A Detailed Review of Beal's Conjecture

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Abstract

The Beal's conjecture has been one of the most interesting problems that existed in the in number theory since the end of the last century. It was discovered by Andrew Beal during his works on Fermat's Last Theorem. In this paper a detailed review of the conjecture is proposed to end in a possible proof.

1. Introduction

In 1997, Andrew Beal [1] announced the following conjecture :

Conjecture 1. *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{1}$$

then $A, B,$ and C have a common factor.

In this paper, we give a complete proof of the Beal Conjecture. Our idea is to construct a polynomial $P(x)$ of three order having as roots A^m, B^n and $-C^l$ with the condition (1). The paper is organized as follows. In Section 2 of preliminaries, we begin with the trivial case where $A^m = B^n$. Then we consider the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$. We express the three roots of $P(x) = x^3 - px + q = 0$ in function of two parameters ρ, θ that depend of A^m, B^n, C^l. The Section 3 is the main part of the paper. We write that $A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3}$. As A^{2m} is an integer, it follows that $cos^2\frac{\theta}{3}$ must be written as $\frac{a}{b}$ where a, b are two positive coprime integers. We discuss the conditions of divisibility of p, a, b so that the expression of A^{2m} is an integer. Depending on each individual case, we obtain that A, B, C have or not a common factor. In the last Section, three numerical examples are presented. We finish with the conclusion.

2. Preliminaries

We begin with the trivial case when $A^m = B^n$. The equation (1) becomes:

$$2A^m = C^l \tag{2}$$

then $2|C^l \Longrightarrow 2|C \Longrightarrow \exists\, c \in N^* /\ C = 2c$, it follows $2A^m = 2^l c^l \Longrightarrow A^m = 2^{l-1}c^l$. As $l > 2$, then $2|A^m \Longrightarrow 2|A \Longrightarrow 2|B^n \Longrightarrow 2|B$. The conjecture (**??**) is verified.

We suppose in the following that $A^m > B^n$.

2.1. ***General Case***

Let $m, n, l \in N^* \ > 2$ and $A, B, C \in N^*$ such:

$$A^m + B^n = C^l \tag{3}$$

We call:

$$P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - x^2(A^m + B^n - C^l)$$
$$+x[A^mB^n - C^l(A^m + B^n)] + C^lA^mB^n \tag{4}$$

Using the equation (3), $P(x)$ can be written:

$$\boxed{P(x) = x^3 + x[A^mB^n - (A^m + B^n)^2] + A^mB^n(A^m + B^n)} \tag{5}$$

We introduce the notations:

$$p = (A^m + B^n)^2 - A^mB^n \tag{6}$$

$$q = A^mB^n(A^m + B^n) \tag{7}$$

As $A^m \neq B^n$, we have :

$$p > (A^m - B^n)^2 > 0 \tag{8}$$

Equation (5) becomes:

$$P(x) = x^3 - px + q \tag{9}$$

Using the equation (4), $P(x) = 0$ has three different real roots : A^m, B^n and $-C^l$.

Now, let us resolve the equation:

$$P(x) = x^3 - px + q = 0 \tag{10}$$

To resolve (10) let:

$$x = u + v \tag{11}$$

Then $P(x) = 0$ gives:

$$P(x) = P(u+v) = (u+v)^3 - p(u+v) + q = 0 \Longrightarrow u^3 + v^3 + (u+v)(3uv - p) + q = 0 \tag{12}$$

To determine u and v, we obtain the conditions:

$$u^3 + v^3 = -q \tag{13}$$

$$uv = p/3 > 0 \tag{14}$$

Then u^3 and v^3 are solutions of the second ordre equation:

$$X^2 + qX + p^3/27 = 0 \tag{15}$$

Its discriminant Δ is written as :

$$\Delta = q^2 - 4p^3/27 = \frac{27q^2 - 4p^3}{27} = \frac{\bar{\Delta}}{27} \tag{16}$$

Let:

$$\begin{aligned}\bar{\Delta} = 27q^2 - 4p^3 &= 27(A^mB^n(A^m + B^n))^2 - 4[(A^m + B^n)^2 - A^mB^n]^3 \\ &= 27A^{2m}B^{2n}(A^m + B^n)^2 - 4[(A^m + B^n)^2 - A^mB^n]^3 \end{aligned} \tag{17}$$

Noting :

$$\alpha = A^mB^n > 0 \tag{18}$$

$$\beta = (A^m + B^n)^2 \tag{19}$$

we can write (17) as:

$$\bar{\Delta} = 27\alpha^2\beta - 4(\beta - \alpha)^3 \tag{20}$$

As $\alpha \neq 0$, we can also rewrite (20) as :

$$\bar{\Delta} = \alpha^3 \left(27\frac{\beta}{\alpha} - 4\left(\frac{\beta}{\alpha} - 1\right)^3\right) \tag{21}$$

We call t the parameter :

$$t = \frac{\beta}{\alpha} \tag{22}$$

$\bar{\Delta}$ becomes :

$$\bar{\Delta} = \alpha^3(27t - 4(t-1)^3) \tag{23}$$

Let us calling :

$$y = y(t) = 27t - 4(t-1)^3 \tag{24}$$

Since $\alpha > 0$, the sign of $\bar{\Delta}$ is also the sign of $y(t)$. Let us study the sign of y. We obtain $y'(t)$:

$$y'(t) = y' = 3(1+2t)(5-2t) \tag{25}$$

t	$-\infty$		$-1/2$		$5/2$		4		$+\infty$
1+2t		-	0	+			+		
5-2t		+		+	0		-		
y'(t)		-	0	+	0		-		
y(t)	$+\infty$	↘	0	↗	54	↘	0	↘	$-\infty$

Figure 1: The table of variation

$y' = 0 \Longrightarrow t_1 = -1/2$ and $t_2 = 5/2$, then the table of variations of y is given below:
The table of the variations of the function y shows that $y < 0$ for $t > 4$. In our case, we are interested for $t > 0$. For $t = 4$ we obtain $y(4) = 0$ and for $t \in]0, 4[\Longrightarrow y > 0$. As we have $t = \frac{\beta}{\alpha} > 4$ because as $A^m \neq B^n$:

$$(A^m - B^n)^2 > 0 \Longrightarrow \beta = (A^m + B^n)^2 > 4\alpha = 4A^m B^n \tag{26}$$

Then $y < 0 \Longrightarrow \bar{\Delta} < 0 \Longrightarrow \Delta < 0$. Then, the equation (15) does not have real solutions u^3 and v^3. Let us find the solutions u and v with $x = u + v$ is a positive or a negative real and $u.v = p/3$.

2.2. ***Demonstration***

Proof. The solutions of (15) are:

$$X_1 = \frac{-q + i\sqrt{-\Delta}}{2} \tag{27}$$

$$X_2 = \overline{X_1} = \frac{-q - i\sqrt{-\Delta}}{2} \tag{28}$$

We may resolve:

$$u^3 = \frac{-q + i\sqrt{-\Delta}}{2} \tag{29}$$

$$v^3 = \frac{-q - i\sqrt{-\Delta}}{2} \tag{30}$$

Writing X_1 in the form:

$$X_1 = \rho e^{i\theta} \tag{31}$$

with:

$$\rho = \frac{\sqrt{q^2 - \Delta}}{2} = \frac{p\sqrt{p}}{3\sqrt{3}} \tag{32}$$

$$\text{and } \ sin\theta = \frac{\sqrt{-\Delta}}{2\rho} > 0 \tag{33}$$

$$cos\theta = -\frac{q}{2\rho} < 0 \tag{34}$$

Then $\theta\,[2\pi] \in]+\frac{\pi}{2}, +\pi[$, let:

$$\boxed{\frac{\pi}{2} < \theta < +\pi \Rightarrow \frac{\pi}{6} < \frac{\theta}{3} < \frac{\pi}{3} \Rightarrow \frac{1}{2} < cos\frac{\theta}{3} < \frac{\sqrt{3}}{2}} \tag{35}$$

and:

$$\boxed{\frac{1}{4} < cos^2\frac{\theta}{3} < \frac{3}{4}} \tag{36}$$

hence the expression of X_2:

$$X_2 = \rho e^{-i\theta} \tag{37}$$

Let:

$$u = re^{i\psi} \tag{38}$$

$$\text{and } \ j = \frac{-1 + i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \tag{39}$$

$$j^2 = e^{i\frac{4\pi}{3}} = -\frac{1 + i\sqrt{3}}{2} = \bar{j} \tag{40}$$

j is a complex cubic root of the unity $\iff j^3 = 1$. Then, the solutions u and v are:

$$u_1 = re^{i\psi_1} = \sqrt[3]{\rho}e^{i\frac{\theta}{3}} \tag{41}$$

$$u_2 = re^{i\psi_2} = \sqrt[3]{\rho}je^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+2\pi}{3}} \tag{42}$$

$$u_3 = re^{i\psi_3} = \sqrt[3]{\rho}j^2e^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{+i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+4\pi}{3}} \tag{43}$$

and similarly:

$$v_1 = re^{-i\psi_1} = \sqrt[3]{\rho}e^{-i\frac{\theta}{3}} \tag{44}$$

$$v_2 = re^{-i\psi_2} = \sqrt[3]{\rho}j^2e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi-\theta}{3}} \tag{45}$$

$$v_3 = re^{-i\psi_3} = \sqrt[3]{\rho}je^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{2\pi-\theta}{3}} \tag{46}$$

We may now choose u_k and v_h so that $u_k + v_h$ will be real. In this case, we have necessary :

$$v_1 = \overline{u_1} \tag{47}$$

$$v_2 = \overline{u_2} \tag{48}$$

$$v_3 = \overline{u_3} \tag{49}$$

We obtain as real solutions of the equation (12):

$$x_1 = u_1 + v_1 = 2\sqrt[3]{\rho} cos\frac{\theta}{3} > 0 \tag{50}$$

$$x_2 = u_2 + v_2 = 2\sqrt[3]{\rho} cos\tfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\tfrac{\theta}{3} + \sqrt{3} sin\tfrac{\theta}{3}\right) < 0 \tag{51}$$

$$x_3 = u_3 + v_3 = 2\sqrt[3]{\rho} cos\tfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\tfrac{\theta}{3} + \sqrt{3} sin\tfrac{\theta}{3}\right) > 0 \tag{52}$$

We compare the expressions of x_1 and x_3, we obtain:

$$2\sqrt[3]{p} cos\tfrac{\theta}{3} \overbrace{>}^{?} \sqrt[3]{p}\left(-cos\tfrac{\theta}{3} + \sqrt{3} sin\tfrac{\theta}{3}\right)$$

$$3cos\tfrac{\theta}{3} \overbrace{>}^{?} \sqrt{3} sin\tfrac{\theta}{3} \tag{53}$$

As $\frac{\theta}{3} \in]+\frac{\pi}{6}, +\frac{\pi}{3}[$, then $sin\frac{\theta}{3}$ and $cos\frac{\theta}{3}$ are > 0. Taking the square of the two members of the last equation, we get:

$$\frac{1}{4} < cos^2\frac{\theta}{3} \tag{54}$$

which is true since $\frac{\theta}{3} \in]+\frac{\pi}{6}, +\frac{\pi}{3}[$ then $x_1 > x_3$. As A^m, B^n and $-C^l$ are the only real solutions of (10), we consider, as A^m is supposed great than B^n, the expressions:

$$\boxed{\begin{cases} A^m = x_1 = u_1 + v_1 = 2\sqrt[3]{\rho} cos\dfrac{\theta}{3} \\ \\ B^n = x_3 = u_3 + v_3 = 2\sqrt[3]{\rho} cos\dfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\dfrac{\theta}{3} + \sqrt{3} sin\dfrac{\theta}{3}\right) \\ \\ -C^l = x_2 = u_2 + v_2 = 2\sqrt[3]{\rho} cos\dfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\dfrac{\theta}{3} + \sqrt{3} sin\dfrac{\theta}{3}\right) \end{cases}} \tag{55}$$

3. Proof of the Main Theorem

Main Theorem: *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{56}$$

then $A, B,$ and C have a common factor.

PROOF. $A^m = 2\sqrt[3]{\rho} cos\frac{\theta}{3}$ is an integer $\Rightarrow A^{2m} = 4\sqrt[3]{\rho^2} cos^2\frac{\theta}{3}$ is an integer. But:

$$\sqrt[3]{\rho^2} = \frac{p}{3} \tag{57}$$

Then:

$$A^{2m} = 4\sqrt[3]{\rho^2} cos^2\frac{\theta}{3} = 4\frac{p}{3}.cos^2\frac{\theta}{3} = p.\frac{4}{3}.cos^2\frac{\theta}{3} \tag{58}$$

As A^{2m} is an integer, and p is an integer then $cos^2\frac{\theta}{3}$ must be written in the form:

$$\boxed{cos^2\frac{\theta}{3} = \frac{1}{b} \quad or \quad cos^2\frac{\theta}{3} = \frac{a}{b}} \tag{59}$$

with $b \in N^*$, for the last condition $a \in N^*$ and a, b coprime.

3.1. ***Case*** $cos^2\frac{\theta}{3} = \frac{1}{b}$

We obtain :

$$A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p}{3.b} \tag{60}$$

As $\frac{1}{4} < cos^2\frac{\theta}{3} < \frac{3}{4} \Rightarrow \frac{1}{4} < \frac{1}{b} < \frac{3}{4} \Rightarrow b < 4 < 3b \Rightarrow b = 1, 2, 3.$

3.1.1. Case $b = 1$

$b = 1 \Rightarrow 4 < 3$ which is impossible.

3.1.2. Case $b = 2$

$b = 2 \Rightarrow A^{2m} = p.\dfrac{4}{3}.\dfrac{1}{2} = \dfrac{2.p}{3} \Rightarrow 3|p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{2p}{3} = 2.p' \tag{61}$$

But :

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{1}{2}\right) = \frac{p}{3} = \frac{3p'}{3} = p' \tag{62}$$

On the one hand:

$$A^{2m} = (A^m)^2 = 2p' \Rightarrow 2|p' \Rightarrow p' = 2p"^2 \Rightarrow A^{2m} = 4p"^2$$
$$\Rightarrow A^m = 2p" \Rightarrow 2|A^m \Rightarrow 2|A$$

On the other hand:

$B^nC^l = p' = 2p"^2 \Rightarrow 2|B^n$ or $2|C^l$. If $2|B^n \Rightarrow 2|B$. As $C^l = A^m + B^n$ and $2|A$ and $2|B$, it follows $2|A^m$ and $2|B^n$ then $2|(A^m + B^n) \Rightarrow 2|C^l \Leftrightarrow 2|C$.

Then, we have : A,B and C solutions of (3) have a common factor. Also if $2|C^l$, we obtain the same result : A,B and C solutions of (3) have a common factor.

3.1.3. Case $b = 3$

$b = 3 \Rightarrow A^{2m} = p.\dfrac{4}{3}.\dfrac{1}{3} = \dfrac{4p}{9} \Rightarrow 9|p \Rightarrow p = 9p'$ with $p' \neq 1$ since $9 \ll p$ then $A^{2m} = 4p' \Longrightarrow p'$ is not a prime. Let μ a prime with $\mu|p' \Rightarrow \mu|A^{2m} \Rightarrow \mu|A$.

On the other hand:

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = 5p'$$

Then $\mu|B^n$ or $\mu|C^l$. If $\mu|B^n \Rightarrow \mu|B$. As $C^l = A^m + B^n$ and $\mu|A$ and $\mu|B$, it follows $\mu|A^m$ and $\mu|B^n$ then $\mu|(A^m + B^n) \Rightarrow \mu|C^l \Longrightarrow \mu|C$.

Then, we have : A,B and C solutions of (3) have a common factor. Also if $\mu|C^l$, we obtain the same result : A,B and C solutions of (3) have a common factor.

3.2. **Case** $a > 1$, $cos^2\frac{\theta}{3} = \frac{a}{b}$

That is to say:

$$cos^2\frac{\theta}{3} = \frac{a}{b} \tag{63}$$

$$A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p.a}{3.b} \tag{64}$$

and a, b verify one of the two conditions:

$$\boxed{\{3|p \quad and \quad b|4p\}} \quad \text{or} \quad \boxed{\{3|a \quad and \quad b|4p\}} \tag{65}$$

and using the equation (36), we obtain a third condition:

$$\boxed{b < 4a < 3b} \tag{66}$$

In these conditions, respectively, $A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4\frac{p}{3}.cos^2\frac{\theta}{3}$ is an integer.

Let us study the conditions given by the equation (65).

3.2.1. ***Hypothesis:*** $\{3|p \quad and \quad b|4p\}$

3.2.1.1. Case $b = 2$ *and* $3|p$ *:.* $3|p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3b} = \frac{4.p'.a}{2} = 2.p'.a \tag{67}$$

As:

$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{2} < \frac{3}{4} \Rightarrow a < 2 \Rightarrow a = 1 \tag{68}$$

But $a > 1$ then the case $b = 2$ and $3|p$ is impossible.

3.2.1.2. Case $b = 4$ *and* $3|p$ *:.* We have $3|p \Longrightarrow p = 3p'$ with $p' \in N^*$, it follows:

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3 \times 4} = p'.a \tag{69}$$

and:

$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{4} < \frac{3}{4} \Rightarrow 1 < a < 3 \Rightarrow a = 2 \tag{70}$$

But a, b are coprime. Then the case $b = 4$ and $3|p$ is impossible.

3.2.1.3. Case: $b \neq 2, b \neq 4$, $b|p$ and $3|p$:. As $3|p$ then $p = 3p'$ and :

$$A^{2m} = \frac{4p}{3} cos^2 \frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{4 \times 3p'}{3}\frac{a}{b} = \frac{4p'a}{b} \tag{71}$$

We consider the case: $b|p' \Longrightarrow p' = bp"$ and $p" \neq 1$ (if $p" = 1$, then $p = 3b$, see sub-paragraph **II. Case k'=1** of paragraph **3.2.1.8**). Hence :

$$A^{2m} = \frac{4bp"a}{b} = 4ap" \tag{72}$$

Let us calculate $B^n C^l$:

$$B^n C^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = p'\left(3 - 4\frac{a}{b}\right) = b.p".\frac{3b-4a}{b} = p".(3b-4a) \tag{73}$$

Finally, we have the two equations:

$$A^{2m} = \frac{4bp"a}{b} = 4ap" \tag{74}$$

$$B^n C^l = p".(3b-4a) \tag{75}$$

I. Case $p"$ is prime:

From (74), $p"|A^{2m} \Rightarrow p"|A^m \Rightarrow p"|A$. From (75), $p"|B^n$ or $p"|C^l$. If $p"|B^n \Rightarrow p"|B$, as $C^l = A^m + B^n \Rightarrow p"|C^l \Rightarrow p"|C$. If $p"|C^l \Rightarrow p"|C$, as $B^n = C^l - A^m \Rightarrow p"|B^n \Rightarrow p"|B$.

Then A,B and C solutions of (3) have a common factor.

II. Case $p"$ is not prime:

Let λ one prime divisor of $p"$. From (74), we have :

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as}\,\lambda\,\text{is prime then} \quad \lambda|A \tag{76}$$

From (75), as $\lambda|p"$ we have:

$$\lambda|B^nC^l \Rightarrow \lambda|B^n \quad \text{or}\ \lambda|C^l \tag{77}$$

If $\lambda|B^n$, λ is prime $\lambda|B$, and as $C^l = A^m + B^n$ then we have also :

$$\lambda|C^l \quad \text{as}\,\lambda\,\text{is prime, then} \quad \lambda|C \tag{78}$$

By the same way, if $\lambda | C^l$, we obtain $\lambda | B$.

Then: A, B and C solutions of (3) have a common factor.

Let us verify the condition (66) given by:

$$b < 4a < 3b$$

In our case, the last equation becomes:

$$p < 3A^{2m} < 3p \quad with \quad p = A^{2m} + B^{2n} + A^m B^n \tag{79}$$

The condition $3A^{2m} < 3p \Longrightarrow A^{2m} < p$ is verified.
If :

$$p < 3A^{2m} \Longrightarrow 2A^{2m} - A^m B^n - B^{2n} > 0$$

We put $Q(Y) = 2Y^2 - B^n Y - B^{2n}$, the roots of $Q(Y) = 0$ are $Y_1 = -\frac{B^n}{2}$ and $Y_2 = B^n$. $Q(Y) > 0$ for $Y < Y_1$ and $Y > Y_2 = B^n$. In our case, we take $Y = A^m$. As $A^m > B^n$ then $p < 3A^{2m}$ is verified. Then the condition $b < 4a < 3b$ is true.

In the following of the paper, we verify easily that the condition $b < 4a < 3b$ implies to verify $A^m > B^n$ which is true.

3.2.1.4. Case $b = 3$ and $3|p$:. As $3|p \Longrightarrow p = 3p'$ and we write :

$$A^{2m} = \frac{4p}{3} cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{a}{b} = \frac{4 \times 3p'}{3} \frac{a}{3} = \frac{4p'a}{3} \tag{80}$$

As A^{2m} is an integer and that a and b are coprime and $cos^2 \frac{\theta}{3}$ can not be one in reference to the equation (35), then we have necessary $3|p' \Longrightarrow p' = 3p"$ with $p" \neq 1$, if not $p = 3p' = 3 \times 3p" = 9$ but $p = A^{2m} + B^{2n} + A^m B^n > 9$, the hypothesis $p" = 1$ is impossible, then $p" > 1$. hence:

$$A^{2m} = \frac{4p'a}{3} = \frac{4 \times 3p"a}{3} = 4p"a \tag{81}$$

$$B^n C^l = \frac{p}{3}\left(3 - 4cos^2 \frac{\theta}{3}\right) = p'\left(3 - 4\frac{a}{b}\right) = \frac{3p"(9 - 4a)}{3} = p".(9 - 4a) \tag{82}$$

As $\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{3} < \frac{3}{4} \Longrightarrow 3 < 4a < 9 \Longrightarrow a = 2$ as $a > 1$. $a = 2$, we obtain:

$$A^{2m} = \frac{4p'a}{3} = \frac{4 \times 3p\text{''}a}{3} = 4p\text{''}a = 8p\text{''} \tag{83}$$

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = p'\left(3 - 4\frac{a}{b}\right) = \frac{3p\text{''}(9-4a)}{3} = p\text{''} \tag{84}$$

The two last equations give that p" is not prime. Then we use the same methodology described above for the case **3.2.1.3.**, and we have : A,B and C solutions of (3) have a common factor.

3.2.1.5. Case $3|p$ *and* $b = p$ *:.* We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{p}$$

and :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{p} = \frac{4a}{3} \tag{85}$$

As A^{2m} is an integer, this implies that $3|a$, but $3|p \Longrightarrow 3|b$. As a and b are coprime, hence the contradiction. Then the case $3|p$ and $b = p$ is impossible.

3.2.1.6. Case $3|p$ *and* $b = 4p$ *:.* $3|p \Longrightarrow p = 3p'$, $p' \neq 1$ because $3 \ll p$, hence $b = 4p = 12p'$.

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{a}{3} \Longrightarrow 3|a \tag{86}$$

because A^{2m} is an integer. But $3|p \Longrightarrow 3|\left[(4p) = b\right]$, that is in contradiction with the hypothesis a, b are coprime. Then the case $b = 4p$ is impossible.

3.2.1.7. Case $3|p$ *and* $b = 2p$ *:.* $3|p \Longrightarrow p = 3p'$, $p' \neq 1$ because $3 \ll p$, hence $b = 2p = 6p'$.

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{2a}{3} \Longrightarrow 3|a \tag{87}$$

because A^{2m} is an integer. But $3|p \Longrightarrow 3|(2p) \Longrightarrow 3|b$, that is in contradiction with the hypothesis a, b are coprime. Then the case $b = 2p$ is impossible.

3.2.1.8. Case $3|p$ and $b \neq 3$ is a divisor of p :. We have $b = p' \neq 3$, and p is written as:

$$p = kp' \quad with \quad 3|k \Longrightarrow k = 3k' \tag{88}$$

and :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{b} = \frac{4 \times 3.k'p'}{3}\frac{a}{p'} = 4ak' \tag{89}$$

We calculate B^nC^l:

$$B^nC^l = \frac{p}{3}.\left(3 - 4cos^2\frac{\theta}{3}\right) = k'(3p' - 4a) \tag{90}$$

I. Case $k' \neq 1$:

We suppose $k' \neq 1$, we use the same methodology described for the case **3.1.2.3.**, and we obtain: A, B and C solutions of (3) have a common factor.

II. Case $k' = 1$:

We have $k' = 1 \Longrightarrow p = 3b$, then we have:

$$A^{2m} = 4a \Longrightarrow a \quad \text{is even} \tag{91}$$

and :

$$A^mB^n = 2\sqrt[3]{\rho}cos\frac{\theta}{3}.\sqrt[3]{\rho}\left(\sqrt{3}sin\frac{\theta}{3} - cos\frac{\theta}{3}\right) = \frac{p\sqrt{3}}{3}sin\frac{2\theta}{3} - 2a \tag{92}$$

let:

$$A^{2m} + 2A^mB^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} = 2b\sqrt{3}sin\frac{2\theta}{3} \tag{93}$$

The left member of (93) is an integer and b also, then $2\sqrt{3}sin\frac{2\theta}{3}$ can be written in the form:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2} \tag{94}$$

where k_1, k_2 are two coprime integers and $k_2|b \Longrightarrow b = k_2.k_3$.

II.1. Case $k_3 \neq 1$:

We suppose $k_3 \neq 1$. Hence:

$$A^{2m} + 2A^mB^n = k_3.k_1 \tag{95}$$

Let μ is an prime integer such that $\mu|k_3$. If $\mu = 2 \Rightarrow 2|b$, but $2|a$ that is contradiction with a, b coprime. We suppose $\mu \neq 2$ and $\mu|k_3$, then:

$$\boxed{\mu|A^m(A^m + 2B^n) \Longrightarrow \mu|A^m \text{ or } \mu|(A^m + 2B^n)} \tag{96}$$

II.1.1. Case $\mu|A^m$:

If $\mu|A^m \Longrightarrow \mu|A^{2m} \Longrightarrow \mu|4a \Longrightarrow \mu|a$. As $\mu|k_3 \Longrightarrow \mu|b$ and that a, b are coprime hence the contradiction.

II.1.2. Case $\mu|(A^m + 2B^n)$:

If $\mu|(A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu|(A^m + 2B^n)$, we can write:

$$A^m + 2B^n = \mu.t' \quad t' \in N^* \tag{97}$$

It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p, we obtain:

$$p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m) \tag{98}$$

As $p = 3b = 3k_2.k_3$ and $\mu|k_3$ hence $\mu|p \Longrightarrow\ p = \mu\mu'$, so we have :

$$\mu'\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m) \tag{99}$$

then:

$$\boxed{\mu|B^n(B^n - A^m) \Longrightarrow \mu|B^n \text{ or } \mu|(B^n - A^m)} \tag{100}$$

II.1.2.1. Case $\mu|B^n$:

If $\mu|B^n \Longrightarrow \mu|B$ which is in contradiction with case **II.1.2.** above.

II.1.2.2. Case $\mu|(B^n - A^m)$:

If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we obtain:

$$\mu|3B^n \tag{101}$$

II.1.2.2.1. Case $\mu|B^n$:

If $\mu|B^n$, using the result above of **II.1.2.1.** of this paragraph, it is impossible.

II.1.2.2.2. Case $\mu = 3$:

If $\mu = 3 \Longrightarrow 3|k_3 \Longrightarrow k_3 = 3k'_3$, and we have $b = k_2k_3 = 3k_2k'_3$, it follows $p = 3b = 9k_2k'_3$ then $9|p$, but $p = (A^m - B^n)^2 + 3A^mB^n$ then :

$$9k_2k'_3 - 3A^mB^n = (A^m - B^n)^2$$

we write it as :

$$3(3k_2k'_3 - A^mB^n) = (A^m - B^n)^2 \tag{102}$$

hence :

$$\boxed{3|(3k_2k'_3 - A^mB^n) \Longrightarrow\ 3|A^mB^n \Longrightarrow 3|A^m \text{ or } 3|B^n} \tag{103}$$

II.1.2.2.2.1. Case $3|A^m$:

If $3|A^m \Longrightarrow 3|A$ and we have also $3|A^{2m}$, but $A^{2m} = 4a \Longrightarrow 3|4a \Longrightarrow 3|a$. As $b = 3k_2k'_3$ then $3|b$, but a, b are coprime hence the contradiction. Then $3 \nmid A$.

II.1.2.2.2.2. Case $3|B^n$:

If $3|B^n \Longrightarrow 3|B$, but the (102) gives $3|(A^m - B^n)^2 \Longrightarrow 3|(A^m - B^n) \Longrightarrow 3|A^m \Longrightarrow 3|(A^{2m} = 4a) \Rightarrow 3|a$. As $3|b$ then the contradiction with a, b coprime.

Then the hypothesis $k_3 \neq 1$ is impossible.

III. Case $k_3 = 1$:

Now we suppose that $k_3 = 1 \Longrightarrow b = k_2$ and $p = 3b = 3k_2$. We have then:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_1}{b} \tag{104}$$

with k_1, b coprime. We write (104) as :

$$4\sqrt{3}sin\frac{\theta}{3}cos\frac{\theta}{3} = \frac{k_1}{b}$$

Taking the square of the two members and replacing $cos^2\frac{\theta}{3}$ by $\frac{a}{b}$, we obtain:

$$3 \times 4^2.a(b-a) = k_1^2 \tag{105}$$

which implies that :

$$\boxed{3|a \quad or \quad 3|(b-a)} \tag{106}$$

III.1. Case $3|a$:

If $3|a$, as $A^{2m} = 4a \Longrightarrow 3|A^{2m} \Longrightarrow 3|A$ and $3|a$. But $p = (A^m - B^n)^2 + 3A^mB^n$ and that $3|p \Longrightarrow 3|(A^m - B^n)^2 \Longrightarrow 3|(A^m - B^n)$. But $3|A$ hence $3|B^n \Longrightarrow 3|B$, as $m \geq 3 \Longrightarrow 3^2|p$, it follows $3|b$ then the contradiction with a, b coprime.

III.2. Case $3|(b-a)$:

Considering now that $3|(b-a)$. As $k_1 = A^m(A^m + 2B^n)$ by the equation (95) and that $3|k_1 \Longrightarrow 3|A^m(A^m + 2B^n) \Longrightarrow \boxed{3|A^m \; or \; 3|(A^m + 2B^n)}$.

III.2.1. Case $3|A^m$:

If $3|A^m \Longrightarrow 3|A \Longrightarrow 3|A^{2m}$ then $3|4a \Longrightarrow 3|a$. But $3|(b-a) \Longrightarrow 3|b$ hence the contradiction with a, b are coprime.

III.2.2. Case $3|(A^m + 2B^n)$:

If:

$$3|(A^m + 2B^n) \Longrightarrow 3|(A^m - B^n) \tag{107}$$

But $p = A^{2m} + B^{2n} + A^mB^n = (A^m - B^n)^2 + 3A^mB^n$ then $p - 3A^mB^n = (A^m - B^n)^2 \Longrightarrow 9|(p - 3A^mB^n)$ or $9|(3b - 3A^mB^n)$, then $3|(b - A^mB^n)$ but $3|(b-a) \Longrightarrow 3|(a - A^mB^n)$. As $A^{2m} = 4a = (A^m)^2 \Longrightarrow \exists a' \in N^*$ and $a = a'^2 \Longrightarrow A^m = 2a'$. We arrive to:

$$\boxed{3|(a'^2 - 2a'B^n) \Rightarrow 3|a'(a' - 2B^n) \Rightarrow 3|a' \quad or \quad 3|(a' - 2B^n)} \tag{108}$$

III.2.2.1. Case $3|a'$:

If $3|a' \Rightarrow 3|a'^2 \Rightarrow 3|a$, but $3|(b-a) \Rightarrow 3|b$, then the contradiction with a, b coprime.

III.2.2.2. Case $3|(a'-2B^n)$:

Now if $3|(a'-2B^n) \Rightarrow 3|(2a'-4B^n) \Rightarrow 3|(A^m-4B^n) \Rightarrow 3|(A^m-B^n)$, we refind the case **III.2.2.**, equation (107), that has a solution given by the case **2.2.1.** above.

Then, the study of the case **3.2.1.8.** is finished.

3.2.1.9 Case $3|p$ and $b|4p$:. As $3|p \Rightarrow p = 3p'$ and $b|4p \Rightarrow \exists k_1 \in N^*$ and $4p = 12p' = k_1 b$.

I. Case $k_1 = 1$:

If $k_1 = 1$, then $b = 12p'$, ($p' \neq 1$ if not $p = 3 \ll A^{2m} + B^{2n} + A^m B^n$). But $A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{12p'}{3}\frac{a}{b} = \frac{4p'.a}{12p'} = \frac{a}{3} \Rightarrow 3|a$ because A^{2m} is an integer, then the contradiction with a, b coprime.

II. Case $k_1 = 3$:

If $k_1 = 3$, then $b = 4p'$ and $A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{k_1.a}{3} = a$.

Let us calculate $A^m B^n$:

$$A^m B^n = 2\sqrt[3]{\rho}cos\frac{\theta}{3}.\sqrt[3]{\rho}\left(\sqrt{3}sin\frac{\theta}{3} - cos\frac{\theta}{3}\right) = \frac{p\sqrt{3}}{3}sin\frac{2\theta}{3} - \frac{a}{2} \tag{109}$$

Let:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} = 2p'\sqrt{3}sin\frac{2\theta}{3} \tag{110}$$

The left member of the equation (110) is an integer and also p', then $2\sqrt{3}sin\frac{2\theta}{3}$ can be written as :

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_2}{k_3} \tag{111}$$

where k_2, k_3 are two coprime integers and:

$$k_3|p' \Longrightarrow \exists k_4 \in N^* \quad and \quad p' = k_3.k_4 \tag{112}$$

II.1. Case $k_4 \neq 1$:

We suppose that $k_4 \neq 1$, then:

$$A^{2m} + 2A^m B^n = k_2.k_4 \tag{113}$$

Let μ one prime integer with:

$$\mu | k_4 \tag{114}$$

Then :

$$\boxed{\mu | A^m(A^m + 2B^n) \Longrightarrow \mu | A^m \quad \text{or} \quad \mu | (A^m + 2B^n)} \tag{115}$$

II.1.1. Case $\mu | A^m$:

If $\mu | A^m \Longrightarrow \mu | A^{2m} \Longrightarrow \mu | a$. As $\mu | k_4 \Longrightarrow \mu | p' \Rightarrow \mu | (4p' = b)$. But a, b are coprime then the contradiction.

II.1.2. Case $\mu | (A^m + 2B^n)$:

If $\mu | (A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu | (A^m + 2B^n)$, we can write:

$$A^m + 2B^n = \mu.t' \quad t' \in N^* \tag{116}$$

It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p, we obtain:

$$p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^m) \tag{117}$$

As $p = 3p'$ and $\mu | p' \Rightarrow \mu | (3p') \Rightarrow \mu | p$, we can write :$\exists \mu' \in N^*$ and $p = \mu\mu'$, then we obtain :

$$\mu' \mu = \mu(\mu t'^2 - 2t' B^n) + B^n (B^n - A^m) \tag{118}$$

and:

$$\boxed{\mu | B^n (B^n - A^m) \Longrightarrow \mu | B^n \quad \text{or} \quad \mu | (B^n - A^m)} \tag{119}$$

II.1.2.1. Case $\mu|B^n$:

If $\mu|B^n \Longrightarrow \mu|B$ which is in contradiction with the case **II.1.2.** above.

II.1.2.2. Case $\mu|(B^n - A^m)$:

If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we obtain:

$$\boxed{\mu|3B^n} \tag{120}$$

II.1.2.2.1. Case $\mu|B^n$:

If $\mu|B^n$ it is impossible, see the case **II.1.2.1.** above.

II.1.2.2.2 Case $\mu = 3$:

If $\mu = 3 \Longrightarrow 3|k_4 \Longrightarrow k_4 = 3k'_4$, and we obtain $p' = k_3k_4 = 3k_3k'_4$, it follows $p = 3p' = 9k_3k'_4$ then $9|p$, but $p = (A^m - B^n)^2 + 3A^mB^n$, then:

$$9k_4k'_5 - 3A^mB^n = (A^m - B^n)^2$$

that we write :

$$3(3k_4k'_5 - A^mB^n) = (A^m - B^n)^2 \tag{121}$$

then $3|(3k_4k'_5 - A^mB^n) \Longrightarrow\ 3|A^mB^n \Longrightarrow \boxed{3|A^m \quad or \quad 3|B^n}$.

II.1.2.2.2.1. Case $3|A^m$:

If $3|A^m \Longrightarrow 3|A^{2m} \Rightarrow 3|a$, but $3|p' \Rightarrow 3|(4p') \Rightarrow 3|b$, then the contradiction with a, b coprime. Then $3 \nmid A$.

II.1.2.2.2.2. Case $3|B^n$:

If $3|B^n$ and using (116), we have $A^m = \mu t' - 2B^n = 3t' - 2B^n \Longrightarrow 3|A^m \Rightarrow 3|A^{2m} \Rightarrow 3|a$, but $3|p' \Rightarrow 3|(4p') \Rightarrow 3|b$, then the contradiction with a, b coprime.

Then the hypothesis $k_4 \neq 1$ is impossible.

II.2. Case $k_4 = 1$:

We suppose that $\boxed{k_4 = 1} \Longrightarrow p' = k_3 k_4 = k_3$. Then we obtain:

$$2\sqrt{3} sin\frac{2\theta}{3} = \frac{k_2}{p'} \tag{122}$$

with k_2, p' coprime, we write (122) as :

$$4\sqrt{3} sin\frac{\theta}{3} cos\frac{\theta}{3} = \frac{k_2}{p'}$$

Taking the square of the two members and replacing $cos^2\frac{\theta}{3}$ by $\frac{a}{b}$ and $b = 4p'$, we obtain:

$$3.a(b-a) = k_2^2 \tag{123}$$

that implies:

$$\boxed{3|a \quad or \quad 3|(b-a)} \tag{124}$$

II.2.1. Case $3|a$:

If $3|a \Rightarrow 3|A^{2m} \Rightarrow 3|A$, as $p = (A^m - B^n)^2 + 3A^m B^n$ and that $3|p \Longrightarrow 3|(A^m - B^n)^2 \Longrightarrow 9|(A^m - B^n)^2$. But $(A^m - B^n)^2 = p - 3A^m B^n = 3b - 3A^m B^n \Longrightarrow 3|(b - A^m B^n)$. As $3|A^m \Longrightarrow 3|b \Longrightarrow$ the contradiction with a, b coprime.

II.2.2. Case $3|(b-a)$:

We consider that $3|(b-a)$. As $k_2 = A^m(A^m + 2B^n)$ given by the equation (113) and that $3|k_2 \Longrightarrow 3|A^m(A^m + 2B^n) \Longrightarrow \boxed{3|A^m \quad or \quad 3|(A^m + 2B^n)}$.

II.2.2.1. Case $3|A^m$:

If $3|A^m \Longrightarrow 3|A^{2m} \Longrightarrow 3|a$, but $3|(b-a) \Longrightarrow 3|b$ then the contradiction with a, b coprime.

II.2.2.2. Case $3|(A^m + 2B^n)$:

If:

$$3|(A^m + 2B^n) \Longrightarrow 3|(A^m - B^n) \tag{125}$$

but $p = A^{2m} + B^{2n} + A^m B^n = (A^m - B^n)^2 + 3A^m B^n$ then $p - 3A^m B^n = (A^m - B^n)^2 \Longrightarrow 9|(p - 3A^m B^n)$ or $9|(3p' - 3A^m B^n)$, then $3|(p' - A^m B^n) \Rightarrow$

$3|4(p' - 4A^mB^n) \Rightarrow 3|(b - 4A^mB^n)$ but $3|(b-a) \Longrightarrow 3|(a - A^mB^n)$. As $3|(A^{2m} - 4A^mB^n) \Rightarrow \boxed{3|A^m(A^m - 4B^n)}$.

II.2.2.2.1. Case $3|A^m$:

If $3|A^m \Longrightarrow 3|A^{2m} \Longrightarrow 3|a$, but $3|(b-a) \Longrightarrow 3|b$ then the contradiction with a, b coprime.

II.2.2.2.2. Case $3|(A^m - 4B^n)$:

Now if $3|(A^m - 4B^n) \Longrightarrow 3|(A^m - B^n)$, we refind the hypothesis of the beginning (125) above, that has a solution **II.2.2.2.1.**.

III. Case $k_1 \neq 3$ and $3|k_1$:

We suppose $k_1 \neq 3$ and $3|k_1 \Rightarrow k_1 = 3k'1$ with $k'_1 \neq 1$. We have $4p = 12p' = k_1b = 3k'_1b \Rightarrow 4p' = k'_1b$. A^{2m} can be written as :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{3k'_1b}{3}\frac{a}{b} = k'_1a \tag{126}$$

and B^nC^l:

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{k'_1}{4}(3b - 4a) \tag{127}$$

As B^nC^l is an integer, we must have $\boxed{4|(3b - 4a) \quad \text{or} \quad 4|k'_1}$.

III.1. Case $4|(3b - 4a)$:

We suppose that $4|(3b - 4a) \Rightarrow \dfrac{3b - 4a}{4} = c \in N^*$, and we obtain:

$$A^{2m} = k'_1a$$
$$B^nC^l = k'_1c$$

III.1.1. Case k'_1 is prime:

If k'_1 is prime, then $k'_1|A^{2m} \Rightarrow k'_1|A$ and $k'_1|B^nC^l \Rightarrow k'_1|B^n$ or $k'_1|C^l$. If $k'_1|B^n \Rightarrow k'_1|B$, then $k'_1|C^l \Rightarrow k'_1|C$. With the same method if $k'_1|C^l$, we arrive to $k'_1|B$.

We obtain: A,B and C solutions of (3) have a common factor.

III.1.2. Case k'_1 not a prime:

We suppose k'_1 not a prime. Let μ a prime divisor of k'_1, as described in **III.1.1.** above, we obtain : A,B and C solutions of (3) have a common factor.

III.2. Case $4|k'_1$:

Now, we suppose that $4|k'_1$.

III.2.1. Case $k'_1 = 4$:

We suppose $k'_1 = 4$, then $A^{2m} = 4a$ and $B^nC^l = 4c$, It is easy to verify that 2 is a common factor of A, B, C.

We obtain: A,B and C solutions of (3) have a common factor.

III.2.2. Case $k'_1 = 4k"_1$:

If $k'_1 = 4k"_1$ with $k"_1 > 1$. Then, we have:

$$A^{2m} = 4k"_1 a \tag{128}$$

$$B^nC^l = k"_1(3b - 4a) \tag{129}$$

III.2.2.1. Case $k"_1$ prime:

If $k"_1$ is prime, then $k"_1|A^{2m} \Rightarrow k"_1|A$ and $k"_1|B^nC^l \Rightarrow k"_1|B^n$ or $k"_1|C^l$. If $k"_1|B^n \Rightarrow k"_1|B$, then $k"_1|C^l \Rightarrow k"_1|C$. With the same method if $k"_1|C^l$, we arrive to $k"_1|B$.

We obtain: A,B and C solutions of (3) have a common factor.

III.2.2.2. Case $k"_1$ not a prime:

If $k"_1$ not a prime. Let μ a prime divisor of $k"_1$, as described in case **III.2.2.1.** above, we obtain : A,B and C solutions of (3) have a common factor.

3.2.2. ***Hypothesis*** *:* $\{3|a \quad and \quad b|4p\}$

We have :

$$3|a \Longrightarrow \exists a' \in N^* \ / \ a = 3a' \tag{130}$$

3.2.2.1. Case $b = 2$ *and* $3|a$ *:.* A^{2m} is written as :

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{b} = \frac{4p}{3}.\frac{a}{2} = \frac{2.p.a}{3} \tag{131}$$

Using the equation (130), A^{2m} becomes:

$$A^{2m} = \frac{2.p.3a'}{3} = 2.p.a' \tag{132}$$

But $cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2} > 1$ which is impossible, then $b \neq 2$.

3.2.2.2. Case $b = 4$ *and* $3|a$ *:.* A^{2m} is written as :

$$A^{2m} = \frac{4.p}{3}cos^2\frac{\theta}{3} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.p}{3}.\frac{a}{4} = \frac{p.a}{3} = \frac{p.3a'}{3} = p.a' \tag{133}$$

$$\text{and} \quad cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3.a'}{4} < \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} \Longrightarrow a' < 1 \tag{134}$$

which is impossible.

Then the case $b = 4$ is impossible.

3.2.2.3. Case $b = p$ *and* $3|a$ *:.* Then:

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{p} \tag{135}$$

and:

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{3a'}{p} = 4a' = (A^m)^2 \tag{136}$$

$$\exists a" \in N^* \ / \ a' = a"^2 \tag{137}$$

We calculate A^mB^n, hence:

$$A^mB^n = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} - 2a'$$

$$or \quad A^mB^n + 2a' = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} \tag{138}$$

The left member of (138) is an integer and p is also, then $2\frac{\sqrt{3}}{3}sin\frac{2\theta}{3}$ will be written as :

$$2\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2} \tag{139}$$

where k_1, k_2 are two coprime integers and $k_2|p \Longrightarrow p = b = k_2.k_3,\ k_3 \in N^*$.

I. Case $k_3 \neq 1$:

We suppose that $k_3 \neq 1$. We obtain :

$$A^m(A^m + 2B^n) = k_1.k_3 \tag{140}$$

Let us μ a prime integer with $\mu|k_3$, then $\mu|b$ and $\mu|A^m(A^m + 2B^n)$. Hence:

$$\boxed{\mu|A^m \quad or \quad \mu|(A^m + 2B^n)} \tag{141}$$

I.1. Case $\mu|A^m$:

If $\mu|A^m \Longrightarrow \mu|A$ and $\mu|A^{2m}$, but $A^{2m} = 4a' \Longrightarrow \mu|4a' \Longrightarrow (\mu = 2$ but $2|a')$ or $\mu|a'$. Then $\mu|a$ hence the contradiction with a, b coprime.

I.2. Case $\mu|(A^m + 2B^n)$:

If $\mu|(A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. We write $\mu|(A^m + 2B^n)$ as:

$$A^m + 2B^n = \mu.t' \quad t' \in N^* \tag{142}$$

It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p:

$$p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m) \tag{143}$$

Since $p = b = k_2.k_3$ and $\mu|k_3$ then $\mu|b \Longrightarrow \exists \mu' \in N^*$ and $b = \mu\mu'$, so we can write:

$$\mu'\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m) \tag{144}$$

From the last equation, we get $\mu|B^n(B^n - A^m) \Longrightarrow \boxed{\mu|B^n \quad or \quad \mu|(B^n - A^m)}$.

I.2.1. Case $\mu|B^n$:

If $\mu|B^n$ which is contradiction with $\mu \nmid B^n$.

I.2.2. Case $\mu|(B^n - A^m)$:

If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we arrive to:

$$\mu|3B^n \Longrightarrow \begin{cases} \boxed{\mu|B^n} \\ or \\ \boxed{\mu = 3} \end{cases} \tag{145}$$

I.2.2.1. Case $\mu|B^n$:

If $\mu|B^n$ which is contradiction with $\mu \nmid B$ from **I.2. Case** $\mu|(A^m + 2B^n)$.

I.2.2.2. Case $\mu = 3$:

If $\mu = 3$, then $b = 3\mu'$, but $3|a$ then the contradiction with a, b coprime.

II. Case $k_3 = 1$:

We assume now $k_3 = 1$. Hence:

$$A^{2m} + 2A^mB^n = k_1 \tag{146}$$

$$b = k_2 \tag{147}$$

$$\frac{2\sqrt{3}}{3} sin\frac{2\theta}{3} = \frac{k_1}{b} \tag{148}$$

Taking the square of the last equation, we obtain:

$$\frac{4}{3}sin^2\frac{2\theta}{3} = \frac{k_1^2}{b^2}$$

$$\frac{16}{3}sin^2\frac{\theta}{3}cos^2\frac{\theta}{3} = \frac{k_1^2}{b^2}$$

$$\frac{16}{3}sin^2\frac{\theta}{3}.\frac{3a'}{b} = \frac{k_1^2}{b^2}$$

Finally:

$$4^2a'(p-a) = k_1^2 \tag{149}$$

but $a' = a"^2$ then $p - a$ is a square. Let us:

$$\lambda^2 = p - a \tag{150}$$

The equation (149) becomes:

$$4^2 a"^2 \lambda^2 = k_1^2 \Longrightarrow k_1 = 4a"\lambda \tag{151}$$

taking the positive square root. Using (146), we get :

$$k_1 = 4a"\lambda \tag{152}$$

But $k_1 = A^m(A^m + 2B^n) = 2a"(A^m + 2B^n)$, it follows:

$$A^m + 2B^n = 2\lambda \tag{153}$$

Let λ_1 prime $\neq 2$, a divisor of λ (if not, $\lambda_1 = 2|\lambda \Longrightarrow 2|\lambda^2 \Longrightarrow 2|(p-a)$ but a is even, then $2|p \Longrightarrow 2|b$ which is contradiction with a, b coprime).

We consider $\lambda_1 \neq 2$ and :

$$\lambda_1|\lambda \Longrightarrow \lambda_1|\lambda^2 \quad and \quad \lambda_1|(A^m + 2B^n) \tag{154}$$

$$\lambda_1|(A^m + 2B^n) \Longrightarrow \lambda_1 \nmid A^m \quad if\ not \quad \lambda_1|2B^n \tag{155}$$

But $\lambda_1 \neq 2$ hence $\lambda_1|B^n \Longrightarrow \lambda_1|B$, it follows:

$$\lambda_1|(p = b) \quad and \quad \lambda_1|A^m \Longrightarrow \lambda_1|2a" \Longrightarrow \lambda_1|a \tag{156}$$

hence the contradiction with a, b coprime.

II.1. Case $\lambda_1 \nmid A^m$ and $\lambda_1|(A^m + 2B^n)$:

We assume now $\lambda_1 \nmid A^m$. $\lambda_1|(A^m + 2B^n) \Longrightarrow \lambda_1|(A^m + 2B^n)^2$ that is $\lambda_1|(A^{2m} + 4A^mB^n + 4B^{2n})$, we write it as $\lambda_1|(p + 3A^mB^n + 3B^{2n}) \Longrightarrow \lambda_1|(p + 3B^n(A^m + 2B^n) - 3B^{2n})$. But $\lambda_1|(A^m + 2B^n) \Longrightarrow \lambda_1|(p - 3B^{2n})$, as $\lambda_1|(p - a)$ hence by difference, we obtain $\lambda_1|(a - 3B^{2n})$ or $\lambda_1|(3a' - 3B^{2n}) \Longrightarrow \lambda_1|3(a' - B^{2n})$, Then:

$$\boxed{\lambda_1 = 3 \quad or \quad \lambda_1|(a' - B^{2n})} \tag{157}$$

II.1.1. Case $\lambda_1 = 3$:

If $\lambda_1 = 3$ but $3|a$, as $\lambda_1|(p-a) \Longrightarrow 3|(p=b)$ hence the contradiction with a, b coprime.

II.1.2. Case $\lambda_1|(a' - B^{2n})$:

If $\lambda_1|(a' - B^{2n}) \Longrightarrow \lambda_1|(a"^2 - B^{2n}) \Longrightarrow \boxed{\lambda_1|(a" - B^n)(a" + B^n)} \Longrightarrow \lambda_1|(a" + B^n)$ or $\lambda_1|(a" - B^n)$, because $(a" - B^n) \neq 1$, if not, we obtain $a"^2 - B^{2n} = a" + B^n \Longrightarrow a"^2 - a" = B^n - B^{2n}$. The left member is positive and the right member is negative, then the contradiction.

II.1.2.1. Case $\lambda_1|(a" - B^n)$:

If $\lambda_1|(a" - B^n) \Longrightarrow \lambda_1|2(a" - B^n) \Longrightarrow \lambda_1|(A^m - 2B^n)$ but $\lambda_1|(A^m + 2B^n)$ hence $\lambda_1|2A^m \Longrightarrow \lambda_1|A^m$ as $\lambda_1 \neq 2$, it follows $\lambda_1|A^m$ hence the contradiction with (155).

II.1.2.2. Case $\lambda_1|(a" + B^n)$:

If $\lambda_1|(a" + B^n) \Longrightarrow \lambda_1|2(a" + B^n) \Longrightarrow \lambda_1|(2a" + 2B^n) \Rightarrow \lambda_1|(A^m + 2B^n)$. We find the case **II.1.** that has solutions.

Then the case $k_3 = 1$ is impossible.

<u>*3.2.2.4. Case $b|p \Rightarrow\ p = b.p', p' > 1,\ b \neq 2,\ b \neq 4$ and $3|a$:*</u>

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.b.p'.3.a'}{3.b} = 4.p'a' \tag{158}$$

We calculate B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{159}$$

But $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, hence using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = p'(b - 4a') \tag{160}$$

As $p = b.p'$, and $p' > 1$, we have then:

$$B^nC^l = p'(b - 4a') \tag{161}$$

$$\text{and} \quad A^{2m} = 4.p'.a' \tag{162}$$

I. Case λ a prime divisor of p':

Let λ a prime divisor of p' (we suppose p' not prime). From (162), we have:

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as}\,\lambda\,\text{is a prime, then} \;\; \lambda|A \tag{163}$$

From (161), as $\lambda|p'$ we have:

$$\lambda|B^nC^l \Rightarrow \lambda|B^n \quad \text{or } \lambda|C^l \tag{164}$$

If $\lambda|B^n$, λ is a prime $\lambda|B$, but $C^l = A^m + B^n$, then we have also :

$$\lambda|C^l \quad \text{as}\,\lambda\,\text{is a prime, then } \lambda|C \tag{165}$$

By the same way, if $\lambda|C^l$, we obtain $\lambda|B$. then : A, B and C solutions of (3) have a common factor.

II. Case p' is a prime number:

We suppose now that p' is prime, from the equations (161) and (162), we obtain that:

$$p'|A^{2m} \Rightarrow p'|A^m \Rightarrow p'|A \tag{166}$$

and:

$$p'|B^nC^l \Rightarrow p'|B^n \quad \text{or } p'|C^l \tag{167}$$

$$\text{If} \quad p'|B^n \Rightarrow p'|B \tag{168}$$

$$\text{As} \quad C^l = A^m + B^n \quad \text{and that} \;\; p'|A, p'|B \Rightarrow p'|A^m, p'|B^n \Rightarrow p'|C^l$$
$$\Rightarrow p'|C \tag{169}$$

By the same way, if $p'|C^l$, we arrive to $p'|B$.

Hence: A ,B and C solutions of (3) have a common factor.

3.2.2.5. Case $b = 2p$ and $3|a$:. We have:

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{2p} = 2a' \Longrightarrow 2|A^m \Longrightarrow 2|a \Longrightarrow 2|a'$$

Then $2|a$ and $2|b$ which is contradiction with a, b coprime.

3.2.2.6. Case $b = 4p$ and $3|a$:. We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{4p} = a'$$

Calculate A^mB^n, we obtain:

$$A^mB^n = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{2p}{3}cos^2\frac{\theta}{3} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{a'}{2} \Longrightarrow$$
$$A^mB^n + \frac{A^{2m}}{2} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} \quad (170)$$

let:

$$A^{2m} + 2A^mB^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} \quad (171)$$

The left member of (171) is an integer and p is an integer, then $\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3}$ will be written:

$$\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2} \quad (172)$$

where k_1, k_2 are two coprime integers and $k_2|p \Longrightarrow p = k_2.k_3$.

I. Case $k_3 \neq 1$:

Firstly, we suppose that $k_3 \neq 1$. Hence:

$$A^{2m} + 2A^mB^n = k_3.k_1 \quad (173)$$

Let μ a prime integer and $\mu|k_3$, then $\mu|A^m(A^m+2B^n) \Longrightarrow \boxed{\mu|A^m \quad or \quad \mu|(A^m + 2B^n)}$.

I.1. Case $\mu|A^m$:

If $\mu|A^m \Longrightarrow \mu|(A^{2m} = a') \Rightarrow \mu|(3a' = a)$. As $\mu|k_3 \Longrightarrow \mu|p \Rightarrow \mu|(4p = b)$. Then the contradiction with a, b coprime.

I.2. Case $\mu|(A^m + 2B^n)$:

If $\mu|(A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then:

$$\mu \neq 2 \quad and \quad \mu \nmid B^n \tag{174}$$

$\mu|(A^m + 2B^n)$, we write:

$$A^m + 2B^n = \mu.t' \quad t' \in N^* \tag{175}$$

Then :

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

$$\Longrightarrow p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m) \tag{176}$$

As $b = 4p = 4k_2.k_3$ and $\mu|k_3$ then $\mu|b \Longrightarrow \exists \mu' \in N^*$ that $b = \mu\mu'$, we obtain:

$$\mu'\mu = \mu(4\mu t'^2 - 8t'B^n) + 4B^n(B^n - A^m) \tag{177}$$

The last equation implies $\mu|4B^n(B^n - A^m)$, but $\mu \neq 2$ then $\boxed{\mu|B^n \quad or \quad \mu|(B^n - A^m)}$.

I.2.1. Case $\mu|B^n$:

If $\mu|B^n$ then the contradiction with (174).

I.2.2. Case $\mu|(B^n - A^m)$:

If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we obtain:

$$\boxed{\mu|3B^n \Longrightarrow \mu|B^n \quad or \quad \mu = 3} \tag{178}$$

I.2.2.1. Case $\mu|B^n$:

If $\mu|B^n$ it is contradiction with (174).

I.2.2.2. Case $\mu = 3$:

If $\mu = 3$, then $b = 3\mu'$, but $3|a$ which is contradiction with a, b coprime.

II. Case $k_3 = 1$:

We assume now $k_3 = 1$. Hence:

$$A^{2m} + 2A^m B^n = k_1 \tag{179}$$

$$p = k_2 \tag{180}$$

$$\frac{2\sqrt{3}}{3} sin \frac{2\theta}{3} = \frac{k_1}{p} \tag{181}$$

Taking the square of the last equation, we obtain:

$$\frac{4}{3} sin^2 \frac{2\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3} sin^2 \frac{\theta}{3} cos^2 \frac{\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3} sin^2 \frac{\theta}{3} . \frac{3a'}{b} = \frac{k_1^2}{p^2}$$

Finally:

$$a'(4p - 3a') = k_1^2 \tag{182}$$

but $a' = a"^2$ then $4p - 3a'$ is a square. Let us:

$$\lambda^2 = 4p - 3a' = 4p - a = b - a \tag{183}$$

The equation (182) becomes:

$$a"^2 \lambda^2 = k_1^2 \Longrightarrow k_1 = a" \lambda \tag{184}$$

taking the positive square root. Using (179), we get :

$$k_1 = a" \lambda \tag{185}$$

But $k_1 = A^m(A^m + 2B^n) = a"(A^m + 2B^n)$, it follows:

$$(A^m + 2B^n) = \lambda \tag{186}$$

Let λ_1 prime $\neq 2$, a divisor of λ (if not $\lambda_1 = 2|\lambda \Longrightarrow 2|\lambda^2$. As $2|(b = 4p) \Longrightarrow 2|(a = 3a')$ which is contradiction with a, b coprime).

We consider $\lambda_1 \neq 2$ and :

$$\lambda_1|\lambda \Longrightarrow \lambda_1|(A^m + 2B^n) \tag{187}$$

$$\Longrightarrow \lambda_1 \nmid A^m \quad if\ not \quad \lambda_1|2B^n \tag{188}$$

But $\lambda_1 \neq 2$ hence $\lambda_1|B^n \Longrightarrow \lambda_1|B$, it follows:

$$\lambda_1|(b = 4p) \quad and \quad \lambda_1|A^m \Longrightarrow \lambda_1|2a" \Longrightarrow \lambda_1|a \tag{189}$$

hence the contradiction with a, b coprime.

II.1. Case $\lambda_1 \nmid A^m$, $\lambda_1 \nmid B^n$ and $\lambda_1|(A^m + 2B^n)$:

We assume now $\lambda_1 \nmid A^m$, $\lambda_1 \nmid B^n$. $\lambda_1|(A^m + 2B^n) \Longrightarrow \lambda_1|(A^m + 2B^n)^2$ that is $\lambda_1|(A^{2m} + 4A^mB^n + 4B^{2n})$, we write it as $\lambda_1|(p + 3A^mB^n + 3B^{2n}) \Longrightarrow \lambda_1|(p + 3B^n(A^m + 2B^n) - 3B^{2n})$. But $\lambda_1|(A^m + 2B^n) \Longrightarrow \lambda_1|(p - 3B^{2n})$, as $\lambda_1|(4p - a)$ hence by difference, we obtain $\lambda_1|(a - 3(B^{2n} + p))$ or $\lambda_1|(3a' - 3(B^{2n} + p)) \Longrightarrow \lambda_1|3(a' - B^{2n} - p) \Longrightarrow \boxed{\lambda_1 = 3 \text{ or } \lambda_1|(a' - (B^{2n} + p))}$.

II.1.1. Case $\lambda_1 = 3$:

If $\lambda_1 = 3|\lambda \Rightarrow 3|\lambda^2 \Rightarrow 3|b - a$ but $3|a \Longrightarrow 3|(p = b)$ hence the contradiction with a, b coprime.

II.1.2. Case $\lambda_1|(a' - (B^{2n} + p))$:

If $\lambda_1 \neq 3$ and $\lambda_1|(a' - B^{2n} - p) \Longrightarrow \lambda_1|(A^mB^n + B^{2n}) \Longrightarrow \lambda_1|B^n(A^m + 2B^n) \Longrightarrow \boxed{\lambda_1|B^n \quad or \quad \lambda_1|(A^m + 2B^n)}$.

II.1.2.1. Case $\lambda_1|B^n$:

If $\lambda_1|B^n$ that is in contradiction with the hypothesis $\lambda_1 \nmid B$ cited above case II.1.

II.1.2.2. Case $\lambda_1|(A^n + 2B^n)$:

If $\lambda_1|(A^n + 2B^n)$. We refind this condition in the case II.1.

Then the case $k_3 = 1$ is impossible.

3.2.2.7. Case $3|a$ and $b = 2p'$ $b \neq 2$ with $p'|p$:. $3|a \Longrightarrow a = 3a'$, $b = 2p'$ with $p = k.p'$, hence:

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.k.p'.3.a'}{6p'} = 2.k.a' \tag{190}$$

Calculate B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{191}$$

But $\sqrt[3]{\rho^2} = \frac{p}{3}$ hence en using $cos^2\frac{\theta}{3} = \frac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p' - 2a') \tag{192}$$

As $p = b.p'$, and $p' > 1$, we have then:

$$B^nC^l = k(p' - 2a') \tag{193}$$

$$\text{and} \quad A^{2m} = 2k.a' \tag{194}$$

I. Case λ is a prime divisor of k:

We suppose that λ is a prime divisor of k (we suppose k not a prime). From (194), we have:

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as}\, \lambda \,\text{is prime then} \quad \lambda|A \tag{195}$$

From (193), as $\lambda|k$, we have:

$$\lambda|B^nC^l \Rightarrow \lambda|B^n \quad \text{or} \quad \lambda|C^l \tag{196}$$

If $\lambda|B^n$, λ is prime $\lambda|B$, and as $C^l = A^m + B^n$ then we have also:

$$\lambda|C^l \quad \text{as}\, \lambda \,\text{is prime then} \quad \lambda|C \tag{197}$$

By the same way, if $\lambda|C^l$, we obtain $\lambda|B$. Then : A, B and C solutions of (3) have a common factor.

II. Case k is prime:

Now, we suppose now that k is prime, from the equations (193) and (194), we obtain:

$$k|A^{2m} \Rightarrow k|A^m \Rightarrow k|A \tag{198}$$

and:

$$k|B^nC^l \Rightarrow k|B^n \quad \text{or } k|C^l \tag{199}$$

$$\text{if} \quad k|B^n \Rightarrow k|B \tag{200}$$

$$\text{as} \quad C^l = A^m + B^n \quad \text{and that } k|A, k|B \Rightarrow k|A^m,\ k|B^n \Rightarrow k|C^l$$

$$\Rightarrow k|C \tag{201}$$

By the same way, if $k|C^l$, we arrive to $k|B$.

Hence: A ,B and C solutions of (3) have a common factor.

3.2.2.8. Case $3|a$ and $b = 4p'$ $b \neq 2$ with $p'|p$:. $3|a \Longrightarrow a = 3a'$, $b = 4p'$ with $p = k.p'$, $k \neq 1$, if not, $b = 4p$ a case that has been studied (paragraph **3.2.2.6**), then we have :

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.k.p'.3.a'}{12p'} = k.a' \tag{202}$$

Writing B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{203}$$

But $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, hence en using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p' - a') \tag{204}$$

As $p = b.p'$, and $p' > 1$, we have:

$$B^nC^l = k(p' - 2a') \tag{205}$$

$$\text{and} \quad A^{2m} = 2k.a' \tag{206}$$

I. Case λ a prime divisor of k:

Let λ a prime divisor of k (we suppose k not a prime). From (206), we have:

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as}\, \lambda \,\text{is prime then} \;\; \lambda|A \tag{207}$$

From (205), as $\lambda|k$ we obtain:

$$\lambda|B^nC^l \Rightarrow \boxed{\lambda|B^n \quad or \quad \lambda|C^l} \tag{208}$$

I.1 Case $\lambda|B^n$ or $\lambda|C^n$:

If $\lambda|B^n$, λ is a prime, then $\lambda|B$, and as $\lambda|A \Rightarrow \lambda|(A^m+B^n=C^l) \Rightarrow \lambda|C$. By the same way if $\lambda|C^l$, we obtain $\lambda|B$. Then : A, B and C solutions of (3) have a common factor.

II. Case k is prime:

We suppose now that k is prime, from the equations (205) and (206), we have:

$$k|A^{2m} \Rightarrow k|A^m \Rightarrow k|A \tag{209}$$

and:

$$k|B^nC^l \Rightarrow k|B^n \quad \text{or } k|C^l \tag{210}$$

$$\text{if} \quad k|B^n \Rightarrow k|B \tag{211}$$

$$\text{as} \quad C^l = A^m + B^n \quad \text{and that } k|A, k|B \Rightarrow k|A^m,\, k|B^n \Rightarrow k|C^l$$

$$\Rightarrow k|C \tag{212}$$

By the same way if $k|C^l$, we arrive to $k|B$.

Hence: A ,B and C solutions of (3) have a common factor.

3.2.2.9. Case $3|a$ and $b|4p$:. $a = 3a'$ and $4p = k_1 b$ with $k_1 \in N^*$. As $A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{3a'}{b} = k_1 a'$ and B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3a'}{b}\right) = \frac{k_1}{4}(b-4a') \tag{213}$$

As B^nC^l is an integer, we must have $\boxed{4|k_1 \quad or \quad 4|(b-4a')}$.

I. Case $k_1 = 1$:

If $k_1 = 1 \Rightarrow b = 4p$: it is the case (**3.2.2.6**) above.

II. Case $k_1 = 4$:

If $k_1 = 4 \Rightarrow p = b$: it is the case (**3.2.2.3**) above.

III. Case $4|k_1$:

We suppose that $4|k_1$ with $k_1 > 4 \Rightarrow k_1 = 4k'_1$, then we have:

$$A^{2m} = 4k'_1a'$$

$$B^nC^l = k'_1(b - 4a')$$

By discussing k'_1 is a prime integer or not, we arrive easily to: A ,B and C solutions of (3) have a common factor.

III.1. Case $4 \nmid (b - 4a')$ and $4 \nmid k'_1$:

If $4 \nmid (b - 4a')$ and $4 \nmid k'_1$ it is impossible.

III.2. Case $4|(b - 4a')$:

If $4|(b - 4a') \Rightarrow (b - 4a') = 4c$, with $c \in N^*$, then we obtain:

$$A^{2m} = k_1a'$$

$$B^nC^l = k_1c$$

By discussing k_1 is a prime integer or not, we arrive easily to: A ,B and C solutions of (3) have a common factor.

The main theorem is proved.

4. Numerical Examples

4.1. Example 1:

We consider the example:

$$6^3 + 3^3 = 3^5 \tag{214}$$

with $A^m = 6^3$, $B^n = 3^3$ and $C^l = 3^5$. With the notations used in the paper, we obtain:

$$p = 3^6 \times 73, \tag{215}$$

$$q = 8 \times 3^{11}, \tag{216}$$

$$\bar{\Delta} = 4 \times 3^{18}(3^7 \times 4^2 - 73^3) < 0, \tag{217}$$

$$\rho = \frac{p\sqrt{p}}{3\sqrt{3}} = \frac{3^8 \times 73\sqrt{73}}{3}, \tag{218}$$

$$cos\theta = -\frac{4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \tag{219}$$

As $A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} \Longrightarrow cos^2\frac{\theta}{3} = \frac{3A^{2m}}{4p} = \frac{3 \times 2^4}{73} = \frac{a}{b} \Longrightarrow a = 3 \times 2^4,\ b = 73;$ then:

$$cos\frac{\theta}{3} = \frac{4\sqrt{3}}{\sqrt{73}} \tag{220}$$

$$p = 3^6 b \tag{221}$$

Let us verify the equation (219) using the equation (220):

$$cos\theta = cos3(\theta/3) = 4cos^3\frac{\theta}{3} - 3cos\frac{\theta}{3} = 4\left(\frac{4\sqrt{3}}{\sqrt{73}}\right)^3 - 3\frac{4\sqrt{3}}{\sqrt{73}} = -\frac{4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \tag{222}$$

That's OK. For this example, we can use the two conditions of (65) as $3|p$,$b|4p$ and $3|a$. The cases **3.2.1.3** and **3.2.2.4** are respectively used. We find for both cases that A^m, B^n and C^l of the equation (214) have a common prime factor which is true.

4.2. Example 2:

Let the second example:

$$7^4 + 7^3 = 14^3 \Rightarrow 2401 + 343 = 2744 \tag{223}$$

With the notations of the paper, we take:

$$A^m = 7^4 \tag{224}$$

$$B^n = 7^3 \tag{225}$$

$$C^l = 14^3 \tag{226}$$

We obtain:

$$p = 57 \times 7^6 = 3 \times 19 \times 7^6 \tag{227}$$

$$q = 8 \times 7^{10} \tag{228}$$

$$\overline{\Delta} = 27q^2 - 4p^3 = 27 \times 4 \times 7^{18}(16 \times 49 - 19^3)$$
$$= -27 \times 4 \times 7^{18} \times 6075 < 0 \tag{229}$$

$$\rho = \tfrac{p\sqrt{p}}{3\sqrt{3}} = 19 \times 7^9 \times \sqrt{19} \tag{230}$$

$$cos\theta = \frac{-q}{2\rho} = -\frac{4 \times 7}{19\sqrt{19}} \tag{231}$$

As $A^{2m} = \dfrac{4p}{3}.cos^2\dfrac{\theta}{3} \Longrightarrow cos^2\dfrac{\theta}{3} = \dfrac{3A^{2m}}{4p} = \dfrac{7^2}{4 \times 19} = \dfrac{a}{b} \Longrightarrow a = 7^2,\ b = 4 \times 19;$ then:

$$cos\frac{\theta}{3} = \frac{7}{2\sqrt{19}} \tag{232}$$

$$3|p \quad and \quad b|(4p) \tag{233}$$

Let us verify the equation (231) using the equation (232):

$$cos\theta = cos3(\theta/3) = 4cos^3\frac{\theta}{3} - 3cos\frac{\theta}{3} = 4\left(\frac{7}{2\sqrt{19}}\right)^3 - 3\frac{7}{2\sqrt{19}} = -\frac{4 \times 7}{19\sqrt{19}} \tag{234}$$

It is the same value of (231)!

Now, from (233), we have $3|p \Rightarrow p = 3p'$, $b|(4p)$ with $b \neq 2, 4$ then $12p' = k_1 b = 3 \times 7^6 b$. It concerns the paragraph **3.2.1.9.** of the first hypothesis. As $k_1 = 3 \times 7^6 = 3k_1'$ with $k_1' = 7^6 \neq 1$. It is the case **III.**, with the two conditions: $4|(3b - 4a)$ or $4|k_1'$. We take $4|(3b - 4a)$. Let us calculate $3b - 4a$:

$$3b - 4a = 3 \times 4 \times 19 - 4 \times 7^2 = 32 \Longrightarrow 4|(3b - 4a) \tag{235}$$

Then it is the sous-case **III.1.** with $A^{2m} = 7^8 = 7^6 \times 7^2 = k_1'.a$ with k_1' not a prime, we find the sous-case **III.1.2** with the result that A, B and C have a common factor namely the prime number 7 a divisor of $k_1' = 7^6$!.

4.3. Example 3:

Let the third example:

$$7^2 + 2^5 = 3^4 \tag{236}$$

with:

$$A^m = 7^2; B^n = 2^5; C^l = 3^4$$

We obtain:

$$p = 4999 \quad a\, prime\, number \tag{237}$$

$$q = 2^5 \times 7^2 \times 3^4 = 127008 \gg p \tag{238}$$

As $q \gg p$, we find that :

$$\overline{\Delta} = 27q^2 - 4p^3 > 0 \tag{239}$$

Then we cannot use the results of our proof because in this example, $m = 2 < 3$. We remark that in all the proof, we don't encountered that m, n or l must be great than 2. Then the condition that $m, n, l > 2$ is important in (1).

5. Conclusion

As seen above, the examples confirm the results of the proof. In conclusion, we can announce the theorem:

Theorem 1. Considering the previous point, we propose: *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{240}$$

then $A, B,$ and C have a common factor.

References

R. Daniel Mauldin. *A Generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem.* Notice of AMS, Vol44, $n°11, 1997.$