Intuitive explanation of the Riemann hypothesis

1. Characterisation of the nontrivial zeroes of ζ.

There is a unique (canonical) one-form α on $\mathbb{H}$ invariant under $\Gamma(2)$ with a pole of residue 1 at the image of $i\infty$ and a pole of residue -1 at the image of 1. Under the embedding $\mathbb{H} \to \mathbb{C}$ with τ the coordinate on $\mathbb{C}$ the ratio $[\alpha : i\pi d\tau]$ tends to one at the upper end of the interval $(0, i\infty)$. Let T be the connected real multiplicative group and consider the multiplication actions

$$\begin{cases} \mu_+ : T \times \mathbb{H} \to \mathbb{H} \\ (g, z) \mapsto \sqrt{g} z \end{cases}$$

$$\begin{cases} \mu_- : T \times \mathbb{H} \to \mathbb{H} \\ (g, z) \mapsto \frac{1}{\sqrt{g}} z \end{cases}$$

1. Theorem. For each unitary character ω of T and each real number c with $0 < c < 1$, the differential two-form

$$g^{2-2c}\omega(g)\mu_+^*(\alpha - i\pi d\tau) \wedge \mu_-^*(\alpha - i\pi d\tau)$$

is real and integrable (rapidly decreasing, that is 'Schwartz') on $T \times (0, i\infty)$. Among rapidly decreasing forms, it is exact if and only if $\zeta(c + i\omega_0)$ is zero where ζ is Riemann's zeta function and ω_0 is the real number corresponding to ω under the rule $\omega(g) = g^{i\omega_0}$.

Proof. It is real because the factors besides $\omega(g)$ are anti-symmetric with respect to interchanging μ_+ and μ_- which matches the reversal of orientation of T. The two-form integrates to the squared absolute value of a holomorphic integral, namely $\int g^{1-c}\omega(g)(\alpha - i\pi d\tau)$. In turn, it is easy to calculate the holomorphic *definite* integral; it is $-L(s, \chi)\Gamma(s)\pi^{1-s}$ L is the L series for sums of four squares, χ is the Dirichlet sign character and $s = c + i\omega_0$. The rule $\omega(g_1 g_2^{-1}) = \omega(g_1)\omega(g_2)^{-1}$ is all that is needed.

2. Remark about the dynamical interpretation.

Here is an intuitive way of integrating the two-form let, us call it A_s for $s = c + i\omega_0$. Let $\tau = ie^t$. By 'integration by parts'

$$\int e^{(c-1)t+i\omega_0 t} d\ log(\frac{\lambda}{q}) = -(c-1+i\omega_0) \int e^{(c-1)t+i\omega_0 t} log(\frac{\lambda}{q}(ie^t))dt.$$

Therefore

$$\int\int A_s = |(s-1)|\ ^2|\int_{-\infty}^{\infty} e^{i\omega_0 t} e^{(c-1)t} log(\frac{\lambda}{q}(ie^t))dt|^2.$$

The second term on the right is the squared magnitude of the Fourier tranform value at frequency ω_0 of the real function

$$e^{(c-1)t} log(\frac{\lambda}{q}(ie^t)).$$

A disk spinning with angular rate ω_0 with pivot point held by a pair of opposing movable bearings, if we move the bearings in a line according to this function (of time), the limiting radius of the circle traced by the initial pivot point will be the magnitude and

2. Theorem.

$$\frac{\pi}{|s-1|^2} \int A_s = \text{ area inside final circle.}$$

3. Lie actions.

Whenever A_s is a Lie derivative, meaning $A_s = \delta B$ for some rapidly decreasing B, under the action of a vector-field δ, then A_s can be obtained by multiplying B by a suitable divergence ratio; put differently $A_s = d\ i_\delta B$ which is an exact form; this can only happen if $\zeta(s) = 0$ (still assuming $0 < 1 < c$).

4. The action of $\frac{\partial}{\partial c}$.

A vector field which does not preserve $T \times (0, i\infty)$ is the partial derivative with respect to c. If $g = e^t$ it sends A_s to $2tA_s$.

2. Question. For $0 < c < 1/2$, is the partial derivative $\frac{\partial}{\partial c} \int A_{c+i\omega_0}$ non-positive?

An affirmative answer would imply that A_s is non-exact, and $\zeta(c + i\omega_0) \neq 0$, for all c in the same range. The reason is that for each value of ω_0 the dependence on c would be a non-increasing real analytic function $(0, 1/2) \to [0, \infty)$. Such a function cannot take the value of zero.

Let's attempt to estimate the partial derivative to see if we can start to answer the question. Let

$$h(r, v) = e^{2(c-1)v} log(\frac{\lambda}{q}(v + r/2)) log(\frac{\lambda}{q}(v - r/2)$$

This has the properties that for $0 < c < 1/2$

$$\begin{cases} h(r, v) > 0 & \text{for all } r, v \\ h(r, v) - h(r, -v) < 0 & \text{for all } r \text{ and all } v > 0 \end{cases}$$

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For each fixed c and r let

$$\gamma(c, r) = \frac{\int v h(r, v) dv}{\int h(r, v) dv}$$

This is the mean value of $h(r, v)$ as a function of v.

Now

$$\frac{\partial}{\partial c}\int A_s = \frac{\partial}{\partial c}\ ((c-1)^2+\omega_0^2)\int\int cos(r\omega_0)h(r,v)dvdr.$$

$$= (2c-2)\int cos(r\omega_0)\int h(r,v) + ((c-1)^2+\omega_0^2)(2v)h(r,v)dvdr$$

$$= (2c-2)\int cos(r\omega_0)\int h(r,v)dvdr$$

$$+((c-1)^2+\omega_0^2)\int 2\gamma(c,r)cos(r\omega_0)\int h(r,v)dvdr$$

The integral $\int\int cos(r\omega_0)h(r,v)dvdr$ is semi-positive since it is the squared magnitude of a complex number. Each of the coefficients $2c-2$ and $\gamma(c,r)$ are negative when $0 < c < 1/2$.

Let

$$\rho(c+i\omega_0) = \frac{\int \gamma(c,r)cos(\omega_0 r)\int h(r,v)dvdr}{\int cos(\omega_0 r)\int h(r,v)dvdr}$$

so our integral is

$$= (2c-2) + 2((c-1)^2+\omega_0^2)\rho(s))\int cos(r\omega_0)\int h(r,v)dvdr$$

$$= 2(\frac{c-1}{(c-1)^2+\omega_0^2} + \rho(s)\ )\int A_s.$$

Removing the leading factor of -1 in $-\ L(s,\chi)\Gamma(s)\pi^{-s}$ which has no effect, and removing our leading factor of 2 which relates the real part of the logarithmic derivative with our integral, we obtain

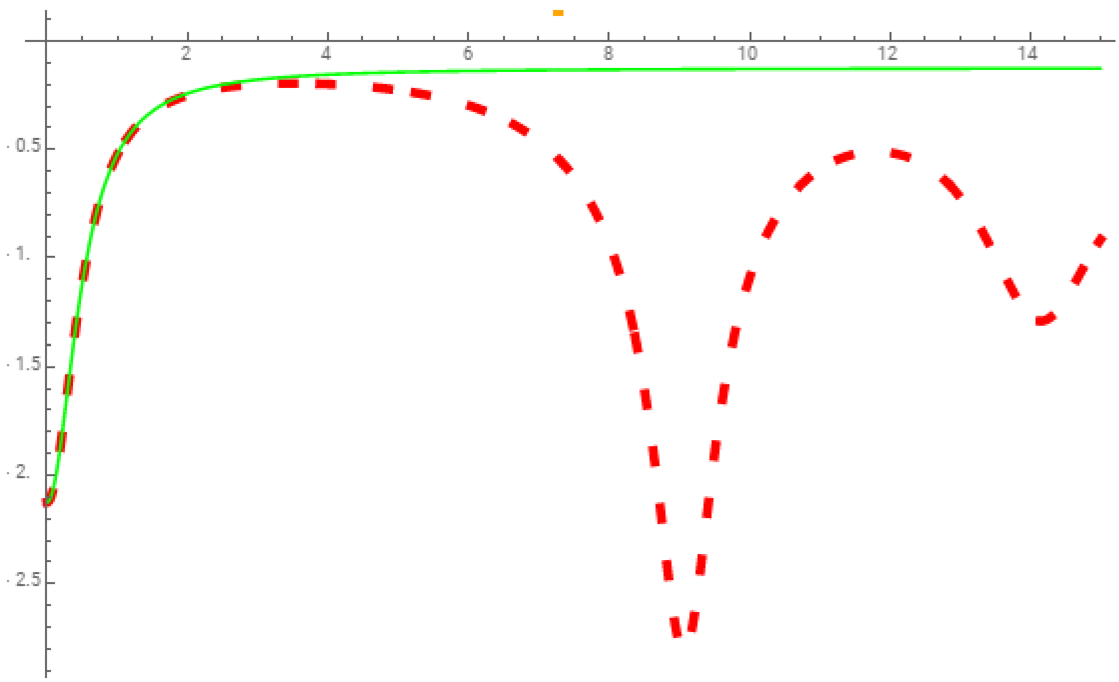
$$Re\ \frac{d}{ds}\ log\ (L(s,\chi)\Gamma(s)\pi^{-s}) = \frac{Re(s-1)}{|s-1|^2} + \rho(s).$$

The logarithmic derivative of the gamma function is the digamma function $\Psi(s)$. With this notation then

$$Re\ \frac{d}{ds}\ log\ L(s,\chi) = \frac{Re(s-1)}{|s-1|^2} + \rho(s) + log(\pi) - Re\ \Psi(s).$$

From this we can write an expression for the real part of the logarithmic derivative of $L(s)$ itself and $\zeta(s)\zeta(s-1)$.

When $c = 1/2$ a first approximation of $\rho(s)$ would just be the constant $log(\frac{log(16)}{\pi})$. Here the real part of the logarithmic derivative of $L(s,\chi)\Gamma(s)\pi^{-s}$ as a red graph, and the $\frac{Re(s-1)}{|s-1|^2} + log(\frac{log(16)}{\pi})$ as a green graph as a function of ω_0 when $c = 1/2$.



This graph is not much evidence as we don't know why the actual value departs from the approximation.

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John Atwell Moody
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