ON Q-LAPLACE TRANSFORMS AND MITTAG-LEFFLER TYPE FUNCTIONS

KISHAN SHARMA

ABSTRACT. n the present paper, the author derived the results based on q-Laplace transform of the K-Function introduced by Sharma[7]. Some special cases of interest are also discussed.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 44A10, 44A20.

KEYWORDS AND PHRASES.q-Laplace transforms, M-series, Generalized M-series, Mittag-Leffler function, Generalized Mittag-Leffler function, K-Function.

1. INTRODUCTION

The q-Laplace transform was defined by Hahn[18] as

$$L(f(t); s) = \int_0^\infty e^{-st} f(t)dt \tag{1.1}$$

by means of the two q-integrals given below

$${}_qL_s(f(t)) = \frac{1}{(1-q)} \int_0^{s^{-1}} E_q(qst) f(t) d(t; q), \tag{1.2}$$

and

$${}_qL_s(f(t)) = \frac{1}{(1-q)} \int_0^\infty e_q(-st) f(t) d(t; q); Re(s) > 0. \tag{1.3}$$

The q-integrals(see Gasper and Rahman[4]) of a function is defined by

$$\int_0^t f(x)d(x; q) = t(1-q) \sum_{k=0}^\infty q^k f(tq^k), \tag{1.4}$$

* drkishansharma2006@rediffmail.com.

$$\int_{t}^{\infty} f(x)d(x;q) = t(1-q)\sum_{k=1}^{\infty} q^{-k} f(tq^{-k}), \tag{1.5}$$

$$\int_{0}^{\infty} f(x)d(x;q) = (1-q)\sum_{k=-\infty}^{\infty} q^{k} f(q^{k}). \tag{1.6}$$

In the theory of q-calculus [5], for $0 < |q| < 1$, the q-shifted factorial or q-analogue of Pochhammer symbol is defined by

$$(a;q)_k = \begin{cases} \prod_{j=0}^{k-1}(1-aq^j), & \text{if} \quad k>0 \\ 1, & \text{k=0} \\ \prod_{j=0}^{\infty}(1-aq^j), & \text{if} \quad k\to\infty \end{cases}$$

Or equivalently

$$(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}, k \in \mathbb{N} \tag{1.7}$$

and for any complex number α,

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \tag{1.8}$$

where the principal value of q^α is taken.

$$(q^\alpha;q)_n = \frac{(q^\alpha;q)_\infty}{(q^{\alpha+n};q)_\infty}$$

The q-analogue of the power function is defined as

$$\text{(a-b)}_\alpha = a^\alpha (b/a;q)_\alpha$$

$$= a^\alpha \prod_{j=0}^{\infty} \frac{1-(b/a)q^j}{1-(b/a)q^{j+\alpha}} = a^\alpha \frac{(b/a;q)_\infty}{(q^\alpha b/a;q)_\infty}, a \neq 0. \tag{1.9}$$

The q-gamma function is defined as

$$\Gamma_q(\alpha)=\frac{G(q^{\alpha})}{G(q)}(1-q)^{1-\alpha}=(1-q)_{\alpha-1}(1-q)^{1-\alpha},\alpha\in \text{R/0,-1,-2,...} \tag{1.10}$$

where

$$G(q^{\alpha})=\frac{1}{(q^{\alpha};q)_{\infty}}. \tag{1.11}$$

$$\Gamma_q(\alpha)=\frac{(q;q)_{\infty}}{(q^{\alpha};q)_{\infty}}(1-q)^{1-\alpha}$$

where

$$\alpha\neq 0,-1,-2,...$$

The q-binomial series [5] is given by

$${}_1\phi_0[-;-;q,x]=\frac{1}{(x;q)_{\infty}} \tag{1.12}$$

$${}_1\phi_0[\alpha;-;q,x]=\sum_{n=0}^{\infty}\frac{(\alpha;q)_n}{(q;q)_n}x^n=\frac{(\alpha x;q)_{\infty}}{(x;q)_{\infty}}. \tag{1.13}$$

The q-Binomial coefficiens are given by [4] as

$$C_q(n,k)=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{1.14}$$

The K-function was introduced by Sharma[7] as

$${}_AK_B^{\alpha,\beta;\gamma}(x)$$

$$=\sum_{r=0}^{\infty}\frac{\prod_{i=0}^{A}(a_i)_r(\gamma)_r x^r}{\prod_{j=0}^{B}(b_j)_r r!\Gamma(\alpha r+\beta)} \tag{1.15}$$

where

$$\alpha, \beta, \gamma \in \mathbb{C}; Re(\alpha), Re(\beta), Re(\gamma) > 0.$$

Further details of this function are given in [7].

2. Relationship of the K-function with another special functions

The function defined by (1.15) related with the following special functions as: (i) If we set $\gamma = 1$, we get

$$_AK_B^{\alpha,\beta;\gamma}(x) =_A M_B^{\alpha,\beta}(x) \tag{2.1}$$

where $_AM_B^{\alpha,\beta}(x)$ is the generalized M-series introduced by Sharma and Jain[8].

(ii) If we take $\beta = \gamma = 1$, we arrive at

$$_AK_B^{\alpha,1;1}(x) =_A M_B^{\alpha,1}(x) =_A M_B^{\alpha}(x) \tag{2.2}$$

where $_AM_B^{\alpha}(x)$ is the M-series given by Sharma[9].
(iii) If we put $A = B = 0$, we arrive at

$$_0K_0^{\alpha,\beta;\gamma}(x) = E_{\alpha,\beta}^{\gamma}(x) \tag{2.3}$$

where $E_{\alpha,\beta}^{\gamma}(x)$ is the generalized Mittag-Leffler function given by Prabhakar[15].
(iv) If we put $A = B = 0, \gamma = 1$, we arrive at

$$_0K_0^{\alpha,\beta;1}(x) = E_{\alpha,\beta;1}^{\gamma}(x) = E_{\alpha,\beta}^{\gamma}(x) \tag{2.4}$$

where is the generalized Mittag-Leffler function given by Wiman[1].
(v) If we put $A = B = 0, \beta = \gamma = 1$, we arrive at

$$_0K_0^{\alpha,1;1}(x) = E_{\alpha,1}^{1}(x) = E_{\alpha,1}(x) = E_{\alpha}(x) \tag{2.5}$$

where $E_{\alpha}(x)$ is the Mittag-Leffler function given by Mittag-Leffler[5,6].

(vi) If we put $A = B = 0, \alpha = \beta = \gamma = 1$, we arrive at

$$_0K_0^{1,1;1}(x) = E_{1,1}^{1}(x) = E_{1,1}(x) = E_1(x) = e^x \tag{2.6}$$

where e^x is the exponential function given by [2].

3. Main Results

In this section we investigate the q-Laplace transform of the K-function.

Theorem 3.1. *Let* $\alpha, \beta, \gamma \in \mathbb{C}; Re(\alpha), Re(\beta), Re(\gamma) > 0,$ *then*

$$ {}_qL_s(x^{\lambda}{}_AK_B^{\alpha,\beta;\gamma}(x)) $$

$$ = \frac{(q;q)_\infty}{s^{1+\lambda}} \times \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{A}(a_i)_r(\gamma)_r}{\prod_{j=0}^{B}(b_j)_r r!\Gamma(\alpha r+\beta)s^r(q^{\lambda+r};q)_\infty} \tag{3.1} $$

Proof:
With the help of (1.4), equation(1.2) can be written as

$$ {}_qL_s(f(t)) = \frac{(q;q)_\infty}{s} \times \sum_{k=0}^{\infty} \frac{q^k f(s^{-1}q^k)}{(q;q)_k} \tag{3.2} $$

Taking $f(x) = x^{\lambda}{}_AK_B^{\alpha,\beta;\gamma}(x)$ in the above equation and making the use of definition(1.15), we get

$$ {}_qL_s(x^{\lambda}{}_AK_B^{\alpha,\beta;\gamma}(x)) $$

$$ = \frac{(q;q)_\infty}{s^{1+\lambda}} \times \sum_{j=0}^{\infty} \frac{q^{\lambda_j}}{(q;q)_j} \times \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{A}(a_i)_r}{\prod_{j=0}^{B}(b_j)_r} \times \frac{(\gamma)_r}{r!\Gamma(\alpha r+\beta)}(s^{-1}q^j)^r \tag{3.3} $$

On interchanging the order of summations and then summing the resulting with the help of (1.12), the right hand side of (3.3) converted to

$$ \frac{(q;q)_\infty}{s^{1+\lambda}} \times \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{A}(a_i)_r}{\prod_{j=0}^{B}(b_j)_r} \times \frac{(\gamma)_r}{r!\Gamma(\alpha r+\beta)s^r(q^{\lambda+r};q)_\infty} \tag{3.4} $$

which is the desired result.

4. SPECIAL CASES

Theorem(3.1) leads to the q-Laplace transform of generalized M-series[8], M-series[9], generalized Mittag-Leffler functions[1,15], Mittag-Leffler function[5,6] and exponential function[2] after implementing the necessary changes in the values of A, B, α, β and γ as mentioned in the section 2.

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Acknowledgement

The author is highly grateful to the learned and renowned referee for making valuable suggestions and comments which led to a better presentation of the manuscript.

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Kishan Sharma

Department of Applied Mathematics,

Amity University Madhya Pradesh, Gwalior.

Address: B-3, Krishna Puri, Taraganj, Lashkar, Gwalior(M.P.)-474001,INDIA

e-mail: drkishansharma1@gmail.com,drkishansharma2006@rediffmail.com