The Golden Ratio in Atomic Theory

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The Golden ratio defined as, $\phi = [1 + (5)^{1/2}]/2 = 1.618$, is an amazing number that is found to govern many natural phenomena[1]. It is shown here for the first time in the context of atomic theory that ϕ is the ratio of the energy of the proton to that of the electron in the hydrogen atom.

An atom of hydrogen consists of an electron and a proton of electric charges –e and +e respectively. The masses m_e and m_p, the magnetic momenta μ_e and μ_p and the angular momenta of the electron and proton are related as per Larmor's relations, as ponited out elsewhere[2].

In the model of atomic hydrogen suggested by Bohr, the distance between the electron and proton (the Bohr radius), a_B, is given by[3]

$$a_B = e^2/(2kW_H) = e/(2kI_H) \tag{1}$$

where $k = 4\pi\varepsilon_0$ is the dielectric constant, ε_0 is the permittivity of vacuum, $W_H = eI_H$ is the total energy of the atom and I_H is the ionization potential. On using the standard values of e, ε_0 and I_H, one finds that $a_B = 0.052946$ nm. (Note: $e^2/k = \alpha ch/2\pi$, where α, c and h are fine structure constant, velocity of electromagnetic radiation in vacuum and Planck's constant respectively.)

Bohr's model was subsequently modified by Sommerfeld, who assumed that the electron follows an elliptical path around the proton located at one of the foci (as in Keplerian motion of planets)[3]. In the Sommerfeld ellipse, the Bohr radius given by equation (1) is the length of the major axis, and the distance between the electron and proton varies periodically. However, on comparing the above value of a_B with the atomic radius[4], $a_B > R_{cov}$ (= 0.037 nm), the covalent radius, which is taken as half the interatomic distance in the molecule, H_2. Therefore, the significance of the Bohr radius was re-examined in the light of equation (1) and the results are presented here.

The total energy of the hydrogen atom given by equation (1) can be written as,

$$W_H = (1/2)e^2/(ka_B) = (1/2)eV_H = (1/2)e(V_p + V_e) = W_p + W_e \qquad (2)$$

where V_H (= e/ka_B) is the sum of the electrostatic potentials V_p and V_e of the proton and electron respectively, $W_p = (1/2)eV_p$ and $W_e = (1/2)eV_e$ are the corresponding energies and a_B (= e/kV_H), the major axis of the Sommerfeld ellipse is equal to the sum of the distances $d_e + d_p$ of the electron and proton, located at the foci F and F' respectively, to any point P on the ellipse. The locus of the point P is thus the surface S of an ellipsoid at potential V_H. The distance between the electron and proton at the two foci is $d_H = ea_B$, where e is the eccentricity of the ellipse. (Note the difference: the electron was assumed to follow an elliptical orbit by Sommerfeld.)

An estimate of the eccentricity of the ellipse using the recent[5] value, 0.0327 nm, for d_H gives $e = 1/\phi$, as for the Golden ellipse.

Since $V_H = e/ka_B = (V_p + V_e)$ at any point P on the surface S, where $V_e = -e/kd_e$ and $V_p = e/kd_p$, one obtains the relation,

$$1/a_B = (1/d_p) - (1/d_e) \qquad (3)$$

On expressing equation (3) in terms of the ratio of distances, (d_e/d_p) and noting that $d_e + d_p = a_B$, one arrives at the equation,

$$\phi^2 - \phi - 1 = 0; \ \phi = (d_e/d_p) = (-V_p/V_e) = (-W_p/W_e) \qquad (4a,b)$$

The positive root of equation (4a) is the Golden ratio, ϕ. Thus d_e and d_p are Golden sections of a_B, as shown below,

$$d_e = a_B/\phi = 0.618a_B \text{ and } d_p = a_B/\phi^2 = 0.382a_B \qquad (5)$$

As a_B, d_e and d_p have fixed values, the point P_ϕ corresponding to the Golden ratio must lie on the circumference of a circle which is a cross section (perpendicular to the major axis) of the ellipsoid S. Since $e = 1/\phi$ and $d_e = d_H$, $FF'P_\phi$ is a Golden isosceles triangle.

The spectral term values for hydrogen[3], corresponding to energies $W_{H,n} = W_H/n^2$, where n is the principal quantum number, pertain to potentials $V_{H,n} = V_H/n^2$ on ellipsoids with major axes equal to $a_{H,n} = n^2a_B = n^2(d_e + d_p)$ and the electron and proton at the foci separated by $d_H = ea_B$ (which is independent of n). For these states, the ratio of the distances n^2d_e and n^2d_p from the electron and proton to any point $P_{H,n}$ on the circle at potential $V_{H,n}$, remains constant at ϕ.

References

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