Question 449 : Some Definite Integrals

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abstract

This note presents some definite integrals.

1. Introduction.

This note presents some definite integrals , example:

$$\int_0^{\sqrt{W(2)/2}} \left(e^{-x^2} + \sqrt{-\ln x} \right) dx = \frac{\sqrt{\pi}}{2} + \frac{W(2)}{2} \tag{1}$$

where $W(x)$ is the Lambert function: $W(x)e^{W(x)} = x$.

2. Integrals.

$$\int_0^u \left(\frac{1}{1+x^2} + \sqrt{\frac{1-x}{x}} \right) dx = \frac{\pi}{2} + u^2 \tag{2}$$

where

$$u = 3^{-2/3} \left(\frac{9+\sqrt{93}}{2} \right)^{1/3} - \left(\frac{2}{3\left(9+\sqrt{93}\right)} \right)^{1/3}$$

The number u is root of the equation: $u^3 + u - 1 = 0$.

$$\int_0^u \left(\frac{1}{1+x^3} + \sqrt[3]{\frac{1-x}{x}} \right) dx = \frac{2\pi}{3\sqrt{3}} + u^2 \tag{3}$$

where

$$u = \frac{1}{2\sqrt[3]{6}} \left(-\sqrt{v} + \sqrt{\frac{12}{\sqrt{v}} - v} \right)$$

$$v=\left(2\left(9+\sqrt{849}\right)\right)^{1/3}-8\left(\frac{3}{9+\sqrt{849}}\right)^{1/3}$$

The number u is root of the equation: $u^4+u-1=0$.

$$\int_0^u\left(\frac{1}{1+x^4}+\sqrt[4]{\frac{1-x}{x}}\right)dx=\frac{\pi}{2\sqrt{2}}+u^2 \tag{4}$$

where

$$u=\frac{1}{3}\left(-1+\left(\frac{25-3\sqrt{69}}{2}\right)^{1/3}+\left(\frac{25+3\sqrt{69}}{2}\right)^{1/3}\right)$$

The number u is root of the equation: $u^5+u-1=0$.

$$\int_0^u\left(\sqrt{\cot x}-\tan^{-1}x^2\right)dx=\frac{\pi}{\sqrt{2}}-\frac{\pi}{2}u+u^2 \tag{5}$$

where $u=0.895206...$, is root of the equation: $u^2\tan u=1$.

$$\int_0^u\left(\sinh^{-1}\left(\cos x\right)+\cos^{-1}\left(\sinh x\right)\right)dx=G+u^2 \tag{6}$$

where $G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$ is Catalan's constant, and $u=0.703290...$ is root of the equation: $\sinh u=\cos u$.

$$\int_0^u\left(\tan^{-1}\left(e^{-x}\right)-\ln\left(\tan x\right)\right)dx=G+u^2 \tag{7}$$

where G is Catalan's constant, and $u=0.531390...$ is root of the equation: $\tan u=e^{-u}$.

$$\int_0^u\left(\frac{1}{\cosh x}+\cosh^{-1}\frac{1}{x}\right)dx=\frac{\pi}{2}+u^2 \tag{8}$$

where $u=0.765009...$ is root of the equation: $u\cosh u=1$.

$$\int_0^u\left(\sin^{-1}\left(e^{-x}\right)-\ln\left(\sin x\right)\right)dx=\frac{\pi}{2}\ln 2+u^2 \tag{9}$$

where $u=0.588532...$ is root of the equation: $\sin u=e^{-u}$.

$$\int_0^u\left(\tan^{-1}\frac{1-x}{1+x}+\frac{1-\tan x}{1+\tan x}\right)dx=\frac{\ln 2}{2}+u^2 \tag{10}$$

where $u=0.402628...$ is root of the equation: $u=\dfrac{1-\tan u}{1+\tan u}$.

$$\int_0^u\left((\cos x)^2+\cos^{-1}\sqrt{x}\right)dx=\frac{\pi}{4}+u^2 \qquad (11)$$

where $u=0.641714...$ is root of the equation: $\cos u=\sqrt{u}$.

$$\int_0^{1/\sqrt{\phi}}\left(\sqrt{1-x^4}+\sqrt[4]{1-x^2}\right)dx=\frac{\left(\Gamma(1/4)\right)^2}{6\sqrt{2\pi}}+\frac{1}{\phi} \qquad (12)$$

where $\phi=\dfrac{1+\sqrt{5}}{2}$.

$$\int_0^u\left(\left(1-x^2\right)^{3/2}+\left(1-x^{2/3}\right)^{1/2}\right)dx=\frac{3\pi}{16}+u^2 \qquad (13)$$

where

$$u=\frac{1}{6}\left(\left(2\left(9+\sqrt{93}\right)\right)^{1/3}-2\left(\frac{3}{9+\sqrt{93}}\right)^{1/3}\right)^{3/2}$$

$$\int_0^u\left(\frac{1}{\sqrt{\sin x}}+\sin^{-1}\left((1+x)^{-2}\right)\right)dx=\frac{\left(\Gamma(1/4)\right)^2}{2\sqrt{2\pi}}-\frac{\pi}{2}+u+u^2 \qquad (14)$$

where $u=0.476521...$ is root of the equation: $\sin u=(1+u)^{-2}$.

$$\int_0^u\left(\frac{1}{\cosh x^2}+\sqrt{\cosh^{-1}\frac{1}{x}}\right)dx=u^2+\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{2n+1}} \qquad (15)$$

where $u=0.814312...$ is root of the equation: $u\cosh u^2=1$.

$$\int_0^u\left(\frac{e^{-x}}{\sqrt{x}}+\frac{W\left(2x^{-2}\right)}{2}\right)dx=\sqrt{\pi}+u^2 \qquad (16)$$

where $u=\dfrac{3}{2}W\left(\dfrac{2}{3}\right)$, and $W(x)$ is the Lambert function.

$$\int_0^u\left(\left(\ln\left(\tan\frac{x}{2}\right)\right)^2+\frac{1}{2}\tan^{-1}\sqrt{e^{\sqrt{x}}}\right)dx=\frac{\pi^3}{8}+\frac{u^2}{4} \qquad (17)$$

where $u=2.260324...$ is root of the equation: $u=2\tan^{-1}\sqrt{e^{\sqrt{u}}}$.

$$\int_0^u \left(e^{\cos^{-1} x} + \cos\left(\ln(1+x)\right)\right) dx = u + u^2 + \frac{e^{\pi/2} - 1}{2} \tag{18}$$

where $u = 0.824574...$ is root of the equation: $u = \cos\left(\ln(1+u)\right)$.

$$\int_0^u \left(e^{-\sin^{-1} x} - \sin\left(\ln\left(x + e^{-\pi/2}\right)\right)\right) dx = \frac{1 - e^{-\pi/2}}{2} + u^2 + u\, e^{-\pi/2} \tag{19}$$

where $u = 0.431912...$ is root of the equation: $u + \sin\left(\ln\left(u + e^{-\pi/2}\right)\right) = 0$.

$$\int_0^u \left(\sinh\left(\cos^{-1} x\right) + \cos\left(\sinh^{-1} x\right)\right) dx = \frac{1}{2}\cosh\frac{\pi}{2} + u^2 \tag{20}$$

where $u = 0.762718...$ is root of the equation: $u = \cos\left(\sinh^{-1} u\right)$.

$$\int_0^u \left(\cosh\left(\cos^{-1} x\right) + \cos\left(\cosh^{-1}(1+x)\right)\right) dx = \frac{1}{2}\sinh\frac{\pi}{2} + u + u^2 - \frac{1}{2} \tag{21}$$

where $u = 0.541461...$ is root of the equation: $u = \cos\left(\cosh^{-1}(1+u)\right)$.

$$\int_0^u \left(\ln\frac{4+x^2}{1+x^2} + \sqrt{\frac{4-e^x}{e^x-1}}\right) dx = \pi + u^2 \tag{22}$$

where $u = 0.947875...$ is root of the equation: $e^u\left(1+u^2\right) = 4 + u^2$.

$$\int_0^u \left(\frac{1-x^6}{1+x^6} + \sqrt[6]{\frac{1-x}{1+x}}\right) dx = \frac{\pi}{3} + \frac{\ln\left(2+\sqrt{3}\right)}{\sqrt{3}} - 1 + u^2 \tag{23}$$

where $u = 0.732464...$ is root of the equation: $u^7 + u^6 + u - 1 = 0$.

$$\int_0^u \left(\frac{1}{\sqrt[4]{7+\cosh x}} + \cosh^{-1}\left(x^{-4} - 7\right)\right) dx = \frac{\sqrt[4]{6}}{3\sqrt{\pi}}\left(\Gamma(1/4)\right)^2 + u^2 \tag{24}$$

where $u = 0.591305...$ is root of the equation: $\left(7 + \cosh u\right) u^4 = 1$.

$$\int_0^u \left(\frac{1-\sin x}{1+\sin x} + \sin^{-1}\frac{1-x}{1+x}\right) dx = 2 - \frac{\pi}{2} + u^2 \tag{25}$$

where $u = 0.420362...$ is root of the equation: $\frac{1-\sin u}{1+\sin u} = u$.

$$\int_0^u \left(\frac{1}{x^2 + \sqrt{1+x^4}} + \sqrt{\frac{1-x^2}{2x}}\right) dx = \frac{\left(\Gamma(1/4)\right)^2}{6\sqrt{\pi}} + u^2 \tag{26}$$

where $u=\frac{1}{6}\left(-1+\left(53-6\sqrt{78}\right)^{1/3}+\left(53+6\sqrt{78}\right)^{1/3}\right)$.

$$\int_0^u\left(\sqrt{1+x^4}+\sqrt{\frac{1-x^2}{2x}}\right)dx=\frac{\left(\Gamma(1/4)\right)^2}{6\sqrt{\pi}}+u^2+\frac{u^3}{3} \qquad (27)$$

where $u=\frac{1}{6}\left(-1+\left(53-6\sqrt{78}\right)^{1/3}+\left(53+6\sqrt{78}\right)^{1/3}\right)$.

$$\int_0^u\left(\left(1-(1-x)^3\right)^{-1/2}-\left(1-(1+x)^{-2}\right)^{1/3}\right)dx=\frac{\left(\Gamma(1/3)\right)^3}{2\pi\sqrt{3}\sqrt[3]{2}}+u^2-1 \qquad (28)$$

where $u=0.273518...$ is root of the equation: $u^5-u^4-2u^3+3u^2+3u-1=0$.

$$\int_0^u\left(\left(1-(1-x)^4\right)^{-1/2}-\left(1-(1+x)^{-2}\right)^{1/4}\right)dx=\frac{\left(\Gamma(1/4)\right)^2}{4\sqrt{2\pi}}+u^2-1 \qquad (29)$$

where $u=0.234344...$ is root of the equation: $u^6-2u^5-u^4+4u^3-2u^2-4u+1=0$.

$$\int_0^u\left(\tan\sqrt{\frac{\pi^2}{16}-x}-\left(\tan^{-1}x\right)^2\right)dx=G-\frac{\pi\ln 2}{4}+u^2-\frac{\pi^2}{16}u \qquad (30)$$

where G is Catalan's constant , and $u=0.442968...$ is root of the equation:

$$u=\tan^{-1}\left(\frac{1-u}{1+u}\right)\tan^{-1}\left(\frac{1+u}{1-u}\right)=\frac{\pi^2}{16}-\left(\tan^{-1}u\right)^2$$

$$\int_0^u\left(\tan^{-1}\left(\frac{1-x}{1+x}\right)\tan^{-1}\left(\frac{1+x}{1-x}\right)+\tan\sqrt{\frac{\pi^2}{16}-x}\right)dx=G-\frac{\pi\ln 2}{4}+u^2 \qquad (31)$$

where G is Catalan's constant, and $u=0.442968...$ is defined by (30).

$$\int_0^u\left(\tan^{-1}(1-x)-\tan x\right)dx=\frac{\pi}{4}-\frac{\ln 2}{2}+u^2-u \qquad (32)$$

where $u=0.479731...$ is root of the equation: $\tan u=1-u$.

$$\int_0^u\left(\frac{1}{1+x^{10}}+\sqrt[10]{\frac{1-x}{x}}\right)dx=\frac{\phi\pi}{5}+u^2 \qquad (33)$$

where $u=0.844397...$ is root of the equation: $u^{11}+u-1=0$.

$$\int_0^u\left(\sqrt{\frac{-1}{\ln(1-x)}}-e^{-1/x^2}\right)dx=\sqrt{\pi}+u^2-u \qquad (34)$$

where $u=0.794714...$ is root of the equation: $u=1-e^{-1/u^2}$.

$$\int_0^u \left(\tan^{-1}\left(2e^{-x}-1\right)-\ln\left(1+\tan x\right)\right)dx=\frac{\pi}{8}\ln 2+u^2-u\ln 2 \qquad (35)$$

where $u=0.367475...$ is root of the equation: $1+\tan u=2e^{-u}$.

$$\int_0^u \left(\frac{1}{1+\left(\tan x\right)^m}+\tan^{-1}\left(\left(\frac{1}{x}-1\right)^{1/m}\right)\right)dx=\frac{\pi}{4}+u^2 \qquad (36)$$

where $m>0$, and u is root of the equation: $u\left(1+\left(\tan u\right)^m\right)=1, 0<u<1$.

$$\int_0^u \left(\ln\Gamma\left(x\right)+\Gamma^{-1}\left(e^x\right)\right)dx=\ln\sqrt{2\pi}+u^2 \qquad (37)$$

where $u=0.524921...$ is root of the equation: $e^u=\Gamma\left(u\right)$. $\Gamma^{-1}\left(x\right)$ is the inverse function of the Gamma function $\Gamma\left(x\right)$.

$$\gamma=u^2-v^2+v-\int_0^u\left(e^{-e^x}+\ln\left(\ln\frac{1}{x}\right)\right)dx-\int_0^v\left(e^{-e^{-x}}+\ln\left(\ln\frac{1}{1-x}\right)\right)dx \qquad (38)$$

where $u=0.269874...$,is root of the equation: $u=e^{-e^u}$, and $v=0.466059...$, is root of the equation: $v=1-e^{-e^{-v}}$. the number $\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}-\ln n\right)=0.577215...$ is the Euler-Mascheroni constant.

Remark:

$$u=e^{-e^{e^{-e^{e^{-\cdots}}}}} \qquad (39)$$

$$v=1-e^{-e^{-1+e^{-e^{-1+e^{-e^{-1+\cdots}}}}}} \qquad (40)$$

References

1. Boros, G. and Moll, V.: Irresistible Integrals.Cambridge, University Press,2004.
2. Gradshteyn, I.S. and Ryzhik, I.M.: Table of Integrals, Series, and Products. seventh edition, ed. A. Jeffrey and D. Zwillinger. Academic Press, 2007.