ON SURFACE MEASURES OF CONVEX BODIES AND GENERALIZATIONS OF KNOWN TANGENTIAL IDENTITIES

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Abstract

One theme of this paper is to extend known results from polygons and balls to the general convex bodies in $n-$ dimensions. An another theme stems from approximating a convex surface with a polytope surface. Our result gives a sufficient and necessary condition for an natural approximation method to succeed (in principle) in the case of surfaces of convex bodies. Thus, Schwartz's paradox does not affect our method. This allows us to define certain surface measures on surfaces of convex bodies in a novel and simple way.

Keywords: Convex Geometry, Polytope Approximation, Surface Measures

1 Introduction

Archimedes discovered the 2-dimensional formula for a disk $D(x, r)$

$$\frac{|A|}{|\partial A|} = \frac{1}{2}R$$

In the early millenia Apostol and Mnatsakanian extended above equality to the class of polygons that they called "tangential" [1]. We will prove that if we extend the definition on tangentiality we obtain in n - dimensions (for that class of polytopes)

$$\frac{|A|}{|\partial A|} = \frac{1}{n}R,$$

where again R is the radius of the inscribed maximal ball.
We will extend above results to general convex bodies. This is achieved via defining a surface measure on convex bodies that we will call a normal measure. In addition we will show that a convex body can be defined as a countable intersection of half-spaces.

2 Some definitions and known results

A convex body is a compact and convex set. We expect always that $0 \in K$. For any measurable set $K \in \mathbf{R}^n$ we use $|K|$ as sign of the Lebesgue measure of that set. For sets $A, B \in \mathbf{R}^n$ we define the Minkowski addition as

$$A + B = \{a + b : a \in A \quad b \in B\}.$$

We define the set dilation as

$$\lambda A = \{\lambda a : a \in A\}.$$

We always use B_2^n an unit ball in n dimensions. The surface area of an convex body K is defined as

$$|\partial(K)| := \lim_{\epsilon \to 0} \frac{|K + \epsilon B_2^n| - |K|}{\epsilon} \tag{1}$$

We use H_θ for the $n-1$ dimensional hyperplane going through the origo and orthogonal to $\theta \in S^n$. In our sense a polyhedra is an intersection of finite number of closed half-spaces containing origo. A polytype is an polyhedra with a finite Lebesque measure. See for example [5] and references therein. For convex bodies in John's position, the inradius R is defined as the radius of the John's ball [2]. We need the following theorem.

Theorem 1. *If (X, d) is a compact metric space, then $(P(X), dP)$ is a compact metric space, where dp is the Prokhorov metric of weak convergence.*

A proof can be found in [3].

3 On polytopes and normal points

A convex body can be defined as a compact set

$$K = \bigcap_{i \in I} H',$$

where each H' is the half space containing origo that is bounded by the corresponding hyperplane H [5]. It can be proved that in each $n-1$-dimensional hyperplane H there exists an unique point x with the vector $\mathbf{x}$ orthogonal to the hyperplane H [5]. These points x_i are the unique solutions to the minimization problems

$$\min_{y \in H} ||y|| = ||x_i||,$$

where for each hyperplane $y \in H$, and H is the boundary of a half space used to define K. We call these points normal points. They depend on the position of K. They exists as long as the origo belongs to the interior of K. We prove that the convex body inherits the normal points from the half spaces used to define it.

Theorem 2. *Let K be a convex set containing origo. If the half-space H is used in the definition of K, then the normal point of H belongs $K \cap H$.*

Proof. If x_i is defined as the solution to

$$\langle y, x_i \rangle = 0,$$

then a closed origo centered ball of radius $||\mathbf{x}_i||$ contains only that point from the hyperplane H, and because the ball has that common supporting hyperplane with K, we have the claim. □

In light of the theorem the next corollary is clear.

Corollary 1. *Let P be a polytope. Then each facet of P contains a normal point and the number of normal points is less than the number of facets.*

We define minimal normal points as follows. They are the normal points of minimal value $|| \centerdot ||_2$ norm. We will use the following definition.

Definition 1. *A convex body is tangential iff every normal point is minimal.*

With respect to any normal point x_i we denote F_i as the corresponding facet. That is the facet that is orthogonal to $\mathbf{x_i}$ and containing x_i. For each K we define the Borel space $(K, \mathcal{X})$. We consider the finite intersection of half spaces used to define K. For each intersection $\bigcap_{i=1}^{m} H'$ of half spaces forming a polytope and used to define K we define a probability measure

$$\mu_j := \frac{\sum_{i=1}^{m} |F_i| \delta_{x_i}}{\sum_{i=1}^{m} |F_i|},$$

where every δ_{x_i} is a Dirac measure.

Theorem 3. *Let $x_1, ..., x_m$ be the normal points of a polytope P, then*

$$\frac{1}{n}\mathbf{E}(||x||) = \frac{|A|}{|\partial A|},$$

where $||x||$ is the discrete random variable obtaining values $||x_1||, .., ||x_m||$ with probabilities $|F_1|, ..., |F_m|$, respectively.

Proof. We know that a volume of an n-dimensional cone A_i is given by

$$|A_i| = \frac{|\text{base}| * \text{height}}{n}.$$

The theorem 2 implies that the polytope P‘s volume is

$$|P| = \frac{1}{n}\sum_{i=1}^{m} ||x_i|||F_i|,$$

where F_i is the facet corresponding to the normal point x_i. Thus,

$$\frac{|P|}{|\partial P|} = \frac{\sum_{i=1}^{m} ||x_i|||F_i|}{n\sum_{i=1}^{m} |F_i|} = \frac{1}{n}\mathbb{E}(||x||).$$

□

Corollary 2. *Let a polytope P be tangential, then*

$$\frac{1}{n}R = \frac{|A|}{|\partial A|},$$

where R is the inradius of K.

Naturally we wan‘t to extend the theorem and it‘s corollary to the general convex bodies. We will first do that for the corollary. For that we need a lemma.

Lemma 1. *For any point $x \in \partial K$, there exists a normal point x' s.t $x \in H_{x'}$.*

Proof. Now, x belongs to some supporting hyperplane. To this supporting hyperplane there corresponds at least one normal point and via theorem 2 the normal point belongs to ∂K. □

Next we proof that the space of normal points N is separable as a metric compact space.

Theorem 4. *The space of normal points is closed in the subspace topology and the space is separable.*

Proof. Suppose that there exists a Cauchy sequence of normal points (x_j) s.t

$$\lim_{j=1} x_j = x.$$

It holds that $x \in \partial K$ because the surface is closed. Thus, via lemma 1 it holds that $x \in H_{x'}$, where x' is a normal point. Now, if $y \in H_{x'}$ it holds that

$$\lim_{j \to \infty} \langle x_j, x' - y \rangle = \langle x, x' - y \rangle = 0$$

because of the continuity of the inner product. Thus, $H_{x'}$ is orthogonal to $\mathbf{x}$ and $x' = x$. In addition it‘s well known that every compact metric space is separable [10]. □

For any sequence of normal points (x_j) we denote the corresponding sequence of nested polyhedras to be the sequence

$$(\cap_{j=1}^{m} H_{x_j}) := (P_m).$$

The following theorem is a key theorem.

Theorem 5. *Let (x_j) be any sequence of normal points. Then for the corresponding nested sequence of polyhedras (P_m) it holds that*

$$\bigcap_{m=1}^{\infty} P_m = K,$$

iff the set of the corresponding normal points is dense in the space of normal points of K.

Proof. Let the set of normal points corresponding to the sequence (x_j) be dense in N. Because $\bigcap_{m=1}^{\infty} P_m$ is some convex body, the set of it‘s normal points is closed. Thus, it follows from the definition of K and from theorem 2 that $K = \bigcap_{m=1}^{\infty} P_m$. On the other hand, if the sequence of normal points is not dense in N, then there exist a normal point of K and a halfspace used in the definition of K that has a boundary that is not a supporting hyperplane to $\bigcap_{m=1}^{\infty} P_m$. Thus,

$$\bigcap_{i=1}^{\infty} P_m \neq K.$$

□

We proved that the space of normal points is separable so it follows from 5 that we can define polytope as an countable intersection of suitable half-spaces. Next we prove a corollary to 5.

Corollary 3. *It holds that*

$$\lim_{m\to\infty} |P_m| = |K|$$

iff the set of normal points corresponding to the polyhedras is dense in N.

Proof. We need only to prove the following. If the set of normal points is not dense then

$$K| < |\bigcap_{i=1}^{\infty} P_m|.$$

Via the theorem 5 the sets are not the same. So there exists $x \in \bigcap_{i=1}^{\infty} P_m \cap K^{\complement}$. Now,

$$\min_{y_i \in K} ||x - y_i|| = ||x - y|| > 0,$$

because there exists a hyperplane separating K and x. It follows that

$$|K| < |\mathrm{conv}(x, K)| \leq |\bigcap_{i=1}^{\infty} P_m|,$$

because there exists a small ball $B(x, \epsilon) \subset \bigcap_{i=1}^{\infty} P_m \cap K^{\complement}$. □

4 On normal measures on convex bodies

In this section we will define a surface measure on convex surfaces in $n-$ dimensions. In general the measure is not continuous, because it can be discrete in the segments of a convex body (for example on the ends of a cylinder). First we will prove the existence of the measure. We use the sequences of measures defined in the last section and we will prove that the measures convergence strongly to a measure.

Lemma 2. *For each Borel set $A \subset K$, the sequence of $(u_m(A))$ is a Cauchy sequence.*

Proof. Let the Borel set A be fixed. W.L.O.G we can assume that there exist m such that

$$\mu_m(A) > 0. \tag{2}$$

Let us denote

$$\mu_m(A) = \frac{|\partial P'_m|}{|\partial P_m|}$$

and

$$\mu_{m+1}(A) = \frac{|\partial P'_{m+i}|}{|\partial P_{m+1}|},$$

where $\partial P'_m$ and $\partial P'_{m+1}$ are the unions of those facets that contain the normal points of $A \cap P_m$ and $A \cap P_{m+1}$, respectively. Now, let us define

$$B' := P'_m - P'_{m+1}.$$

If $P'_m = P'_{m+1} \cup B'$ it can be proved (for example considering that the $n-1$ dimensional Hausdorff-measure of the facets agree with the definition (1)) that

$$0 < |\partial P'_m| = |\partial(P'_{m+1} \cup B')| = |\partial P'_{m+1}| + |\partial B'| - |\partial(P'_{m+1} \cap B')|, \tag{3}$$

where the first inequality above follows from our asumption (2). It follows that B' is not the empty set. So we have

$$|\partial(P'_{m+1} \cap B')| = |\partial P'_{m+1} \cap \partial B'|.$$

Thus, we can rewrite (3) as

$$0 < |\partial P'_m| = |\partial P'_{m+1}| + |\partial B'| - |\partial P'_{m+1} \cap \partial B'|,$$

It follows that

$$|\partial P'_{m+1}| \le |\partial P'_m|.$$

Thus $|\partial P'_m|$ converges to, say, $|K'|$. It follows that

$$u_m(A) = \frac{|\partial P'_m|}{|\partial P_m|}$$

converges. □

The total variation distance of probability measures is defined as

$$||\mu - v||_{TV} := 2\sup_A |\mu(A) - v(A)|.$$

Above is an stronger convergence than

$$\lim_{m\to\infty} \mu_m(A) = \mu(A),$$

for any Borel set $A \subset K$.

Theorem 6. *The set function defined by* $\lim_{j\to\infty} u_j = \mu$ *is a probability measure on* $(K, \mathcal{X})$*, where the metric is understood to be the Prokhorov metric of weak convergence.*

Proof. By lemma 2 $\lim_{j\to\infty} u_j(A) \geq 0$ is a Cauchy sequence. Thus, there is a set function μ such that $\lim_{j\to\infty} u_j = \mu$. We just proved that μ_j converges strongly to μ. Because

$$||\mu_i - \mu_j||_{TV} := 2\sup_A |\mu_i(A) - \mu_j(A)| \leq 2|K|$$

is uniformly bounded and the space is Polish, the strong convergence implies the weak convergence. Thus μ is a probability measure via the theorem 1. □

We need the following theorem.

Theorem 7. *For any measurable function f on K it holds that*

$$\int_K f\mu = \lim_{m\to\infty} \int_K f\mu_m. \tag{4}$$

Proof. The claim follows straight from the lemma for Borel sets. Thus, it follows for simple functions. If f is a nonnegative measurable function then there exists a monotonically growing sequence of simple functions such that

$$\int_K f\mu - \sum_{i=1}^{n} \alpha_i \mu(A_i)| < \epsilon.$$

Thus,

$$|\int_K f\mu + \lim_{m\to\infty}\sum_{i=1}^{n}\alpha_i\mu_m(A_i)| < \epsilon$$

for all $\epsilon > 0$ and we have the claim.

□

We need to prove that the measure does not depend on the sequence of polytopes, if the sequence of normal points is dense.

Theorem 8. *If A is Borel set it holds that*

$$\lim_{j\to\infty}\int_K 1_A\mu_j = \int_K 1_A\mu = \lim_{l\to\infty}\int_K 1_A v_l = \int_K 1_A v.$$

Proof. If

$$v(A) < \mu(A),$$

then there exists k s.t

$$v_l(A) < \mu(A),$$

when $l > k$. Thus via previous theorem 7 this is a contradiction.

□

For each convex body K we call the measure just defined the normal measure of K.

Next we generalize the result 3 from polytopes to general convex bodies.

Theorem 9. *For any convex body it holds that*

$$\frac{1}{n}\mathbb{E}||x|| = \frac{|K|}{|\partial K|},$$

where $x \in N$ are the normal points of K and the expected value is understood with respect to the measure given by the theorem 6.

Proof. Now, via theorem 7

$$\lim_{j\to\infty}\int 1_A\mu_j \geq \int 1_A\mu$$

for all open sets. It follows that

$$\mathbb{E}(|f|)\mu_j \to \mathbb{E}(|f|)\mu$$

for all bounded continuous functions.

□

The tangential case is an easy corollary.

Corollary 4. *For tangential convex body K, it holds that*

$$\frac{R}{n} = \frac{|K|}{|\partial K|},$$

where R is the inradius of K.

The next theorem is an analogue of the corollary 3 from the last section.

Theorem 10. *It holds that*

$$\lim_{m\to\infty} |\partial P_m| = |\partial K|$$

iff the set of normal points corresponding to the polyhedras is dense in N.

Proof. Via the theorem 5 we need only to prove that if

$$|\partial K| = |\partial \bigcap_{i=1}^{\infty} P_m|,$$

then the set of normal points is dense. Now, if it holds that

$$|K| < |\bigcap_{i=1}^{\infty} P_m|.$$

It follows that

$$|\partial(\bigcap_{i=1}^{\infty} P_m)| - |\partial K|$$
$$= \lim_{\epsilon\to 0} \frac{|\bigcap_{i=1}^{\infty} P_m + \epsilon B_2^n| - |\bigcap_{i=1}^{\infty} P_m| - |K + \epsilon B_2^n| + |K|}{\epsilon} < 0.$$

which is a contradiction.

□

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