VORTEX EQUATION IN HOLOMORPHIC LINE BUNDLE OVER NON-COMPACT GAUDUCHON MANIFOLD

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Abstract. In this paper, by the method of heat flow and the method of exhaustion, we prove an existence theorem of Hermitian-Yang-Mills-Higgs metrics on holomorphic line bundle over a class of non-compact Gauduchon manifold.

1. Introduction

Let X be a complex manifold of dimension n and g a Hermitian metric with associated Kähler form ω. g is called Kähler if $\mathrm{d}\omega = 0$; balanced if $\mathrm{d}\omega^{n-1} = 0$; Gauduchon if $\partial\bar{\partial}\omega^{n-1} = 0$. A Hermitian-Yang-Mills-Higgs metric is a Hermitian metric on holomorphic vector bundle E over X satisfying the following vortex equation which was introduced by Bradlow [3]:

$$2\sqrt{-1}\Lambda_\omega F_H + \phi \otimes \phi^{*H} - \tau \cdot \mathrm{Id}_E = 0, \tag{1.1}$$

where Λ_ω is the contraction operator with respect to ω, F_H is the curvature of the Chern connection with respect to the metric H, ϕ is a non-trivial holomorphic section of E, and τ is a real number. The vortex equation (1.1) is a generalization of the Hermitian-Yang-Mills equation (i.e. $\phi = 0$). By the classical Donaldson-Uhlenbeck-Yau theorem [6, 23], the existence of the solution to the Hermitian-Yang-Mills equation relates to the stability of the underlying bundle over compact Kähler manifold. The classical Donaldson-Uhlenbeck-Yau theorem has many interesting and important generalizations [1, 2, 4, 5, 8, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 25, 26].

Let L be a holomorphic line bundle over complex manifold (X, g), given any initial metric H_0, let $H = e^f H_0$, then solving the vortex equation (1.1) is equivalent to solving the following equation:

$$\widetilde{\Delta} f - |\phi|^2_{H_0} e^f - (2\sqrt{-1}\Lambda_\omega F_{H_0} - \tau \cdot \mathrm{Id}_L) = 0, \tag{1.2}$$

where $\widetilde{\Delta}$ is the complex Laplace operator, acting on functions given by $\widetilde{\Delta} f = -2\sqrt{-1}\Lambda_\omega \bar{\partial}\partial f$. As usual, we denote the Beltrami-Laplcaian operator by Δ. It is well known that the difference of the two Laplacians is given by a first order differential operator as follows

$$(\widetilde{\Delta} - \Delta) f = \langle V_g, \mathrm{d}f\rangle_g, \tag{1.3}$$

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where V_g is the unique 1-form satisfying $\mathrm{d}\omega^{n-1} = \omega^{n-1} \wedge V_g$. Clearly, $\widetilde{\Delta} f$ and Δf coincide provided X is balanced. Once we set $h = -|\phi|^2_{H_0}$ and $k = 2\sqrt{-1}\Lambda_\omega F_{H_0} - \tau \cdot \mathrm{Id}_L$, (1.2) is just

$$\widetilde{\Delta} f + he^f - k = 0. \tag{1.4}$$

When (X, g) is a Kähler manifold, of course X is balanced in this setting, then Eq. (1.4) turns out to be precisely of the form considered in [9, 10]:

$$\Delta f + he^f - k = 0, \tag{1.5}$$

which is known as Kazdan-Warner equation. In [15], Liu and Yao solved Eq. (1.4) on compact Gauduchon manifolds by means of the upper and lower solution method. Then the solvability of the vortex equation (1.1) in holomorphic line bundle over compact Gauduchon manifolds stands out. Later, Wang and Zhang discussed Eq. (1.5) on a class of non-compact Riemannian manifold by the method of heat flow [24]. As an application, they solved the vortex equation (1.1) in holomorphic line bundle over a class of non-compact Kähler manifold. Then, one may well ask: can we solve the vortex equation (1.1) in holomorphic line bundle over the non-compact Gauduchon manifold? Before we state the answer of this question, we need some requirements.

In the following, we always suppose that (X, g) is a Gauduchon manifold unless otherwise stated. Following [21], we will make the following three assumptions:

Assumption 1. (X, g) has finite volume.

Assumption 2. There exists a non-negative exhaustion function ψ with $\sqrt{-1}\Lambda_\omega \partial\bar{\partial}\psi$ bounded.

Assumption 3. There is an increasing function $a : [0, +\infty) \to [0, +\infty)$ with $a(0) = 0$ and $a(x) = x$ for $x > 1$, such that if f is a bounded positive function on X with $\sqrt{-1}\Lambda_\omega \partial\bar{\partial} f \geq -B$ then

$$\sup_X |f| \leq C(B) a\Big(\int_X |f| \frac{\omega^n}{n!}\Big).$$

Furthermore, if $\sqrt{-1}\Lambda_\omega \partial\bar{\partial} f \geq 0$, then $\sqrt{-1}\Lambda_\omega \partial\bar{\partial} f = 0$.

We fix a background metric H_0 in the bundle L over X, and suppose that

$$\sup_X |\Lambda_\omega F_{H_0}|_{H_0} < +\infty.$$

Following [21], define the analytic degree of L to be the real number

$$\deg_\omega(L, H_0) = \sqrt{-1} \int_X \Lambda_\omega F_{H_0} \frac{\omega^n}{n!}.$$

In this paper, we prove the following theorem:

Theorem 1.1. *Let (X, g) be a non-compact Gauduchon manifold satisfying Assumptions 1,2,3 and $|\mathrm{d}\omega^{n-1}|_g \in L^2(X)$. Let L be a holomorphic line bundle with a nontrivial holomorphic section ϕ on X. Suppose that there exists a Hermitian metric H_0 satisfying that $\sup_X |\Lambda_\omega F_{H_0}| < +\infty$, $\sup_X |\phi|^2_{H_0} < +\infty$ and $\deg_\omega(L, H_0) < \frac{\tau}{2}\mathrm{Vol}(X)$. Then there exists a unique Hermitian metric H satisfying the vortex equation (1.1).*

To prove Theorem 1.1, we only need to solve (1.4) on the non-compact Gauduchon manifold. We will use the method of heat flow and the method of exhaustion to solve (1.4). We can not directly apply Wang and Zhang's approach in [24] since the complex Laplace makes a huge difference (1.3). By considering the heat flow satisfying the Dirichlet boundary condition, we solve the perturbed equation $\widetilde{\Delta} f - \varepsilon f + he^f - k = 0$ with $\varepsilon \geq 0$ on any exhaustion subset X_φ of X. Noting that the C^0-bound of the solution f_φ on each X_φ depends only on ε^{-1} and the initial date. One can pass to limit and eventually obtain a solution on the whole manifold X provided $\varepsilon > 0$. At last, we complete the proof by showing that the C^0-bound of the solution of the perturbed equation on X is independent of ε. The approach used in this paper is more natural and it can be used to deal with the equation (1.5). This approach arises from the study of the Hermitian-Einstein equation discussed in [25], in which the Hermitian-Einstein equation is different from the vortex equation (1.1).

2. Kazdan-Warner type equation on the non-compact manifold

The aim of this section is to solve Eq. (1.4) on the non-compact manifold. We first solve the Dirichlet problem for Eq. (1.4) on a compact Hermitian manifold with non-empty boundary. To be specific, we prove the following theorem.

Proposition 2.1. *Let (M, g) be a compact Hermitian manifold with non-empty boundary ∂M. Suppose that $h, k \in C^\infty(M)$, then for any function $\widetilde{f}$ on the restriction to ∂M, there is a unique function $f \in C^\infty(M)$ which satisfies the equation $\widetilde{\Delta} f = \varepsilon f - he^f + k$ for $\varepsilon \geq 0$ and $f = \widetilde{f}$ on ∂M.*

Proof. We consider the following heat flow with Dirichlet boundary condition:

$$\begin{cases} \frac{\partial f}{\partial t} = \widetilde{\Delta} f - \varepsilon f + he^f - k, \quad \varepsilon \geq 0, \\ f(0) = 0, \\ f|_{\partial M} = \widetilde{f}. \end{cases}$$

This is a parabolic equation, so we have a short-time solution. Suppose that the solution $f(\cdot, t)$ exists for $[0, T)$. Direct calculation shows that

$$\begin{aligned} (\widetilde{\Delta} - \frac{\partial}{\partial t})(\widetilde{\Delta} f - \varepsilon f + he^f - k)^2 &= 2|\mathrm{d}(\widetilde{\Delta} f - \varepsilon f + he^f - k)|^2 \\ &\quad - 2he^f(\widetilde{\Delta} f - \varepsilon f + he^f - k)^2 \\ &\quad + 2\varepsilon(\widetilde{\Delta} f - \varepsilon f + he^f - k)^2 \\ &\geq 0. \end{aligned} \tag{2.1}$$

On the other hand, $(\widetilde{\Delta} f - \varepsilon f + he^f - k)^2|_{\partial M} = 0$. By the maximum principle, we have

$$\max_M |\widetilde{\Delta} f - \varepsilon f + he^f - k| \leq \max_M (|h| + |k|) < +\infty. \tag{2.2}$$

From Eq. (2.2), we know that $|\frac{\partial f}{\partial t}|$ is bounded uniformly in t. Then it is easy to conclude that $f(\cdot, t)$ converge in C^0 to a continuous function $f(T)$ as $t \to T$. On the other hand, from Eq. (2.2), we have

$$\sup_M |\widetilde{\Delta} f|(\cdot, t) \leq C_1, \tag{2.3}$$

where C_1 is a constant depending only on ε, T and $\max_M(|h| + |k|)$. Elliptic estimates with boundary condition show that $f(\cdot, t)$ is bounded in C^1 and also bounded in L_2^p (for any $1 < p < +\infty$) uniformly in $[0, T)$. We can apply Hamilton's method ([7]) to deduce that $f(\cdot, t) \to f(\cdot, T)$ in C^∞, and the solution can be continued past T. That is, the flow has a solution defined for all time. Let f_1 and f_2 be two solutions of the flow with the boundary condition, one can easily obtain

$$(\widetilde{\Delta} - \frac{\partial}{\partial t})(f_1 - f_2)^2 \geq 0.$$

Then by $(f_1 - f_2)^2|_{\partial M} = 0$ and the maximum principle, we have the uniqueness of the long-time solution.

From Eq. (2.1), we have

$$(\widetilde{\Delta} - \frac{\partial}{\partial t})|\widetilde{\Delta} f - \varepsilon f + he^f - k| \geq 0. \tag{2.4}$$

By [22, Chapter 5, Proposition 1.8], one can solve the following Dirichlet problem on M:

$$\begin{cases} \widetilde{\Delta} v = -|h - k|, \\ v|_{\partial M} = 0. \end{cases}$$

Set

$$w(x, t) = \int_0^t |\widetilde{\Delta} f - \varepsilon f + he^f - k|(x, \rho)\mathrm{d}\rho - v(x).$$

From the boundary condition satisfied by f implies that, for $t > 0$, $|\widetilde{\Delta} f - \varepsilon f + he^f - k|(x, t)$ vanishes on the boundary of M. Then, combining (2.4), it is easy to check that $w(x, t)$ satisfies

$$(\widetilde{\triangle} - \frac{\partial}{\partial t})w(x, t) \geq 0, \quad w(x, 0) = -v(x), \quad w(x, t)|_{\partial M} = 0.$$

By the maximum principle, we have

$$\int_0^t |\widetilde{\Delta} f - \varepsilon f + he^f - k|(x, \rho)\mathrm{d}\rho \leq \sup_{y \in M} v(y), \tag{2.5}$$

for any $x \in M$, and $0 < t < +\infty$. Let $t_1 \leq t$, then

$$|f(t) - f(t_1)| \leq \int_{t_1}^t |\frac{\partial f(\rho)}{\partial \rho}|\mathrm{d}\rho = \int_{t_1}^t |\widetilde{\Delta} f - \varepsilon f + he^f - k|(x, \rho)\mathrm{d}\rho \to 0 \quad \text{as} \quad t_1 \to +\infty,$$

which means $f(t)$ converge in the C^0 topology to some continuous function f_∞. Then, the standard elliptic theory implies that there exists a subsequence $f(t) \to f_\infty$ in C^∞ topology. From Eq. (2.5), we know that f_∞ is the desired function satisfying the boundary condition. Let f_1 and f_2 be two solutions of the elliptic equation satisfying the boundary condition. One can check that $\widetilde{\Delta}(f_1 - f_2)^2 \geq 0$, then by the maximum principle we prove the uniqueness. □

Let (X, g) be a non-compact Gauduchon manifold with finite volume and a non-negative exhaustion function ψ. Fix a number φ, let X_φ denote the compact space $\{x \in X | \psi(x) \leq \varphi\}$, with smooth boundary ∂X_φ. By Proposition 2.1, we know that the following Dirichlet problem is solvable on X_φ, i.e.

$$\begin{cases} \widetilde{\Delta} f_\varphi - \varepsilon f_\varphi + h e^{f_\varphi} - k = 0, \quad \forall x \in X_\varphi, \\ f_\varphi(x)|_{\partial X_\varphi} = 0. \end{cases}$$

By simple calculations, we have

$$\begin{aligned} \widetilde{\Delta} |f_\varphi|^2 &\geq 2\varepsilon |f_\varphi|^2 - 2|k||f_\varphi| - 2h f_\varphi e^{f_\varphi} \\ &\geq 2\varepsilon |f_\varphi|^2 - 2|k||f_\varphi| - 2h f_\varphi \\ &\geq 2|f_\varphi|\big(\varepsilon |f_\varphi| - (|k| + |h|)\big). \end{aligned}$$

We assume that $\varepsilon > 0$, the maximum principle implies:

$$\max_{X_\varphi} |f_\varphi| \leq \frac{1}{\varepsilon} \max_{X_\varphi} (|k| + |h|).$$

By $\partial\bar{\partial}\omega^{n-1} = 0$ and $f_\varphi|_{\partial X_\varphi} = 0$, we have

$$\begin{aligned} \int_{X_\varphi} |\mathrm{d} f_\varphi|^2 \frac{\omega^n}{n!} &= - \int_{X_\varphi} f_\varphi \tilde{\Delta} f_\varphi \frac{\omega^n}{n!} \\ &\leq - \int_{X_\varphi} \big(\varepsilon f_\varphi^2 + (k - h) f_\varphi\big) \frac{\omega^n}{n!} \\ &\leq \frac{1}{\varepsilon} \max_{X_\varphi} (|k| + |h|)^2 \mathrm{Vol}(X_\varphi). \end{aligned}$$

Then, by using the standard elliptic estimates, we can prove that, by choosing a subsequence, f_φ converge in C^∞_{loc}-topology to a solution on whole X, i.e. we prove the following proposition.

Proposition 2.2. *Let (X, g) be a non-compact Gauduchon manifold with finite volume and a non-negative exhaustion function ψ. Suppose that $h, k \in C^\infty(X)$ and $\sup_X (|h| + |k|) < +\infty$. For any $\varepsilon > 0$, there is a function $f \in C^\infty(X)$ which satisfies the equation*

$$\widetilde{\Delta} f = \varepsilon f - h e^f + k \tag{2.6}$$

with

$$\sup_X |f| \leq \frac{1}{\varepsilon} \sup_X (|h| + |k|) \tag{2.7}$$

and

$$\int_X |\mathrm{d} f|^2 \frac{\omega^n}{n!} \leq \frac{1}{\varepsilon} \Big(\sup_X (|h| + |k|) \Big)^2 \mathrm{Vol}(X). \tag{2.8}$$

Now we are ready to solve the Kazdan-Warner type equation on the non-compact Gauduchon manifold.

Theorem 2.3. *Let (X,g) be a non-compact Gauduchon manifold satisfying Assumptions 1,2,3 and $|\mathrm{d}\omega^{n-1}|_g \in L^2(X)$. Suppose that $h,k \in C^\infty(X)$ and $\sup\limits_X(|h|+|k|) < +\infty$. If $h \le 0$, h is not identically zero and $\int_X k < 0$, then there is a function $f \in C^\infty(X)$ which satisfies* (1.4) *with* $\sup_X |f| < +\infty$.

Proof. From Proposition 2.2, for any $\varepsilon > 0$, we have a solution f of the equation (2.6) and f satisfies (2.7). By direct calculations, we have

$$\begin{aligned}\widetilde{\Delta}\log(e^f+e^{-f}) &= \frac{e^f-e^{-f}}{e^f+e^{-f}}\widetilde{\Delta}f + \frac{4}{(e^f+e^{-f})^2}|\mathrm{d}f|^2\\ &\ge \frac{e^f-e^{-f}}{e^f+e^{-f}}(\widetilde{\Delta}f - \varepsilon f + he^f - k) + \frac{e^f-e^{-f}}{e^f+e^{-f}}\varepsilon f\\ &\quad - \frac{e^f-e^{-f}}{e^f+e^{-f}}h(e^f-1) + \frac{e^f-e^{-f}}{e^f+e^{-f}}(k-h)\\ &\ge -|\widetilde{\Delta}f - \varepsilon f + he^f - k| - (|h|+|k|)\\ &\ge -\sup_X(|h|+|k|).\end{aligned}$$

On the other hand, the following is well-known:

$$|f| \le \log(e^f+e^{-f}) \le |f| + \log 2.$$

Then by Assumption 3, we have

$$\sup_X |f_\varepsilon| \le \sup_X \log(e^{f_\varepsilon}+e^{-f_\varepsilon}) \le \widetilde{C}_1 \int_X |f_\varepsilon|^2 + \widetilde{C}_2, \tag{2.9}$$

where constants $\widetilde{C}_1$ and $\widetilde{C}_2$ depend only on $\sup_X(|h|+|k|)$ and $\mathrm{Vol}(X)$.

In the following, we will use a contradiction argument to prove that $\|f_\varepsilon\|_{C^0}$ is uniform bounded. If $\|f_\varepsilon\|_{C^0}$ is unbounded, then there exists a subsequence $\varepsilon_i \to 0$, $i \to +\infty$, such that $\iota_i := \|f_{\varepsilon_i}\|_{L^2} \to +\infty$. Set

$$f_i := f_{\varepsilon_i}, \quad u_i := u_{\varepsilon_i} = \frac{f_{\varepsilon_i}}{\|f_{\varepsilon_i}\|_{L^2}}.$$

It follows that

$$\|u_i\|_{L^2} = 1 \quad \text{and} \quad \sup_X |u_i| < \widetilde{C}_3 < +\infty,$$

where $\widetilde{C}_3$ is a uniform constant depending only on $\sup_X(|h|+|k|)$ and $\mathrm{Vol}(X)$.

Let us recall a useful lemma.

Lemma 2.4 ([21, Lemma 5.2], [25, Lemma 2.5])**.** *Suppose (X,g) is a non-compact Gauduchon manifold admitting an exhaustion function ϕ with $\int_X |\widetilde{\Delta}\phi|\frac{\omega^n}{n!} < +\infty$, and suppose η is a $(2n-1)$-form with $\int_X |\eta|^2\frac{\omega^n}{n!} < +\infty$. Then if $\mathrm{d}\eta$ is integrable,*

$$\int_X \mathrm{d}\eta = 0.$$

Using the conditions $\partial\bar{\partial}\omega^{n-1} = 0$, $|\mathrm{d}\omega^{n-1}|_g \in L^2(X)$, (2.7), (2.8), and Lemma 2.4, one can check that

$$\int_X f_i \widetilde{\Delta} f_i \frac{\omega^n}{n!} = -\int_X |\mathrm{d}f_i|^2 \frac{\omega^n}{n!}. \tag{2.10}$$

Substituting the perturbed equation into (2.10), we have

$$\int_X |\mathrm{d}f_i|^2 = -\int_X f_i(\varepsilon_i f_i + k - he^{f_i})\frac{\omega^n}{n!}, \tag{2.11}$$

which implies

$$\int_X |\mathrm{d}u_i|^2 \frac{\omega^n}{n!} \leq -\varepsilon_i - \frac{1}{\|f_i\|_{L^2}} \int_X u_i(k-h)\frac{\omega^n}{n!}.$$

Then, by passing to a subsequence, we have that u_i converge weakly to u_∞ in L_1^2 as $i \to +\infty$, and u_∞ is constant almost everywhere. Note that for any relatively compact $Z \subset X$, $L_1^2 \to L^2(Z)$ is compact. So

$$\int_Z |u_i|^2 \to \int_Z |u_\infty|^2.$$

Recalling $\sup_X |u_i| < \widetilde{C}_3 < +\infty$ and X has finite volume, so for a small $\epsilon > 0$, we have

$$\int_{X\setminus Z} |u_i|^2 < \epsilon,$$

when Z is big enough. Thus $1 \geq \int_Z |u_\infty|^2 \geq 1-\epsilon$. So, we have

$$u_\infty = \text{const.} \neq 0 \quad a.e..$$

In the following, we will follow Wang and Zhang's arguments in [24].

Suppose $u_\infty = C^* > 0$. Choose $0 < \epsilon < C^*$ and a non-negative smooth function $\zeta : \mathbb{R} \to \mathbb{R}$ by

$$\zeta(x) = \begin{cases} 1, & x \geq C^*; \\ 0, & x \leq \epsilon. \end{cases}$$

Choose ι_0 such that $\iota_0 > \frac{2}{\epsilon}$ and $\epsilon\frac{e^{\iota_0\epsilon}}{\iota_0^2} > \sup(\zeta)$. Then for $x > \epsilon$ and $\iota \geq \iota_0$, we have

$$\zeta(x) \leq \sup(\zeta) < \epsilon\frac{e^{\iota_0\epsilon}}{\iota_0^2} < \epsilon\frac{e^{\iota\epsilon}}{\iota^2} < x\frac{e^{\iota x}}{\iota^2}.$$

Clearly, the above inequality holds for $0 < x \leq \epsilon$. Having in mind that $u_\infty > 0$, thus $u_i > 0$ provided i is sufficiently large. Thus we have

$$\zeta(u_i) \leq u_i\frac{e^{\iota_i u_i}}{\iota_i^2},$$

for large enough i. Therefore,

$$\int_X \zeta(u_i)(-h) \leq \int_X (-h)u_i\frac{e^{\iota_i u_i}}{\iota_i^2} \leq -\int_X \frac{u_i k}{\iota_i^2},$$

where we used (2.11). Since $L^2_1(Z)\subset L^2_{0,b}(Z)$ is a compact embedding, where $L^2_{0,b}=\{f\in L^2(Z)||f|\le b\ \ a.e.\}$ and Z is an arbitrary subset of X. Then $u_i\to u_\infty$ strongly in $L^2_{0,b}(Z)$ for some b. Therefore

$$\begin{aligned}\int_Z(-h)&=\int_Z\zeta(u_\infty)(-h)\\&=\lim_{i\to+\infty}\int_Z\zeta(u_i)(-h)+\lim_{i\to+\infty}\int_Z\{\zeta(u_\infty)-\zeta(u_i)\}(-h)\\&\le\lim_{i\to+\infty}\int_X\zeta(u_i)(-h)\\&\le 0.\end{aligned}$$

Since $h\le 0$ and h is not identically zero. We thus get a contradiction, so we must have $u_\infty=C^*<0$.

On the other hand, from (2.11), we have

$$\int_X C^*k=\int_X u_\infty k=\lim_{i\to+\infty}\int_X u_ik\le\lim_{i\to+\infty}\int_X hu_ie^{u_it_i}\to 0.$$

This contradicts the assumption $\int_X k<0$. So we have proved that $\|f_\varepsilon\|_{C^0}$ is bounded uniformly when ε goes to zero. By standard elliptic estimates, we obtain, by choosing a subsequence f_ε must converge to a smooth function f_∞ in C^∞_{loc}-topology as $\varepsilon\to 0$, and f_∞ satisfies the equation (1.4). This completes the proof of Theorem 2.3.

□

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