

Kaluza and Klein assumed $\gamma_{55} = 1$ or the positive sign $\gamma_{55} > 0$ for the condition of the 5th dimension to ensure that the 5th dimension is metrically space-like. In geometric terms, one can interpret $x^5$ as an angle variable, so that all values of $x^5$ differ by an integral multiple of $2\pi$ corresponding to the same point of the 5D space, if the values of the $x^{\mu}$ are the same. Greek indices $\mu, \nu$ run from 1 to 4, and Latin indices $i, k$ run from 1 to 5 and for this specific case, each point of the 5D space passes exactly one geodesic curve which returns to the same point. In this case, there always exists a perpendicular coordinate system in which $\gamma_{55} = 1$ and

$$\frac{\partial \gamma_{5\mu}}{\partial x^5} = 0. \qquad (9)$$

It follows from those properties that $g_{\mu\nu}$ and $\gamma_{ik}$ can be made analogous so that $g_{\mu\nu} = \gamma_{ik}$ then

$$\gamma^{55} = 1 + \gamma^{\mu\nu}\gamma_{\mu 5}\gamma_{\nu 5} \qquad (10a)$$

(also see Eq. (7)) and

$$\gamma^{\mu 5} = -g^{\mu\nu}\gamma_{\mu 5}\,. \qquad (10b)$$

The gauge-like form alone is analogous to the gauge group, which suggests the identification of $\gamma_{\mu 5}$ with the electromagnetic potential, $\phi_{\mu}$. We can write an expression for an antisymmetric tensor,

$$\frac{\partial\ \gamma_{\mu 5}}{\partial x^{\mu}} - \frac{\partial\ \gamma_{\mu 5}}{\partial x^{\nu}} = f_{\mu\nu} \qquad (11)$$

which is an invariant with respect to the "gauge transformation". (Chap. 8)

We now use the independence of $\gamma_{ik}$ of $x_5$ or $\partial\,\gamma_{ik}/\partial x^5 = 0$. The geodesics of the metric in five-space can be interpreted by the expression

$$\frac{dx^5}{ds} + \gamma_{\mu 5}\frac{dx^{\nu}}{ds} = C \qquad (12)$$

where $C$ is a constant and $s$ is a distance parameter. If we consider the generalized 5D curvature tensor, and using the form for $f_{\mu\nu}$ we can express it in terms of $F_{\mu\nu}$, the electromagnetic field strength,

$$f_{\mu\nu} = \sqrt{\frac{16\pi G}{c^4}}F_{\mu\nu} \qquad (13)$$

where $\sqrt{G/c^4} = 1\big/\sqrt{F}$ where $F$ is the quantized force introduced by Rauscher [3,8-10] which relates to the driving force for the perceived expansion of the universe. This is the Rauscher force term that appears in the stress energy term in Einstein's field equations [11]. Then we can write,

$$\gamma_{\mu 5} = \sqrt{\frac{16\pi G}{c4}}\phi_{\mu}\,. \qquad (14)$$

The integration constant, above, can be identified as proportional to the ratio $e/m$ of charge to mass of a particle traveling geodesics in the Kaluza-Klein space [1-3].

Under the specific conditions of the conformal mappings in the complex Minkowski space, one can represent twistors in terms of spinors. The spinor is said to "represent" the twistor. The twistor is described as a complex two-plane in the complex Minkowski space (see Section 3 and [3] for references on twistor theory and the spinor calculus. Twistors and spinors can be easily related by the general Lorentz conditions in such a manner as to retain the condition that all signals are luminal in real 4-space.

The conformal invariance of the tensor field, which can be Hermitian, can be defined in terms of twistors and these fields can be identified with particles [11,12].

It is through the representation of spinors as twistors in complex Minkowski space that we can relate the complex eight-space model to the Kaluza-Klein geometries and to the grand unification or GUT theory. See Chap. 13. In the 5D Kaluza-Klein geometries, the XD is considered to be a spatial rotational dimension in terms of $\gamma_{\mu 5}$ . The Hanson-Newman [6], and Rauscher [4,5] complex Minkowski space has introduced with it an angular momentum or helix or spiral dimension called a twistor which is expressed in terms of spinors [7].

The spinor formalism was used by Dirac to define the Schrödinger equation in a relativistic invariant form so that the complex scalar time dependent field of Schrödinger is in terms of a two-component spinor field. Using this formalism Dirac obtained a 2-valued solution which predicted the observed electron and positron pair. The spinor field or spinor variable, utilized in the Kaluza-Klein geometry, directly relates to the spin degrees of freedom that are observed by the Zeeman Effect in atomic spectra. The spin degrees of freedom appear to be fundamental to quantum theory and to relativity and are a good starting point to treat spin in a fundamental manner. The Lorentz four-space representation of relativity can be reduced to the direct product of two two-dimensional complex representations. The spinor variable is the most fundamental representation of a relativistically invariant theory and spin degrees of freedom may be formulated relativistically and, hence, also in a possible "quantum gravity" picture which applies to the Dirac equation. This approach may be applicable to the Penrose twistor (Chap. 12).

This approach appears to fit well with the spinor approach in the Dirac formalism in the quantum domain, that is, that the Lorentz conditions applied by Einstein in relativity may be the origin of the spinor and, hence, be the fundamental theory that yields the spinor formalism and the role of spin. Other implications of the relationship between the Penrose twistor formalism and the complex Minkowski space, which includes anticipatory systems related to causality and spatial and temporal nonlocality, are given in references [12-18].

## 3. The Penrose Twistor, Harmonic Sequencing and Particle Spin

Interest in the twistor program has been in the form of quantizing gravity in order to unify the physics of the micro- and macro-cosmos in 1971 and 2005. Such a procedure has been taken by Penrose, et al and is based on the concept of a more general theory that has limits in the quantum theory and the relativistic theory [15]. In addition, there have been approaches to the underlying structure of spacetime in the quantum [11] and structural regime [8]. A structured and/or quantized spacetime [1] may allow a formalism that unequally relates the electromagnetic fields with the gravitational metric [9]. Feynman [13] and Penrose graphs [11,12] may overcome the divergences of such an approach. In order to translate the equations of motion and Lagrangians from spinors to twistors, one can use the eigenfunctions of the Casimir operators of the Lie algebra of $U(2,2)$.

For the simplest case of a zero rest mass field (photon-like) for $n/2$ spin for $n \neq 0$, we can write

$$\nabla_{AA'}\,\varphi^{A\ldots N} = 0 \qquad (15)$$

for $A,\ldots,N$ written in terms of $N$ indices, and for $N=1$, we have the Dirac equation for massless particles. For a spin zero field, we have the Klein-Gordon equation

$$\nabla^{AA'}\,\nabla_{AA'}\,\varphi = 0 \qquad (16)$$

and in Eq. (15) for $n=2$, we have the source-free Maxwell equation $\Box F^{\mu\nu}=0$ for spin 1 or $U_1$ fields, and for $n=4$, we have the spin free Einstein field equations, $R_{\mu\nu}=0$. The indices $\mu$ and $\nu$ run 0 to 3. For a system with charge, then $\Box F^{\mu\nu}=J_{\mu\nu}-J_{\nu\mu}$, or this can be written as $\frac{F_{\mu\nu}}{\partial x_\nu}=J_\mu$ and then we can write

$$\gamma_{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x_\nu}=J_\mu . \qquad (17)$$

In this section, we outline a program to relate the twistor topology to the spinor space and specifically to the Dirac spinors. Both Fermi-Dirac and Bose-Einstein statistics are considered. The relationship between twistor theory and the Dirac "string trick" model is further discussed in our chapter on the complexification of the Dirac equation (Chap. 12).The Penrose spin approach is designed to facilitate the calculation of angular momentum states for SL(2). The spinor formalism, in the Dirac equation, established spinors within quantum theory. The twistor formalisms are related to the structure of spacetime and the relation of the spinors and twistors is also of interest because it identifies a relationship between quantum mechanics and relativity [11,12,18,19].

Twistor theory has been related to conformal field theory and string theory [20]. Also, twistor theory has been related to quaternions and complex quaternionic manifolds [21,22]. The projective twistor space, PT, corresponds to two copies of the associated complex projective space of $CP^3$ or $CP^3\times CP^3$ . It is through the conformal geometry of surfaces in $S^4$, utilizing the fact that $CP^3$ is an $S^2$ bundle over $S^4$, that quaternions can be related to twistors [23].

We can demonstrate a useful relationship between the complex eight-space and the Penrose twistor topology; the twistor is derived from the imaginary part of the spinor field. The Kerr Theorem results naturally from this approach in which twisting is shear free in the limit of asymptotic flat space. The twistor is described as a two-plane in complex Minkowski space, $M^4$. Twistors define the conformal invariance of the tensor field, which can be identified with spin or spinless particles. For particles with a specific intrinsic spin, $s$, we have $\mathrm{Z}^\alpha\overline{\mathrm{Z}}_\alpha=2s$, and for zero spin, such as the photon, $\mathrm{Z}^\alpha\overline{\mathrm{Z}}_\alpha=0$ where $\overline{\mathrm{Z}}_\alpha$ is the Hermitian conjugate of $\mathrm{Z}^\alpha$, and $\mathrm{Z}^\alpha$ and $\mathrm{Z}_\alpha$ can be regarded as canonical variables such as $\underline{x}$, $\underline{p}$ in the quantum theory phase space analysis. The twist free conditions, $\mathrm{Z}^\alpha\overline{\mathrm{Z}}_\alpha$, hold precisely when $\mathrm{Z}^\alpha$ is a null twistor. The upper case Latin indices are used for spinors, and the Greek indices for twistors. The spinor field of a twistor is conformally invariant and independent of the choice of origin [24]. For the spinor, the indexes $A$ and $A'$ take on values 1, 2 [11,12]). We briefly follow along the lines of Hanson and Newman in the formalism relating the complex Minkowski space to the twistor algebra.

Penrose states regarding the Robinson congruence:

*I had, earlier, worked out the geometry of a general Robinson congruence: in each time-slice t = constant of M, the projections of the null directions into the slice are the tangents to a twisting family of linked circles (stereographically projected Clifford parallels on $S^4$ – a picture with which I was well familiar), and the configuration moves with the speed of light in the (negative) direction of the one straight line among the circles.*

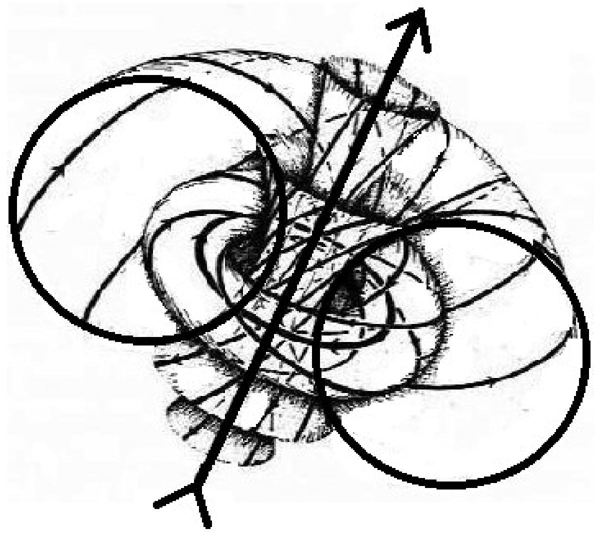

**Figure 1** A Penrose twistor known as the Robinson congruence representing the propagation of a photon or a time-slice (t = 0) of a Robinson congruence. Redrawn from [11].

> *I decided that the time had come to count the number of dimensions of the space* R *of Robinson congruences. I was surprised to find, by examining the freedom involved that the number of real dimensions was only six (so of only three complex dimensions) whereas the special Robinson congruences, being determined by single rays, had five real dimensions. The general Robinson congruences must twist either right-handedly or left-handedly, so* R *had two disconnected components* $R^+$ *and* $R^-$, *these having a common five-dimensional boundary* S *representing the special Robinson congruences. The complex 3-space of Robinson congruences was indeed divided into two halves* $R^+$ *and* $R^-$ *by* S.
>
> *I had found my space! The points of* S *indeed had a very direct and satisfyingly relevant physical interpretation as "rays", i.e. as the classical paths of massless particles. And the "complexification" of these rays led, as I had decided that I required, to the adding merely of one extra real dimension to* S, *yielding the complex 3-manifold PT* = S *U* $R^-$ *U* $R^+$ [11].

Twistors and spinors are related by the general Lorentz conditions in such a manner as to retain the fact that all signals are luminal in the real four-space, which does not preclude superluminal signals in an $n > 4\text{D}$ space. The twistor $Z^\alpha$ can be expressed in terms of a pair of spinors, $\omega^A$ and $\pi_A$, which are said to represent the twistor. We write

$$Z^\alpha = \left(\omega^A, \pi_{A'}\right) \tag{18}$$

where $\omega^A = i\, r^{AA'} \pi_{A'}$

Every twistor $Z^\alpha$ is associated with a point in complex Minkowski space, which yields an associated spinor, $\omega^A$, $\pi_{A'}$. The spinor is associated with a tensor which can be Hermitian or not. The spinor can be written equivalently as a bivector forming antisymmetry. In terms of spinors $\omega^A$ and $\pi_{A'}$, they are said to represent the twistor $Z^{\alpha}$ as $Z^{\alpha} = \left(\omega^{A}, \pi_{A'}\right)$ (see Eq. (18)). In terms of components of the twistor space in Hermitian form, $\varphi$ for $\varphi_{AA'} = \varphi_{A'A}$, we have,

$$\varphi\left(Z^{\alpha} Z^{\beta}\right) = \overline{Z^{0} Z^{2}} + \overline{Z^{1} Z^{3}} + \overline{Z^{2} Z^{0}} + \overline{Z^{3} Z^{1}} \qquad (19)$$

where the $\alpha$ index runs 0 to 3. The components of $Z^{\alpha}$ are $Z^{0}, Z^{1}, Z^{2}, Z^{3}$ and are identifiable with a pair of spinors, $\omega^{A}$ and $\pi_{A'}$, so that

$$\omega' = Z^{1},\ \pi_{0'} = Z^{2},\ \pi_{1'} = Z^{3} \qquad (20)$$

so that we have

$$Z\overline{Z}_{\mu} = \mu^{0}\overline{\pi}_{0} + \mu'\overline{\pi}_{1} + \pi_{0}{}'\overline{\mu}^{0'} + \pi_{1'}\overline{\mu}^{1'}. \qquad (21)$$

Note that the spinor $\omega^{A}$ is the more general case of $\mu^{A}$. This approach ensures that the transformations on the spin space preserve the linear transformations on twistor space, which preserves the Hermitian form, $\varphi$.

The underlying concept of twistor theory is that of conformal invariance or the invariance of certain fields under different scalings of the metric $g_{\mu\nu}$. Related to the Kerr theorem, for asymptotic shear-free null flat space, the analytic functions in the complex space of twistors may be considered a twisting of shear-free geodesics. In certain specific cases, shear inclusive geodesics can be accommodated.

Twistors are formally connected to the topology of certain surfaces in complex Minkowski space, $M^{4}$. This space, the complex space, $C^{4}$, is the cover space of $R^{4}$, the 4D Riemannian space. On the Riemann surface, one can interpret spinors as roots of the conformal tangent plane of a Riemann surface into $R^{3}$. This approach is significant because it ensures the diffeomorphism of the manifold. Complexification is formulated as $Z^{\mu} = X^{\mu}_{\mathrm{Re}} + X^{\mu}_{\mathrm{Im}}$, which constitutes the complexification of the Minkowski space, $M^{4}$. The usual form Minkowski space is a submanifold of complex Minkowski space. Twistors are spacetime structures in Minkowski space, which is based upon the representation of twistors in terms of a pair of spinors as we have shown [4,14]. Twistors provide a unique formulation of complexification. The interpretation of twistors in terms of asymptotic continuation accommodate curved spacetime. One feature of this approach to quantum theory in twistor space leads to a quantum gravity theory [14].

This spinor representation of a twistor makes it possible to interpret a twistor as a two-plane in complex Minkowski space, $M^{4}$. Then we can relate $\omega^{A}$ and $\pi_{B'}$ so that $\xi^{AA'}$ is a solution as

$$\omega^{A} = i\xi^{AB'}\pi_{B'} \qquad (22)$$

for the position vector $\xi^{AB'}$ in the complex Minkowski space. We can also consider the relationship of $Z^{AA'}$ and $\pi^{A'}$ to a complex position vector as

$$Z^{AA'} = \xi^{AA'} + \omega^{A}\pi^{A'} \qquad (23)$$

where $\omega^A$ is a variable spinor. Just as in the conformal group on Minkowski space, spin space forms a two-valued representation of the Lorentz group. Note that $SU_2$ is the four-value covering group of C $(1, 2)$, the conformal group of Minkowski space. The element of a 4D space can be carried over to the complex eight-space.

For spin, $n$ the Dirac spinor space is a covering group of $SO_n$ where this cohomology theory will allow us to admit spin structure and can be related to the $SU_2$ Lie group. Now let us consider the spin conditions associated with the Dirac equation and further formulate the manner in which the Dirac "string trick" relates to the electron path having chirality. For a spin, $s = \frac{1}{2}$ particle, the spin vector $u(p)$ is written as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for spin up and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for spin down for momentum, $p$. For a particle with mass we have for $c \neq 1$,

$$\left( -i\hbar c\alpha_\mu \frac{\partial}{\partial x_\mu} + \beta mc^2 \right) \psi = 0 \qquad (24)$$

for the time independent equation, and we can divide Eq. (24) by $i\hbar c$ and have,

$$\left( \gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar} \right) \psi = 0 \qquad (25)$$

where $k_\circ = mc/\hbar$ and $\gamma_\mu = i\hbar c\alpha_\mu$ where indices $\mu$ run 0 to 3. The time dependent Dirac equation is given as,

$$\left( -i\hbar c\alpha_\mu \frac{\partial}{\partial x_\mu} + \beta mc^2 \right) \psi + \frac{i}{\hbar} \frac{\partial \psi}{\partial t} = 0 . \qquad (26)$$

The solution to the Dirac equation is in terms of spin $u(p)$ as

$$\psi = u(p) \exp\left[ \frac{i}{\hbar} \left( p \cdot \underline{x} - Et \right) \right] \qquad (27)$$

the Dirac spin matrices $\gamma_\mu = i\hbar c\alpha_\mu$. The spinor calculus is related to twistor algebra, which relates a 2-space to an associated complex 8-space [25].

An example of the usefulness of spinors is in the Dirac equation. For example, we have the Dirac spin matrices, $\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu & 0 \end{pmatrix} = -i\,\beta\alpha_\kappa$ where terms such as $\gamma_\mu(1-\gamma_5)$ come into the electroweak vector-axial vector formalism. The three Dirac spinors (also called Pauli spin matrices) are given as

$$\sigma_x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \sigma_y = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \quad \text{and } \sigma_z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \tag{28}$$

where indices 1,2,3 stand for *x, y, z* and $\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3$ for $\gamma_0 = \beta$ is given as,

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{29}$$

for trace, $\mathrm{tr}\beta = 0$, that is, Eq. (29) can be written as,

$$\gamma_0 = \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \tag{30}$$

where we have the $2 \times 2$ spin matrix as $I_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ where $\mathrm{tr}I_2 = 2$. Note that the Dirac spinors are the standard generators of the Lie algebra of $SU_2$.

The commutation relations of the Dirac spin matrices is given as

$$\left\{\gamma^\mu, \gamma^\nu\right\}_+ = \gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\nu = ig^{\mu\nu} \underset{\sim}{I} \tag{31}$$

and $\det\left|\gamma_{\mu\nu}\right. = \det\left\|g_{\mu\nu}\right|$ where $g_{\mu\nu}$ is the metric tensor. The Dirac spin matrices come into use in the electroweak vector-axial vector model as $\gamma_\mu(1-\gamma_5)$ for $\gamma_5$ as,

$$\gamma_5 = i\,\gamma_0\gamma_1\gamma_2\gamma_3 = i\,\gamma^0\gamma^1\gamma^2\gamma^3 \tag{32}$$

where indices run 0 to 3.

We can also write,

$$\gamma_{\mu\nu}\left(x^5, x^\mu\right) = \sum_{n=-\infty}^{\infty} \gamma_{\mu\nu}^{(n)}\left(x^\nu\right)\, e^{inx^5} \tag{33}$$

which expresses some of the properties of a 5D space having $\gamma_0, \gamma_{1,} \gamma_2, \gamma_3$ and $\gamma_5$. Note that $\gamma_5$ is associated with a 5D metric tensor. This 5D space passes exactly one geodesic curve which returns to the same point with a continuous direction. Note that this is a similar formalism to that of the Dirac string trick $720^\circ$ path which appears to demand a hyperdimensional $n > 4$ space in analogy to the Mobius strip from dimension 2D $\rightarrow$ 3D and the Klein bottle from 3D $\rightarrow$ 4D.

A connection can also be made to the electromagnetic potential; and the metric of the Kaluza-Klein geometry. We can express $\gamma_{\mu 5}$ in terms of a potential $\varphi_\mu$ so that

$$\gamma_{\mu 5} = \sqrt{2\kappa}\phi_\mu \tag{34}$$

where $\kappa \equiv 8\pi/F$ and where $F = c^4/G$ or the quantized cosmological force [8-10] (also see Eq. (34)). Then we have a 5-space vector as,

$$\gamma_{\nu 5} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{35}$$

Through this approach, we can relate covariance and gauge invariance [14]. Using Poisson's equation,

$$\nabla \varphi_\mu = \frac{1}{2}\kappa c^4 \mu_0 \tag{36}$$

where again $\kappa \equiv 8\pi/F$ as above. The electromagnetic field, $F_{\mu\nu}$, can be expressed as,

$$F_{\mu\nu} = \frac{\partial \varphi_\mu}{\partial x^\nu} - \frac{\partial \varphi_\nu}{\partial x^\mu} \tag{37}$$

which yields an interesting relation of the gravitational metric to the electromagnetic field. Also the Lagrangian is given as $L = \frac{1}{2}F^{\mu\nu}F_{\mu\nu}$ so that $\mathsf{L} = L\sqrt{-g}$ for the metric $g$. Note $L = \int \sqrt{g}\, d\tau$, where $d\tau$ represents a 4-space. Now we return to our discussion of twistor algebra and relate it to the spinor calculus. The Penrose twistor space also yields a 5D formalism as is also formulated by the Kaluza-Klein theory.

Both projective and non-projective twistors are considered as images in a complex Riemannian manifold in its strong conformal field condition. Duality, analytic continuation, unitary and other symmetry principles can be incorporated by using appropriate (Bose-Einstein or Fermi-Dirac) spin statistics in analogy to the Hartree-Fock spaces or Fock space (Chap. 3). Particles can be considered as states as the Fock space elements or the "end" of each disconnected portion of the boundary of the manifold. The quanta are associated with a quantum field of particles that carry both momentum and energy. The total energy Hamiltonian can be defined in terms of a number of simple phonon states which can be expressed in terms of $a_n^+$ creation and $a_n$ destruction operator states. Since all creation operators commute, these states are completely symmetric and satisfy Bose-Einstein statistics. Such phonon states, having a definite number of phonons, are called Fock states, which is the vector sum of the momentum of each of the photons in the state. The ground state $|0\rangle$ can be considered the photon vacuum state or Fock state where the photon is taken as a phonon state. The creation and destruction operators commute as $\{a_n, a_n^+\} = \delta_{nn'}$ for the delta function, $\delta_{nn'}$ [26].

In this picture, we can consider an $n$-function as a "twistor wave" function for a state of $n$-particles. Penrose [11] considers a set of $n$-massless particles as a first order approximation. We form a series on a complex manifold as elements of the space $C_n$ as

$$f_0,\ f_1\left(z^\alpha\right),\ f_2\left(z^\alpha, y^\alpha\right),\ f_3\left(z^\alpha, y^\alpha, x^\alpha\right), \ldots \qquad (38)$$

which are, respectively, the 0th function, 1st function, 2nd function, and 3rd function, etc. of the twistor space, which are also elements of $C_n$. We can also consider $f_0,\ f_1,\ f_2,\ f_3, \ldots.$ as the functions of several nested twistors in which $f_0$ is the central term of the wave of the twistor space. The $f_n$ could represent nested tori that can act as a recursive sequence.

Penrose [11,12] suggests that, to a first approximation, $f_1$ corresponds to the amplitude of a massless, spin 1 particle, $f_2$ to a lepton spin ½ particle, and $f_3$ to hadron particle states, and $f_4$ to higher energy and exotic hadron particle states. Mass results from the breaking of conformal invariances for $f_n$ for $n = 2$ or greater; similar to the $S-$matrix approach [27]. The harmonic functions $f_n$ form a harmonic sequence, where $f_n$ for $n = 2$ form the Fermion states, and $f_n$ for $n = 3$ form the Hadron twistor states. Essentially, in the twistor space, we have a center state $f_0$ around which $f_1,\ f_2, \ldots$ occur. Each of these sequences of waves forms a torus, hence, $f_1$ and $f_2$ form a double nested tori set consistent with both spin 1 and spin ½ particle states where all $n$ states are elements of the twistor, $z$, as $n \in z$.

In the specific case of a massless particle with spin for $f_1$, the 2-surface in complex Minkowski space corresponding to the twistor represents the center of mass of the system so that the surface does not intersect the real Minkowski space. This reflects the system's intrinsic spin. We see an analogy to the 3-torus Calabi-Yau M-Theory [28]. Calabi-Yau manifolds (a form of Kahler manifold) preserve the correct supersymmetry for the theory to reproduce the features of the standard model. This form of M-Theory, which features a 3-cycle toroidal symmetry is one of the better M-Theories with $10^{1000}$ or ($10^{\text{googolplex}}$ as sometimes called)[1] possible candidates for the string vacuum. In fact utilizing the continuous-state hypothesis we have been able to derive a unique candidate for the string vacuum [29]. The higher order $f_n$ may describe higher order string modes or oscillations of $Z^\alpha \overline{Z}_\alpha = 0$ or $f_0$. This occurs for the case using $f_1,\ f_2,$ and $f_3$ and, hence, all known particle states.

We can consider the topology of three Penrose projective twistor states which are $PT$, $PT^+$, and $PT^-$. The $PT^+$, and $PT^-$ are meant to represent the domain of $PT$ where we denote these two states in which $(-1,1)$ are elements of $t$ where $\varepsilon$ is small. We denote two line elements which are denoted in terms of twistors as a surface on the sphere $S^3$ as $PT^\pm$ which corresponds to $\underset{-t}{Z^\alpha}\,\overline{\underset{-t}{Z_\alpha}} = 0$ and $\underset{t}{Z^\alpha}\,\overline{\underset{t}{Z_\alpha}} = 0$ for $t = 1-\varepsilon$ for $PT^+$, and $PT^-$ gives $t = 1-\varepsilon = \varepsilon - 1$. These two branches correspond to a transformation matrix,

[1] Googolplex: a googolplex cannot be written out since a googol of '0's will not fit into the observable universe

$$\begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ t & 0 & 1 & 0 \\ 0 & t & 0 & 1 \end{pmatrix}. \tag{39}$$

This gives us a translation formulation for vectors into the states of spinors in terms of $t$, in terms of the spinors

$$\begin{pmatrix} \omega^0_{t\,1} \\ \omega^1_{t} \\ \pi'_{t\,0} \\ \pi'_{t\,1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ t & 0 & 1 & 0 \\ 0 & t & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^0_1 \\ \omega^1 \\ \pi'_0 \\ \pi'_1 \end{pmatrix} \tag{40}$$

which is $Z^\alpha_t$ and $t \sim \pm 1$ since $\varepsilon$ is small. Then in terms of twistors,

$$\hat{\omega}^A = \omega^A + \varepsilon\xi^{AB} \frac{\partial f}{\partial \omega^B} \tag{41}$$

for $\hat{\pi}_{A'} = \pi_{A'}$ where $\omega$ and $\pi$ are orthogonal spinors. The term $\varepsilon\xi^{AB} \dfrac{\partial f}{\partial \omega^B}$ is small compared to $\omega^A$ and $\pi_A$ since $\varepsilon$ is small. The unit spinors or vectors are $\hat{\omega}^A$ and $\hat{\pi}_{A'}$ for both $A, B = 1,2$.

The projective twistor space, $PT$, corresponds to two copies of $CP^3$, which has an associated complex projective space. The $PT$ space is the space which yields the torus topology of the Riemann surface of genus, $g = 1$. The genus-1 topology contains one "hole" or singularity, genus-2, two holes, etc. The two-hole system is a continuous manifold which can represent two connected tori or a double torus producing an equatorial planar membrane. This topology is related to the high-energy plasma dynamics found around black hole ergospheres and their equatorial accretion disks. It is, as well, observed in stars, and gas and dust circulation within galactic disks and halos. Observation of double tori topology at the cosmological level may, as well, be evidence of a structured polarized vacuum interacting with the high energy plasma dynamics at these scales. Haramein and Rauscher utilize torus topology to describe astrophysical objects such as supernovas and astrophysical plasmas [30].

### 11.4 Penrose Twistor Fields, Particles and Nested Tori

We explore some unique features of the torus topology (Recall that one of the popular forms of Calabi-Yau space is a form of dual 3-torus) We consider the relationship between the $T = U_1 \times U_1$ group and the $S^2$ group. An example of the $n$- dimensional manifold, which is not a product of $n$-one-dimensional manifolds, is given by the sphere, $S^n$. When one deals with two or more real or complex variables, there is usually a manifold, $M$, on which these functions are definable. The surface of a sphere of unit radius 3D Euclidian space, $S^2$, can be triangulated on the boundary of a tetrahedron. For the torus, $T$, its triangulation, $K$, consists of seven 0-simplexes and fourteen 2-simplexes. The contractible 1D sub-polyhedron of $K$ contains all vertices of $K$. The two generators commute so that the torus group is generated by the two commuting generators $\simeq Z \oplus Z$ (see Section 5).

The manifold $T^n$ is the $n$D torus. If $n = 2$, then $T^2 = S^1 \times S^1$ defines a torus. The torus is a subset of $R^3$, where $R$ is the topology on the real numbers. The sets $X$ and $Y$ are called the topological space. If $X$ is a set as a discrete topology, then $Y$ can be a collection of all subsets of $X$, i.e., the set $2^x$. Any finite or infinite subcollection $\{Z_\alpha\}$ of the $X_\alpha$ has the property that $\bigcup Z_{\alpha i} \in Y$, or the union of $Z_{\alpha i}$ are elements of $Y$. The torus is a subset of $R^3$, and $T^2 = S^1 \times S^1$ is the Cartesian product of two subsets of $R^2$ so that it is at least a subset of $R^2 \times R^2 = R^4$. The torus, which is in $R^3$, is not flat, but the torus $S^1 \times S^1$ in $R^4$ can be considered flat. The topology of the dual tori are the same, which has to do with the precise definition of flatness and curvature [31].

The definition of curvature depends on the specification of a Riemannian metric [32]. Once we specify the Riemannian metric, then we can define our flatness of $T^2$. This entails the specification of the metric $g_{\mu\nu}$ or $\eta_{\mu\nu}$ which allows us to specify the restrictions that the points in $R^3$ lie on the torus. Then, with respect to the metric, $\eta_{\mu\nu}$ we have a curved space torus. For $T^2 = S^1 \times S^1$, which defines two points $(x, y)$ and $(x', y')$ in $T^2$, the difference is expressed as $\left[(x - x')^2 + (y - y')^2\right]^{1/2}$ for the usual $g_{\mu\nu}$ .See Chap. 2. For this metric, $T^2$ is flat and does not lie on $R^3$. The reason for this condition is that for a 2D compact connected surface to lie in $R^3$, it must have at least one non-zero curvature, where $R_3$ is the topology of real numbers [30,31].

In defining a vector space on a sphere, $S^2$, or torus $T$, we consider a simple observation of a 2D surface in $R^3$. For example, a disk $x^2 + y^2 \le a^2$ for $z = 0$ has a top side and a bottom side, or a sphere $S^2$ has an inside and an outside, as does the torus $T^2$. These 2-sided surfaces are defined as orientable since we can use their two-sided properties to define directions or orientations of vectors projected from their surfaces in $R^3$. Hence, we have two normals at each point, an inward, or outward pointing normal vector, $\hat{n}$. We are guaranteed, in general, a diffeomorphic manifold for a torus in curved space, but not in general, for a spherical topology. Therefore, for any non-Euclidian space, diffeomorphism holds for the torus topology. Hence the Penrose topology is diffeomorphic.

## 11.5. Quaternions, Groups, and Allowable Spatial Structures

The complexified rotational dimensionality of quaternions may be the most appropriate approach to the description of twistor space in the context of a fundamental rotational force embedded in the structure of spacetime itself. We explore some of their interesting and related properties in this section.

### 5.1 THE QUATERNION FORMALISM AND SIMPLE TOPOLOGICAL SPACES

The quaternion group is isomorphic to the group with elements $1, -1, -i, j, k, -k$, and $i^2 = j^2 = k^2 = -1$ and $ij = k,\ jk = i,\ ki = j$. These properties operate similar to complex numbers where $i = \sqrt{-1}$ and $i = -1$. In the case of the quaternions, $i, j, k$ can represent orthogonal dimensions in three-space. The isomorphism condition states that the group elements of two groups can have a one-to-one correspondence, which is preserved under combinations of elements. Then one can construct a group table as a square array; this is only necessary for higher order groups. Quaternion groups have $SU_2$ or $SU_3$ subgroups and can be related to $O_3 +$.

Symmetric groups such as the quaternion group, which is a two-dimensional unimodular unitary group, are simply reducible groups. Following Hamilton, we identify Euclidian four-space with the space of quaternions so that $H = \{\rho + xi + yi + zk\}$ where $\rho, x_o, y, z \in R^4$ are elements of the Riemannian space $R^4$. The Euclidian three-space is the subset of imaginary quaternion, $H_{im} = \{ xi + yi + zk\}$ where $x, y, z \in R^3$ (see Section 3).

## 5.2 QUATERNIONS AND QUANTUM THEORY

The key is that the Dirac string trick represents the properties of the symmetric group which is $SU_2$. The $SU_2$ is isomorphic with the unit length of the quaternion in 5D space. Quaternions, constructed by Hamilton, can represent rotations in three-space, which can be performed without matrices. They also obey non-Abelian algebra. Furthermore, correspondence of quaternions can be made to vectors and tensors. Quaternions are a viable algebra for understanding rotations in 3D and 4D space. Due to symmetry considerations in the Dirac electron theory, a 720° twist is required for the electron to return to the exact same quaternion state, where a 360° rotation will not and must be doubled.

Quaternions are a complex number system with properties similar to the Rauscher [4] and Newman [5] complex eight-space. In the usual notation, we start from any complex number, $a + ib$ where $a$ and $b$ are real, where $a \times 1 = a$ and $ib$ is imaginary. The quaternion is written as $t + ia + jb + kc$ where $t, a, b,$ and $c$ are real and they are multiples of a real unit 1 and imaginary units $i, j,$ and $k$. The following conditions,

$$jk = -kj = i \tag{42a}$$

$$ki = -ik = j \tag{42b}$$

$$ij = -ji = k \tag{42c}$$

and

$$i^2 = j^2 = k^2 = -1 \tag{42d}$$

and

$$ijk = -1 \tag{42e}$$

also

$$i^2 = j^2 = k^2 = ijk = -1 \tag{42f}$$

which yields a set of recursive relationships.

Quaternions also have multiplicative properties similar to the complex Minkowski eight-space. Let $w = t + ia + jb + kc$ , then the conjugate of $w$ is $\bar{w}$ and is given as $\bar{w} = t - ia - jb - kc$, and the modulus is given as, $w\bar{w}$ or,

$$w\bar{w} = t^2 + a^2 + b^2 + c^2 . \tag{43}$$

In fact, quaternions contain all the properties of complex numbers except for commutivity and thus comprise a non-Abelian algebra such as in the quantum theory. Note that we have used a slightly different notation from Hamilton; that is, we write $ia$, $jb$, etc., instead of $ai$, $bj$, etc. Quaternions are comprehensively explored by Kauffman [31] and Rowlands [32].

If $t = 0$, then we have a pure imaginary quaternion or

$$u = ia + jb + kc \tag{44a}$$

and then

$$u^2 = -\left(a^2 + b^2 + c^2\right) \tag{44b}$$

and are of a unit length

$$a^2 + b^2 + c^2 = 1 \tag{45}$$

so that $u^2 = -1$. Also for two pure imaginary quaternions

$$uv = -u \cdot v + u \times v \tag{46}$$

as the dot and cross product of vector-like quantities in three-space. The addition of the scalar component connotates a coordinate in the fourth dimension and hence we see the analogy of quaternions to the 4D Minkowski space, where $t$ is time, and $a$ corresponds to $x$, $b$ to $y$, and $c$ to $z$. What is unique then about the quaternionic "space" is that we have, for example, the permutation relations from Eqs. (42a) to (42f), and thus quaternions form a non-Euclidian set with the properties for pure quaternions $uv$ in Eq. (46). We can form a set of pure quaternions on a 2D sphere of -1 in each of the three quaternion directions $i, j, k$. Note that the complex Minkowski space is formed by one imaginary component $i$, multiplied by $x, y,$ and $z$. Now consider $A$ and $B$ real numbers and $u$ is a unit length of a pure quaternion, then $u^2 = -1$ and the powers of $A + Bu$ occupy the same form as powers of complex numbers. That is, $u$ is indistinguishable from any other $\sqrt{-1} = i$.

Let us now relate the quaternions to a complex number $Z = A + uB$ which we can write as $Z = \cos\theta + R\sin\theta$ or, in general,

$$Z^n = R^n \cos(n\theta) + R^n \sin(n\theta)\, u\,. \tag{47}$$

We can proceed with mapping of the $n^{th}$ roots of the quaternions. Consider a space of $n+1$ dimensions in which we represent $n+1$ space in the form of $A + Bu$, where $A$ is a scalar and $B$ is a real number. Now $u$ is a limit vector in an $N-$space represented as $R^N$ which is a Euclidian $N-$space. The vector-like quantity $u$ belongs to the unit sphere, $S^{N-1}$ about the origin, $R^N$ and is taken to have squares equal to minus one, or $u^2 = -1$ for all vectors $S^{N-1}$. In general, $uv$ is not defined in a HD geometry such as the 8D Minkowski space of Rauscher [4] and Newman [5]. We can, however, create power maps of the form $Z^n + K$ where $K$ is a vector in $R^{N+1}$ and $Z = A + Bu$ for $u^2 = -1$ for all $u$ in $S^{N-1}$. With this approach, we can form classes of hypercomplex iterative processes with incursion in any arbitrary dimensional space. This is the key to Kauffman's ability to relate the hypercomplex interactions formed from quaternions to define HD fractal sets [33].

One of the basic principles of the quaternion twist holds for the Dirac string trick for 720° degree rotation. A half cycle of twist, or 360 degrees, is expressed in terms of quaternions as $ijk = -1$. To return to +1, another twist through 360° must occur. Spin must involve a preferred geometry in space [33]. The geometry of a preferred direction can be constructed by the magnitude of total electron transfer. The Penrose spin approach is utilized to calculate angular momentum and $SL(2)$ .

In terms of complex analysis involving quaternions, a single 180 degree turn is an instance of $i = \sqrt{-1}$ where $i^2 = -1$ and represents a 360 degree right- or left-handed turn. The case for $i^3 = -i$ is a non-trivial rotation and $i^4 = 1$ returns the rotation of the electron and observer to their original states, through the 720 degree rotation – hence, the interpretation of the quaternionic formalism of one square root of $-1$ for every direction in three-dimensional space. The electron moves on the bounded space to have contiguous surfaces at the equatorial plane. In order for the electron to pass through a 720 degree rotation and return the spin and chirality to its original state, the electron path must be different than that of a sphere.

In quantum theory, the symmetry group is the $SU_2$ group rather than the 3D space rotation group such as $O_3^+$ . The $SU_2$ group is isomorphic with the quaternions of unit length in 4D space. In [33], the group theoretic approach that relates spinors, twistors, and quaternions is detailed. A spinor is a vector in two complex variables. Antisymmetric conditions lead to the second twist involving the quaternions to create the cycle of the electron to its original state. The antisymmetric conditions utilizing spin calculations can be conducted with Clebsch-Gordan coefficients, $3j$ and $6j$ symbols and other components of angular momentum [34,35]. Through these means, one can calculate the correct spin interactions involving multi-particle quaternion states. Suffice it to say that the iterative properties, formulated here, have a variety of applications such as scalable inclusive relations from the quantum domain to astrophysical and cosmological systems [10].

## 6. Conclusion

We have demonstrated a unique relationship of the spinor calculus, twistor algebra, the quaternionic formalism and the complex 8-Space. This topology appears to be ubiquitous in Nature. The twistor formalism appears to also occupy a role in unification models through the $E_8$ group utilized in supersymmetry models.

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