Intuitive explanation of the Riemann hypothesis

1. Characterisation of the nontrivial zeroes of ζ.

There is a unique (canonical) one-form α on $\mathbb{H}$ invariant under $\Gamma(2)$ with a pole of residue 1 at the image of $i\infty$ and a pole of residue -1 at the image of 1. Under the embedding $\mathbb{H} \to \mathbb{C}$ with τ the coordinate on $\mathbb{C}$ the ratio $[\alpha : d\tau]$ tends to one at the upper end of the interval $(0, i\infty)$. Let T be the connected real multiplicative group and consider the multiplication actions

$$\begin{cases} \mu_+ : T \times \mathbb{H} \to \mathbb{H} \\ (g, z) \mapsto \sqrt{g} z \end{cases}$$

$$\begin{cases} \mu_- : T \times \mathbb{H} \to \mathbb{H} \\ (g, z) \mapsto \frac{1}{\sqrt{g}} z \end{cases}$$

1. Theorem. For each unitary character ω of T and each real number c with $0 < c < 1$, the differential two-form

$$g^{2-2c}\omega(g)\mu_+^*(\alpha - d\tau) \wedge \mu_-^*(\alpha - d\tau)$$

is real and integrable (rapidly decreasing, that is 'Schwartz') on $T \times (0, i\infty)$. Among rapidly decreasing forms, it is exact if and only if $\zeta(c + i\omega_0)$ is zero where ζ is Riemann's zeta function and ω_0 is the real number corresponding to ω under the rule $\omega(g) = g^{i\omega_0}$.

Proof. It is real because the factors besides $\omega(g)$ are anti-symmetric with respect to interchanging μ_+ and μ_- which matches the reversal of orientation of T. The two-form integrates to the squared absolute value of a holomorphic integral, namely $\int g^{1-c}\omega(g)(\alpha - d\tau)$. In turn, it is easy to calculate the holomorphic *definite* integral; it is $iL(s, \chi)\Gamma(s)\pi^{-s}$ where i is the imaginary unit, and L is the L series for sums of four squares, χ is the Dirichlet sign character and $s = c + i\omega_0$. The rule $\omega(g_1 g_2^{-1}) = \omega(g_1)\omega(g_2)^{-1}$ is all that is needed.

2. Remark about the dynamical interpretation.

Here is an intuitive way of integrating the two-form let, us call it A_s for $s = c + i\omega_0$. if we let $\tau = ie^t$. By 'integration by parts'

$$\frac{1}{i\pi}\int e^{(c-1)t+i\omega_0 t} d\, log(\lambda/q) = \frac{-c+1-i\omega_0}{i\pi}\int e^{(c-1)t+i\omega_0 t} log(\lambda/q(ie^t))dt.$$

Therefore

$$\int\int A_s = (\frac{|(s-1)|}{\pi})^2 |\int_{-\infty}^{\infty} e^{i\omega_0 t} e^{(c-1)t} log(\lambda/q(ie^t))dt|^2.$$

The second term on the right is the squared magnitude of the Fourier tranform value at frequency ω_0 of the real function

$$e^{(c-1)t} log(\lambda/q(ie^t)).$$

A disk spinning with angular rate ω_0 with pivot point held by a pair of opposing movable bearings, if we move the bearings in a line according to this function (of time), the limiting radius of the circle traced by the initial pivot point will be the magnitude and

2. Theorem.

$$\frac{\pi^3}{|s-1|^2}\int A_s = \text{ area inside final circle.}$$

3. Lie actions.

Whenever A_s is a Lie derivative, meaning $A_s = \delta B$ for some rapidly decreasing B, under the action of a vector-field δ, then A_s can be obtained by multiplying B by a suitable divergence ratio; put differently $A_s = d\ i_\delta B$ which is an exact form; this can only happen if $\zeta(s) = 0$ (still assuming $0 < 1 < c$).

4. The action of $\frac{\partial}{\partial c}$.

A vector field which does not preserve $T \times (0, i\infty)$ is the partial derivative with respect to c. If $g = e^t$ it sends A_s to $2tA_s$.

2. Conjecture. For $0 < c < 1/2$, the partial derivative $\frac{\partial}{\partial c} A_{c+i\omega_0}$ is non-positive.

The conjecture implies that A_s is non-exact, and $\zeta(c + i\omega_0) \neq 0$, for all c in the same range, because it would imply that $\frac{\partial}{\partial c} \int A_s \leq 0$. For each value of ω_0 the dependence on c is a non-increasing real analytic function $(0, 1/2) \to [0, \infty)$. Such a function cannot take the value of zero.

Let's attempt to estimate the partial derivative.

3. Lemma. The function

$$h(r, v) = e^{2(c-1)v} log(\frac{\lambda}{q}(v + r/2)) log(\frac{\lambda}{q}(v - r/2)$$

has the properties that for $0 < c < 1/2$

$$\begin{cases} h(r, v) > 0 & \text{for all } r, v \\ h(r, v) - h(r, -v) < 0 & \text{for all } r \text{ and all } v > 0 \end{cases}$$

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Now we calculate

$$\frac{\partial}{\partial c} \int A_s = \frac{\partial}{\partial c} \frac{(c-1)^2 + \omega_0^2}{\pi^2} \int \int e^{(2c-2)v} cos(r\omega_0) h(r, v) dv dr.$$

Symmetrizing and cancelling a factor of $\frac{1}{2}$ gives

$$= \frac{1}{\pi^2} \int cos(r\omega_0) \int (c-1)(h(r, v) + h(r, -v)) + ((c-1)^2 + \omega_0^2)\ v\ (h(r, v) - h(r, -v)) dv dr$$

Since $c-1$ is negative, the lemma implies that the inner integrand, and in fact each of the two terms separately, take strictly negative values for all r and all v.

We can begin to unwind the definition again; the first term corresponds to the multiple by the negative number $2c-2$ of a cosine transform that we already knew evaluates to the magnitude of a complex number. The final contribution from that term is the negative of the magnitude and must be non-positive.

The coefficient of v is more problematic and will require concepts of estimates.

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John Atwell Moody
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