Developments of the Extended Relativity Theory in Clifford Spaces

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Abstract

A brief tour of the developments of the Extended Relativity Theory in Clifford Spaces (C-space) is presented. These include : (i) Novel physical consequences like generalized dispersion relations, energy-dependent speed of light propagation, extended Lorentz transformations, relative locality, generalized Weyl-Heisenberg algebra and uncertainty relations, tensionless branes, superluminality, generalized velocities. (ii) Generalized areal, volume, $\cdots$ metrics and gravitational field equations in C-space. (iii) A unified description of particles, strings and branes. (iv) Clifford gravity based cosmology and dark energy. (v) Moyal deformations of Clifford gauge theories of gravity. (vi) N-ary algebras. We conclude with a brief discussion on symplectic Clifford algebras and generalized geometries.

Keywords : Clifford algebras; Extended Relativity Theory in Clifford Spaces; String theory; M-theory; Generalized geometries.

1 The Extended Relativity Theory in Clifford Spaces

1.1 Introduction

In the past years, the Extended Relativity Theory in C-spaces (Clifford spaces) and Clifford-Phase spaces were developed in [1], [2]. The Extended Relativity theory in

*Dedicated to the memory of Dolly Ramirez, Efrain Jaramillo, mother, son from Bogotá, Colombia

Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory whose generalized coordinates are Clifford polyvector-valued quantities which incorporate the lines, areas, volumes, and hyper-volumes degrees of freedom associated with the collective dynamics of particles, strings, membranes, p-branes (closed p-branes) moving in a D-dimensional target spacetime background. C-space Relativity permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing.

The theory has 2 fundamental parameters : the speed of a light c and a length scale which can be set equal to the Planck length. The role of "photons" in C-space is played by *tensionless* branes. The polyvector valued coordinates

$$x, \quad x^{\mu}, \quad x^{\mu_1\mu_2} \ = \ - \, x^{\mu_2\mu_1}, \quad x^{\mu_1\mu_2\mu_3} \ = \ - \, x^{\mu_2\mu_1\mu_3}, \quad \ldots \tag{1.1}$$

are now linked to the basis generators $\mathbf{1}$, vectors γ^{μ}, bi-vectors generators $\gamma_{\mu} \wedge \gamma_{\nu}$, tri-vectors generators $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$, ... of the Clifford algebra, including the Clifford algebra unit element (associated to a scalar coordinate).

These polyvector valued coordinates can be interpreted as the quenched-degrees of freedom of an ensemble of p-loops associated with the dynamics of closed p-branes, for $p = 0, 1, 2, ..., D-1$, embedded in a target D-dimensional spacetime background. C-space is parametrized not only by 1-vector coordinates x^{μ} but also by the 2-vector coordinates $x^{\mu\nu}$, 3-vector coordinates $x^{\mu\nu\alpha}$, ..., called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops,..., onto the coordinate planes . By p-loop we mean a closed p-brane; in particular, a 1-loop is closed string.

For example, when $\mathbf{X}$ is the Clifford-valued coordinate corresponding to the $Cl(1,3)$ algebra in four-dimensions it can be decomposed as

$$\mathbf{X} \ = \ s \, \mathbf{1} \ + \ x^{\mu} \, \gamma_{\mu} \ + \ x^{\mu\nu} \, \gamma_{\mu} \wedge \gamma_{\nu} \ + \ x^{\mu\nu\rho} \, \gamma_{\mu} \wedge \gamma_{\nu} \wedge \gamma_{\rho} \ + \ x^{\mu\nu\rho\tau} \, \gamma_{\mu} \wedge \gamma_{\nu} \wedge \gamma_{\rho} \wedge \gamma_{\tau} \tag{1.2}$$

where we have omitted combinatorial numerical factors for convenience in the expansion of eq-(1.1). If one imposes the lexicographic ordering of indices $\mu_1 < \mu_2 < \mu_3 < \cdots$ then it is not necessary to include combinatorial numerical factors in the (1.2). To avoid introducing powers of a length parameter L (like the Planck scale L_p), in order to match physical units in the expansion of the polyvector X in eq-(1), we can set it to unity to simplify matters.

The component s is the Clifford scalar component of the polyvector-valued coordinate and $d\Sigma$ is the infinitesimal C-space proper "time" interval

$$(d\Sigma)^2 \ = \ (ds)^2 \ + \ dx_{\mu} \, dx^{\mu} \ + \ dx_{\mu\nu} \, dx^{\mu\nu} \ + \ \ldots \tag{1.3}$$

that is *invariant* under $Cl(1,3)$ transformations and which are the Clifford-algebraic extensions of the $SO(1,3)$ Lorentz transformations [1]. One should emphasize that $d\Sigma$ is *not* equal to the proper time Lorentz-invariant interval $d\tau$ in ordinary spacetime $(d\tau)^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = dx_{\mu}dx^{\mu}$. Generalized Lorentz transformations (poly-rotations) in flat C-spaces were discussed in [1].

Let us provide several examples of generalized Lorentz transformations in C-space. For example, given γ_{02} the transformation involving the rotor $R_1 = cosh(\beta/2) - \gamma_{02} \, sinh(\beta/2)$

corresponds to an ordinary Lorentz boost transformation along the X^2 direction and involving the ordinary temporal variable X^0. The ordinary Lorentz boots generators are given by the bivectors $\gamma_{\mu\nu}$, and which in turn are also expressed as the commutators $[\gamma_\mu, \gamma_\nu]$. The physical significance of the latter commutators is that they represent a "rotation" along the $X^\mu - X^\nu$ directions.

However, since one may also write the bivector γ_{02} as the commutator $[\gamma_{12}, \gamma_{01}] = -2\gamma_{02}$, the transformation involving the above rotor R_1 also corresponds to an *areal* boost along the X^{12} direction but involving the areal temporal coordinate X^{01}. Namely, it is a "rotation" along the $X^{12} - X^{01}$ directions. Whereas the ordinary boost is a "rotation" along the $X^2 - X^0$ directions.

After writing

$$(X^B)' \; \Gamma_B \;=\; (\; cosh(\beta/2) \;-\; \gamma_{02} \; sinh(\beta/2) \;) \; (\; X^A \; \Gamma_A \;) \; (\; cosh(\beta/2) \;+\; \gamma_{02} \; sinh(\beta/2) \;) \tag{1.4}$$

straightforward algebra yields the transformation of the following bivector coordinates

$$(X^{12})' \;=\; X^{12} \; cosh\beta \;+\; X^{01} \; sinh\beta \tag{1.5a}$$

$$(X^{01})' \;=\; X^{01} \; cosh\beta \;+\; X^{12} \; sinh\beta \tag{1.5b}$$

One has a mixing of the spatial and temporal areal bivector coordinates in the new frame of reference.

Furthermore, since $[\gamma_{013}, \gamma_{123}] \sim \gamma_{02}$, the transformation involving the above rotor R_1 also corresponds to a 3-volume boost along the X^{123} direction but involving the 3-volume temporal coordinate X^{013}. Namely, it is a "rotation" along the $X^{123} - X^{013}$ directions giving

$$(X^{123})' \;=\; X^{123} \; cosh\beta \;+\; X^{013} \; sinh\beta \tag{1.6a}$$

$$(X^{013})' \;=\; X^{013} \; cosh\beta \;+\; X^{123} \; sinh\beta \tag{1.6b}$$

One has a mixing of the spatial and temporal trivector coordinates in the new frame of reference. The ordinary Lorentz boosts of the vector coordinates give

$$(X^2)' \;=\; X^2 \; cosh\beta \;+\; X^0 \; sinh\beta \tag{1.7a}$$

$$(X^0)' \;=\; X^0 \; cosh\beta \;+\; X^2 \; sinh\beta \tag{1.7b}$$

while the remaining coordinates remain invariant and such that the quadratic form $X^A X_A = (X^A)'(X_A)'$ remains invariant. Straightforward algebra leads to

$$- \; (X'_0)^2 \;+\; (X'_1)^2 \;-\; L^{-2} \; (X'_{01})^2 \;+\; L^{-2} \; (X'_{12})^2 \;-\; L^{-4} \; (X'_{013})^2 \;+\; L^{-4} \; (X'_{123})^2 \;=$$

$$- \; (X_0)^2 \;+\; (X_1)^2 \;-\; L^{-2} \; (X_{01})^2 \;+\; L^{-2} \; (X_{12})^2 \;-\; L^{-4} \; (X_{013})^2 \;+\; L^{-4} \; (X_{123})^2 \tag{1.8}$$

The quadratic form is defined as

$$< \mathbf{X}^\dagger \, \mathbf{X} > \; = \; X_A \, X^A \; = \; s^2 + X_\mu \, X^\mu + X_{\mu_1\mu_2} \, X^{\mu_1\mu_2} + ... \, X_{\mu_1\mu_2.....\mu_D} \, X^{\mu_1\mu_2....\mu_D} \quad (1.9)$$

where $\mathbf{X}^\dagger$ denotes the reversal operation obtained by reversing the order of the gamma generators in the wedge products. The symbol $< \; \Gamma_A \; \Gamma_B \; >$ denotes taking the scalar part in the Clifford geometric product of $\Gamma_A \; \Gamma_B$. It is the analog of the trace of a product of matrices. Such scalar part can be obtained from the (anti) commutator relations of the Clifford algebra generators as displayed in the Appendix. For example

$$< \; \gamma_\mu \; \gamma^\nu \; > \; = \; \delta^\nu_\mu, \quad < \; \gamma_{\mu_1\mu_2} \; \gamma^{\nu_1\nu_2} \; > \; = \; - \; \delta_{\mu_1\mu_2}^{\;\nu_1\nu_2}$$
$$< \; \gamma_{\mu_1\mu_2\mu_3} \; \gamma^{\nu_1\nu_2\nu_3} \; > \; = \; - \; \delta_{\mu_1\mu_2\mu_3}^{\;\nu_1\nu_2\nu_3}, \quad < \; \gamma_{\mu_1\mu_2\mu_3\mu_4} \; \gamma^{\nu_1\nu_2\nu_3\nu_4} \; > \; = \; \delta_{\mu_1\mu_2\mu_3\mu_4}^{\;\nu_1\nu_2\nu_3\nu_4}, \quad ...... \quad (1.10)$$

One should note the presence of $\pm$ signs in the right hand side of eqs-(1.10). They are connected to the even/odd behavior of the reversal operation $(\gamma_C)^\dagger = \pm\gamma_C$.

The quadratic form is invariant under the isometry transformations

$$\mathbf{X}' \; = \; \mathbf{R} \, \mathbf{X} \, \mathbf{L}^\dagger, \;\; \mathbf{R}^\dagger \, \mathbf{R} \; = \; 1, \;\; \mathbf{L}^\dagger \, \mathbf{L} \; = \; 1 \;\; \Rightarrow \;\; < \; \mathbf{X}'^\dagger \, \mathbf{X}' \; > \; = \; < \; \mathbf{X}^\dagger \, \mathbf{X} \; > \quad (1.11)$$

due to the cyclic property of the scalar part projection

$$< \; \mathbf{X}'^\dagger \, \mathbf{X}' \; > \; = \; < \; \mathbf{L} \, \mathbf{X}^\dagger \, \mathbf{R}^\dagger \, \mathbf{R} \, \mathbf{X} \, \mathbf{L}^\dagger, \; > \; = \; < \; \mathbf{L} \, \mathbf{X}^\dagger \, \mathbf{X} \, \mathbf{L}^\dagger \; > \; =$$
$$< \; \mathbf{L}^\dagger \, \mathbf{L} \, \mathbf{X}^\dagger \, \mathbf{X} \; > \; = \; < \; \mathbf{X}^\dagger \, \mathbf{X} \; > \quad (1.12)$$

where $\mathbf{R}, \mathbf{L}$ are Clifford-valued rotors acting on the right and left respectively.

The second example corresponds to the case when there is a *mixing* of different grades. It involves the commutator $[\gamma_{0123}, \gamma_3] \sim \gamma_{012}$ and such that the transformation involving the rotor $R_2 = cosh(\beta'/2) - \gamma_{012} \; sinh(\beta'/2)$ corresponds to a boost along the spatial X^3 direction but involving now the *temporal* 4-volume polyvector-valued coordinate X^{0123}. The reason being that γ_{012} can be rewritten as the commutator of γ_{0123} and γ_3, so we have now "rotations" along the $X^3 - X^{0123}$ directions. Straightforward algebra yields now the transformation of the following (poly) vector coordinates

$$(X^3)' \; = \; X^3 \; cosh(\beta') \; - \; L^{-3} \; X^{0123} \; sinh(\beta') \quad (1.13a)$$

$$(X^{0123})' \; = \; X^{0123} \; cosh(\beta') \; - \; L^3 \; X^3 sinh(\beta') \quad (1.13b)$$

In this case one has a *mixing* of polyvector-valued coordinates of $different \; grade$. In the new frame of reference the spatial X^3 coordinate and the temporal 4-volume coordinate X^{0123} are mixed.

Furthermore, since $[\gamma_{03}, \gamma_{123}] \sim \gamma_{012}$, the transformation involving the rotor $R_2 = cosh(\beta'/2) - \gamma_{012} \; sinh(\beta'/2)$ also corresponds to a boost along the spatial *trivector* X^{123} direction but involving now the *temporal* bivector coordinate X^{03}. These transformations are

$$(X^{123})' \;=\; X^{123}\; cosh(\beta') \;-\; L\; X^{03}\; sinh(\beta') \qquad (1.14a)$$

$$(X^{03})' \;=\; X^{03}\; cosh(\beta') \;-\; L^{-1}\; X^{123}\; sinh(\beta') \qquad (1.14b)$$

In the above equations we have used the relations (see Appendix)

$$\gamma_{01}^2 = 1, \quad \gamma_{02}^\dagger = -\gamma_{02}, \quad \gamma_{012}^2 = 1, \quad \gamma_{012}^\dagger = -\gamma_{012}$$

$$\{\gamma_{12}, \gamma_{02}\} = 0, \quad [\gamma_{0123}, \gamma_{012}] \;=\; -\,2\;\gamma_3, \quad \{\gamma_{0123}, \gamma_{012}\} \;=\; 0$$

$$\gamma_{02}\;\gamma_{12}\;\gamma_{02} = -\gamma_{12}, \quad [\gamma_{012},\; \gamma_3] \;=\; 2\;\gamma_{0123}, \quad \{\gamma_{012}, \gamma_3\} = 0, ... \qquad (1.15)$$

$$cosh^2(\xi) - sinh^2(\xi) \;=\; 1, \;\; cosh^2(\xi) + sinh^2(\xi) \;=\; cosh(2\xi), \;\; \sinh(2\xi) \;=\; 2\, sinh(\xi)\, cosh(\xi) \qquad (1.16)$$

Given in general a transformation of the form

$$X'^B\;\Gamma_B \;=\; (\; cosh(\beta/2) \;-\; \Gamma_C\; sinh(\beta/2))\;\; X^A\;\Gamma_A\;\; (\; cosh(\beta/2) \;+\; \Gamma_C\; sinh(\beta/2)) \qquad (1.17)$$

one learns that

$$X'^B(\beta, \Gamma_C) \;=\; X^B\; cosh^2(\beta/2) \;-\; X^A\; sinh^2(\beta/2)\; <\; \Gamma_C\;\Gamma_A\;\Gamma_C\;\Gamma^B \;>\; + $$

$$X^A\; cosh(\beta/2)\; sinh(\beta/2)\; <\; [\Gamma_A,\; \Gamma_C]\;\Gamma^B \;> \qquad (1.18)$$

The generator Γ_C of generalized Lorentz boosts is of the form $(\gamma_{0\mu_1\mu_2...\mu_{n-1}})$ with the provision that under the reversal operation it changes sign

$$(\gamma_{0\mu_1\mu_2...\mu_{n-1}})^\dagger \;=\; -\;\gamma_{0\mu_1\mu_2...\mu_{n-1}} \qquad (1.19a)$$

so that $\mathbf{R}\mathbf{R}^\dagger = 1$. This condition will *restrict* the values of n to be $n = 2, 3, 6, ...$and obeying

$$(\gamma_{0\mu_1\mu_2...\mu_{n-1}})^2 \;=\; 1 \qquad (1.19b)$$

Generalized *spatial* rotations don't involve the temporal directions and are generated by $\gamma_{\mu_1\mu_2...\mu_m}$ obeying

$$(\gamma_{\mu_1\mu_2...\mu_m})^\dagger \;=\; -\;\gamma_{\mu_1\mu_2...\mu_m} \qquad (1.20)$$

and

$$(\gamma_{\mu_1\mu_2...\mu_m})^2 \;=\; -\;1 \qquad (1.21)$$

For instance, a generalized rotation in $D > 4$ and generated by $\gamma_{12...6}$ involving the parameter $\alpha^{12....6}$ yields a rotor whose Taylor series expansion becomes

$$\mathbf{R} \ = \ e^{\alpha^{12...6} \ \gamma_{12.....6}} \ = \ cos(\alpha^{12....6}) \ + \ \gamma_{012.....6} \ sin(\alpha^{12...6}) \qquad (1.22)$$

due to the condition $(\gamma_{12.....6})^2 \ = \ - \ 1$ which is similar to having the imaginary unit $i^2 = -1$ and the expression $e^{i\theta} = cos(\theta) + i \ sin(\theta)$. For an earlier discussion of generalized rotations within C-space see [33]. Whereas a generalized Lorentz boost is like having a "rotation" with an imaginary "angle" leading to the hyperbolic functions

$$\mathbf{R} \ = \ e^{\beta^{012...5} \ \gamma_{02.....5}} \ = \ cosh(\beta^{012....5}) \ + \ \gamma_{012.....5} \ sinh(\beta^{12...5}) \qquad (1.23)$$

due to the condition $(\gamma_{012.....5})^2 \ = \ 1$.

Eq-(1.18) only simplifies considerably in the very special case when the values of the polyvector valued indices A, B, C are such that

$$< \ \Gamma_C \ \Gamma_A \ \Gamma_C \ \Gamma^B \ > \ = \ - \ \delta^B_A, \quad < \ [\Gamma_A, \ \Gamma_C] \ \Gamma^B \ > \ = \ \pm 2 \qquad (1.24)$$

and it leads to the type of transformations displayed above. In general, for a given set of values of B, C, one must *sum* over *all* the A indices in eq-(1.18). For this reason the most general expression for X'^B given by eq-(1.18) is more complicated than that given by the above equations. Another special case occurs when

$$< \ \Gamma_C \ \Gamma_A \ \Gamma_C \ \Gamma^B \ > \ = \ \delta^B_A, \quad < \ [\Gamma_A, \ \Gamma_C] \ \Gamma^B \ > \ = \ 0 \qquad (1.25)$$

leading to $X'^B = X^B$ so that these particular polyvector coordinate components remain invariant.

One should emphasize that the functional form of the most general transformations are even *more complicated* than those described in eq-(1.18). Let us write the rotor associated with a "rotation" along the $X^A - X^B$ directions in C-space with parameter α^{AB}, after writing the commutation relations $[\Gamma_A, \Gamma_B] = f_{AB}^{\ \ C} \Gamma_C$, as follows

$$\mathbf{R} \ = \ e^{\alpha^{AB} \ [\Gamma_A, \Gamma_B]} \ = \ e^{\alpha^{AB} \ f^C_{AB} \ \Gamma_C} \ = \ e^{\beta^C \Gamma_C}, \ \ \beta^C \ = \ \alpha^{AB} \ f_{AB}^{\ \ C} \qquad (1.26)$$

where $f_{AB}^{\ \ C}$ are the structure constants of the algebra. There is a summation over the C indices (but not over the A, B indices) in eq-(1.26) and the reversal condition reads

$$[\Gamma_A, \ \Gamma_B]^\dagger \ = \ - \ [\Gamma_A, \ \Gamma_B] \ \Rightarrow \ \mathbf{R} \ \mathbf{R}^\dagger = 1 \qquad (1.27)$$

and which is satisfied in particular when $\Gamma^\dagger_A = -\Gamma_A; \Gamma^\dagger_B = -\Gamma_B$ giving $\Gamma^\dagger_C = -\Gamma_C$. This is a result of the relations $(\Gamma_A \Gamma_B)^\dagger = (\Gamma_B)^\dagger (\Gamma_A)^\dagger = \Gamma_B \Gamma_A$. In the most general case, for arbitrary dimensions, due to the *summation* over the C polyvector indices in eq-(1.26), the rotor $\mathbf{R}$ cannot be expressed in the form displayed in eq-(1.17) after performing a Taylor series expansion of the exponentials. For instance

$$e^{\beta^{01}\gamma_{01} \ + \ \beta^{023}\gamma_{023}} \ \neq \ \Big(cosh(\beta^{01}) \ + \ \gamma_{01} \ sinh(\beta^{01}) \Big) \ \Big(cosh(\beta^{023}) \ + \ \gamma_{023} \ sinh(\beta^{023}) \Big) \qquad (1.28)$$

as a result of the Baker-Campbell-Hausdorf formula. Because $[\gamma_{01}, \gamma_{023}] \neq 0$ the left hand side of eq-(1.28) does not factorize.

1.2 Relative Locality

We learnt from Special Relativity that the concept of simultaneity is *relative*. The typical example arises when a moving observer inside a train sees the front and back doors of a train opening simultaneously. Due to the spatial separation ($\Delta X^3 \neq 0$) between the two doors, an observer at rest in the platform will see the doors opening at *different* times

$$(\Delta X^0)' \;=\; \Delta X^0 \; cosh(\beta) \;+\; \Delta X^3 \; sinh(\beta) \;\neq\; 0, \tag{1.29}$$

despite $\Delta X^0 = 0$ due to the fact that $\Delta X^3 \neq 0$.

Something analogous, and more general, occurs in C-space. Let us denote by $\Delta X^3 = X^3_{(2)} - X^3_{(1)}$, $\Delta X^{0123} = X^{0123}_{(2)} - X^{0123}_{(1)}$ the spatial and 4-volume *separation*, respectively, between two events **1** and **2** in a given frame of reference in a *flat* C-space. From eqs-(1.13) it follows that in the new frame of reference one has

$$(\Delta X^3)' \;=\; \Delta X^3 \; cosh(\beta') \;-\; L^{-3} \; \Delta X^{0123} \; sinh(\beta') \tag{1.30a}$$

$$(\Delta X^{0123})' \;=\; \Delta X^{0123} cosh(\beta') \;-\; L^3 \; \Delta X^3 sinh(\beta') \tag{1.30b}$$

if $\Delta X^{0123} \neq 0$ one has that $(\Delta X^3)' \neq 0$ despite that $\Delta X^3 = 0$. Therefore, because $(\Delta X^3)' \neq 0$ the observer in the new frame of reference does *not* experience events $\mathbf{1}, \mathbf{2}$ at the *same* location.

An "extended" event in C-space described by eqs-(1.30) can be envisaged as follows. An observer assigns to a physical event the coordinate values X^A where the index A spans 2^D values corresponding to the dimension of a Clifford algebra in D-dim. In particular X^3, X^{0123}. Event **1** can be described in terms of a spherical bubble (a closed 3-brane) moving in spacetime whose 4-volume (swept by the 3-brane at a given time $X^0_{(1)}$) is given by $X^{0123}_{(1)}$. The center of mass of such bubble is given by the $X^\mu_{(1)}$ coordinates, in particular $X^3_{(1)}$ represents the z-component. Whereas event **2** is described in terms of another spherical bubble of *different size* in spacetime whose 4-volume at a given time $X^0_{(2)}$ is given by $X^{0123}_{(2)}$. The center of mass of such bubble is given now by $X^\mu_{(2)}$ coordinates, in particular $X^3_{(2)}$. If the centers of mass of the small and large bubble *coincide* one has that $\Delta X^3 = 0$, while $\Delta X^{0123} \neq 0$ since the bubbles are of *different size*. Consequently one learns from eq-(37a) that $(\Delta X^3)' \neq 0$ in the new frame of reference : namely, the centers of mass of the bubbles in the new frame of reference do *no* longer *coincide*.

Concluding, the concept of spacetime locality is *relative* due to the *mixing* of 4-volume coordinates with spacetime vector coordinates under generalized Lorentz transformations in C-space. In the most general case, there will be mixing of all polyvector valued coordinates. This was the motivation to build a unified theory of all extended objects, p-branes, for all values of p subject to the condition $p+1 = D$. Therefore, the Extended Relativity Theory in C-spaces (Clifford spaces) were provides a very different physical explanation of the phenomenon of "relativity of locality" than the one described by the Doubly Special Relativity (DSR) framework [19].

1.3 Generalized Velocities in C-space, Superluminality

We shall now discuss the concept of "photons" and generalized velocities in C-space. Superluminal particles were studied within the framework of the Extended Relativity theory in Clifford spaces (C-spaces) in [8]. As discussed in detailed by [1], [3] one can have tachyonic (superluminal) behavior in ordinary spacetime while having non-tachyonic behavior in C-space. Hence from the C-space point of view there is no violation of causality nor the Clifford-extended Lorentz symmetry. The analog of "photons" in C-space are *tensionless* strings and branes [1].

Let us take the spacetime signature to be $(-,+,+,+,......,+)$ and factorize the C-space interval in eq-(2) as follows by bringing the $c^2(dt)^2$ factor outside the parenthesis

$$(d\Sigma)^2 = c^2(dt)^2 \left(\frac{L^2}{c^2} (\frac{ds}{dt})^2 - 1 + \frac{1}{c^2} (\frac{dX_i}{dt})^2 + \frac{1}{L^2c^2} (\frac{dX_{ij}}{dt})^2 - \frac{1}{L^2c^2} (\frac{dX_{0i}}{dt})^2 .... \right) \tag{1.31}$$

where the spatial index i range is $1,2,...,D-1$. The Clifford space associated with the Clifford algebra in $4D$ is 16-dimensional and has a neutral/split signature of $(8,8)$ [3], [1]. For example, the terms $(dX_{0i})^2$, $(dX_{0ij})^2$, $(dX_{0123})^2$ will appear with a negative sign, while the terms $(dX_{ij})^2, (dX_{ijk})^2$ will appear with a positive sign.

There are many possible combination of numerical values for the 16 terms inside the parenthesis in eq-(1.31). As explained in [3], [1], *superluminal* velocities in ordinary spacetime are possible, while retaining the null interval condition in C-space $(d\Sigma)^2 = 0$, associated with *tensionless* branes. The null interval in C-space $(d\Sigma)^2 = 0$ can be attained, for example, if each term inside the parenthesis is ± 1 respectively. Since there are 8 positive ($+1$) terms and 8 negative (-1) terms one has that $8-8=0$ and the null interval condition $(d\Sigma)^2 = 0$ is still satisfied despite having superluminal speeds.

A very different combination of numerical values, as compared to the previous one, leading also to a null interval condition in C-space $(d\Sigma)^2 = 0$, occurs when

$$\frac{1}{c^2} \left((\frac{dX_1}{dt})^2 + (\frac{dX_2}{dt})^2 + (\frac{dX_3}{dt})^2 \right) = 1 \tag{1.32a}$$

$$\frac{1}{L^2c^2} \left((\frac{dX_{12}}{dt})^2 + (\frac{dX_{13}}{dt})^2 + (\frac{dX_{23}}{dt})^2 \right) = \frac{1}{L^2c^2} \left((\frac{dX_{01}}{dt})^2 + (\frac{dX_{02}}{dt})^2 + (\frac{dX_{03}}{dt})^2 \right) \tag{1.33b}$$

$$\frac{1}{L^4c^2} \left((\frac{dX_{012}}{dt})^2 + (\frac{dX_{013}}{dt})^2 + (\frac{dX_{023}}{dt})^2 \right) = \frac{1}{L^4c^2} (\frac{dX_{123}}{dt})^2 \tag{1.33c}$$

$$\frac{1}{L^6c^2} (\frac{dX_{0123}}{dt})^2 = \frac{L^2}{c^2} (\frac{ds}{dt})^2 \tag{1.33d}$$

Another description of C-space "photons" can then be given in terms of an $effective$ temporal variable T comprised of all the temporal coordinates in the interval of eq-(1.31).

In order to simplify matters let us work with $D = 3$ instead of $D = 4$. The effective temporal variable T is defined as

$$c^2 (dT)^2 \equiv c^2 (dt)^2 + \frac{1}{c^2} (\frac{dX_{01}}{dt})^2 + \frac{1}{c^2} (\frac{dX_{02}}{dt})^2 + \frac{1}{L^2 c^2} (\frac{dX_{012}}{dt})^2 \tag{1.34}$$

so that the C-space interval can be rewritten, after factoring out the $c^2(dT)^2$ term, as

$$(d\Sigma)^2 = - c^2 (dT)^2 \left(1 - \frac{L^2}{c^2} (\frac{ds}{dT})^2 - \frac{1}{c^2} (\frac{dX_1}{dT})^2 - \frac{1}{c^2} (\frac{dX_2}{dT})^2 - \frac{1}{L^2 c^2} (\frac{dX_{12}}{dT})^2 \right) \tag{1.35}$$

The last expression has the same functional form as the ordinary spacetime interval in MInkowski space. Namely one can write the C-space interval $(d\Sigma)^2$ in the form

$$(d\Sigma)^2 = - c^2 (dT)^2 (1 - \frac{V^2}{c^2}) \tag{1.36}$$

where the generalization of the magnitude-squared of the spatial velocity divided by c^2 is

$$\frac{V^2}{c^2} \equiv \frac{L^2}{c^2} (\frac{ds}{dT})^2 + \frac{1}{c^2} (\frac{dX_1}{dT})^2 + \frac{1}{c^2} (\frac{dX_2}{dT})^2 + \frac{1}{L^2 c^2} (\frac{dX_{12}}{dT})^2 \tag{1.37}$$

Another description of C-space Photons is obtained from the null C-space interval condition $(d\Sigma)^2 = 0$ which is equivalent to setting $V^2/c^2 = 1$ in eq-(1.37) and where the velocity squared is defined with respect to the effective temporal variable T.

To finalize let us write down the addition law of generalized velocities based on the extended Lorentz transformations described in this work. Upon defining $\beta = -\beta'$ in eqs-(1.13) and differentiating gives

$$dX'_3 = dX_3 \ cosh\beta + L^{-3} \ dX_{0123} \ sinh\beta \tag{1.38a}$$

$$dX'_{0123} = dX_{0123} \ cosh\beta + L^3 \ dX_3 \ sinh\beta \tag{1.38b}$$

such that

$$\frac{dX'_3}{dX'_{0123}} = \frac{\frac{dX_3}{dX_{0123}} + L^{-3} \ tanh\beta}{1 + L^3 \ \frac{dX_3}{dX_{0123}} \ tanh\beta} \tag{1.39}$$

Using the following definitions of the generalized velocities (in $c = 1$ units)

$$V_3 \equiv \frac{dX_3}{dX_{0123}}, \quad V''_3 \equiv L^{-3} \ tanh\beta, \tag{1.40}$$

corresponding, respectively, to the generalized velocity V_3 of a polyparticle with respect to the temporal 4-volume X^{0123} coordinate (as measured in a given frame of reference) and the generalized velocity V''_3 of a *moving* observer associated with the generalized boost transformation with parameter β. Hence, eq-(1.39) can be rewritten as

$$V'_3 = \frac{V_3 + V''_3}{1 + \frac{V_3 \ V'_3}{L^{-6}}} \tag{1.41}$$

leading to the *addition* law of the generalized velocities. In particular, one can see that if the *maximal* generalized velocity is identified with the quantity cL^{-3}, after restoring the speed of light that was set to unity, we have that the addition/subtraction law of the maximal generalized velocities cL^{-3} yields always the maximal generalized velocity

$$V'_3 \;=\; \frac{V_3 \;\pm\; V''_3}{1 \;\pm\; \frac{V_3\; V''_3}{L^{-6}c^2}} \;=\; \frac{L^{-3}c \;\pm\; L^{-3}c}{1 \;\pm\; \frac{L^{-3}c\; L^{-3}c}{L^{-6}c^2}} \;=\; L^{-3}c\,\frac{1\;\pm\;1}{1\;\pm\;1} \;=\; L^{-3}\,c \qquad (1.42)$$

so that the *maximal* velocity cL^{-3} is never surpassed and it is a C-space relativistic *invariant* quantity. Meaning also that if the velocities of two polyparticles in a given reference frame is maximal cL^{-3}, their relative velocity is also maximal resulting from the subtraction law in eq-(1.42).

Following the same procedure in eqs-(1.14) as performed above one arrives at

$$V'_{123} \;=\; \frac{V_{123} \;+\; V''_{123}}{1 \;+\; \frac{V_{1233}\; V'_{1233}}{L^2\; c^2}},\;\; V_{123} \;=\; c\,\frac{dX_{123}}{dX_{03}},\;\; V''_{123} \;=\; c\,L\,tanh(\beta),\;\; V'_{123} \;=\; c\,\frac{dX'_{123}}{dX'_{03}} \qquad (1.43)$$

where the maximal generalized velocity V_{123} is now cL. In general, the maximal values of the generalized velocities are c and cL^n where n is a positive, negative integer. The case $n = 0$ corresponds to a generalized velocity associated with polyvector-valued coordinates of the *same* grade [1]. Namely, $c\,(dX^{\mu_1\mu_2....\mu_n}/dX^{0\nu_1\nu_2....\nu_{n-1}})$ such that the maximal velocity is the speed of light. More research is warranted to explore many more novel consequences of Clifford Space Relativity. Progress in the construction of generalized gravitational theories in Clifford spaces can be found in [16]. We must remark that one has not been trying to "squeeze" new physics out of Clifford algebras in this work. One the contrary, it is the physics of p-branes that led us to Clifford space relativity in the first place.

1.4 Modified Dispersion Relations, Generalized Uncertainty Principle

Next we will show how the quadratic Casimir invariant in C-space leads to modified wave equations, dispersion laws and to the generalizations of the stringy-uncertainty principle relations. The on-shell mass condition for a *massless* polyparticle in the 2^4-dimensional C-space corresponding to a Clifford algebra in $D = 4$, can be rewritten in terms of the polyvector valued components of a wave polyvector $\mathbf{K}$, after setting $L = 1, \hbar = c = 1$ for simplicity, as

$$k^2 \;+\; K_\mu K^\mu \;+\; K_{\mu_1\mu_2}K^{\mu_1\mu_2} \;+\; ...... \;+\; K_{\mu_1\mu_2....\mu_4}K^{\mu_1\mu_2...\mu_4} \;=\; \mathcal{M}^2 \;=\; 0 \qquad (1.44)$$

A particular *slice* through the 2^4-dimensional C-space can be taken by imposing the set of algebraic conditions

[1] We should note that the coordinate $X^0 \equiv ct$ is chosen to have length dimensions.

$$k^2 \;=\; 0, \;\; K_{\mu_1\mu_2} K^{\mu_1\mu_2} \;=\; \lambda_1 \; (K_\mu K^\mu)^2 \;=\; \lambda_1 \; K^4 \qquad (1.45a)$$

$$K_{\mu_1\mu_2\mu_3} \, K^{\mu_1\mu_2\mu_3} \;=\; \lambda_2 \, (K_\mu K^\mu)^3 \;=\; \lambda_2 \, K^6, \;\; K_{\mu_1\mu_2\mu_3\mu_4} \, K^{\mu_1\mu_2\mu_3\mu_4} \;=\; \lambda_3 \, (K_\mu K^\mu)^4 \;=\; \lambda_3 \, K^8 \qquad (1.45b)$$

where the λ's are numerical parameters. Since k is the Clifford scalar part of the wave polyvector it is invariant under C-space transformations. Hence the condition $k^2 = 0$ will not break the C-space symmetry. However the other slice conditions in eqs-(1.45) will *break* the generalized (extended) Lorentz symmetry in C-space because these conditions are *not* preserved under the most general C-space transformations as described earlier. There will be only the residual standard Lorentz symmetry (in ordinary spacetime) remaining which preserves these conditions/constraints in eqs-(1.45).

Inserting the conditions of eqs-(1.45) into eq-(1.44), after setting $k^2 = 0$, yields the modified dispersion law

$$K^2 \; (\; 1 \; + \lambda_1 \; K^2 \; + \lambda_2 \; K^4 \; + \lambda_3 \; K^6 \;) \;=\; \mathcal{M}^2 - k^2 \;=\; 0 \qquad (1.46)$$

Upon writing explicitly

$$K^2 \;=\; K_\mu \; K^\mu \;=\; |\vec{K}|^2 \; - \; (K_0)^2 \;=\; |\vec{K}|^2 \; - \; (\omega)^2 \qquad (1.47)$$

in eq-(1.46), and solving the algebraic equation for ω in terms of $|\vec{K}|$ obtained from eq-(1.46) leads to $\omega \;=\; \omega(|\vec{K}|)$. Finally, the group velocity (after reinstating c) is given by

$$c(|\vec{K}|) \;=\; \frac{\partial\omega(|\vec{K}|)}{\partial|\vec{K}|} \;=\; c \; + \; ... \qquad (1.48)$$

The group velocity might be greater, smaller or equal to c. From eq-(1.46) one can deduce immediately that one solution is $K^2 = |\vec{K}|^2 \; - \; (\omega)^2 = 0 \Rightarrow \omega = |\vec{K}| \Rightarrow \frac{\partial\omega(|\vec{K}|)}{\partial|\vec{K}|} = 1$ (in $c = 1$ units) and as expected massless particles move at the speed of light. However, there are *other* solutions to eq-(1.46) besides the trivial one leading to *energy* dependent speed of propagation. Setting $K^2 = Z$ leads to a cubic equation inside the parenthesis of eq-(1.46)

$$1 \; + \lambda_1 \; Z \; + \lambda_2 \; Z^2 \; + \lambda_3 \; Z^3 \;=\; 0 \qquad (1.49)$$

that can be solved exactly in terms of the λ's parameters giving 3 roots $z_i(\lambda_1, \lambda_2, \lambda_3), \;\; i = 1, 2, 3$. The roots can be all real, or one real and a pair of complex conjugate roots. In the former case we have (after reinstating c and adjusting the proper units for z_i) the particular solutions are

$$K^2 \;=\; c^2 \; |\vec{K}|^2 \; - \; (\omega)^2 \;=\; z_i(\lambda_1, \lambda_2, \lambda_3), \quad \Rightarrow \; \omega \;=\; \sqrt{c^2|\vec{K}|^2 \; - z_i} \; \Rightarrow$$

$$c(|\vec{K}|) = \frac{\partial\omega(|\vec{K}|)}{\partial|\vec{K}|} \;=\; c \; \frac{c \; |\vec{K}|}{\sqrt{c^2 \; |\vec{K}|^2 \; - z_i}} \;=\; c \; \frac{\sqrt{(\omega)^2 + z_i}}{\omega} \quad i = 1, 2, 3 \qquad (1.50)$$

Therefore, from eq-(1.50) one has an *energy* dependent speed of propagation that can be superluminal if $z_i > 0$, or subluminal if $z_i < 0$, in the case one has 3 real roots to the cubic equation (1.49). One should add that after differentiating $c^2 \ |\vec{K}|^2 \ - \ (\omega)^2 \ = \ z_i$ in eq-(1.50) gives

$$2 \ c^2 \ |\vec{K}| \ d|\vec{K}| \ = \ 2 \ \omega \ d\omega \ \Rightarrow \ c^2 \ = \ \frac{\omega}{|\vec{K}|} \ \frac{d\omega}{d|\vec{K}|} \tag{1.51}$$

leading always to the standard relation $v_{group} \ v_{phase} \ = c^2$ between group and phase velocities for all the possible solutions. The above results were all obtained by setting the Clifford scalar part k of the wave polyvector to zero. The calculations in the simplest $D = 2$ case when $k^2 \neq 0$ can be found in [8] leading also to the possibility of superluminal propagation.

Thus the key *novel* results one obtains from our analysis of wave propagation in C-space when $k^2 = 0$ are :

1. Irrespective of the solutions found in eqs-(1.49,1.50) the standard dispersion relation $K^2 \ = \ c^2 |\vec{K}|^2 \ - \ (\omega)^2 \ = \ 0$ is *always* a solution to eq-(1.46) giving a constant speed of photon propagation. This is a valid solution to choose whether or not an energy-dependent photon speed is found.

2 . Because the *modified* dispersion relation in eq-(1.46) *is Lorentz invariant* since the proper norm $K^2 \ = \ c^2 |\vec{K}|^2 \ - \ (\omega)^2$ is Lorentz invariant, one is able to arrive at the energy-dependent speed of propagation $c(|\vec{K}|)$ in eqs-(1.50) while still *retaining* the Lorentz symmetry. This does *not* occur in DSR nor in other approaches.

The on-shell mass condition for a massive polyparticle moving in the 2^4-dimensional flat C-space, corresponding to a Clifford algebra in $D = 4$, can be written in terms of the polymomentum (polyvector-valued) components, in natural units $L = L_P = 1, \hbar = c = 1$, as

$$\pi^2 \ + \ p_\mu \ p^\mu \ + \ p_{\mu_1 \mu_2} \ p^{\mu_1 \mu_2} \ + \ p_{\mu_1 \mu_2 \mu_3} \ p^{\mu_1 \mu_2 \mu_3} \ + \ p_{\mu_1 \mu_2 .... \mu_4} p^{\mu_1 \mu_2 ... \mu_4} \ = \ - \ \mathcal{M}^2 \tag{1.52}$$

Let us *break* the ordinary Lorentz invariance by imposing the non-Lorentz invariant conditions on the poly-momenta in C-space

$$p_{ij} \ p^{ij} \ = \ \beta_1 \ |\vec{p}|^4, \ \ p_{ijk} \ p^{ijk} \ = \ \beta_2 \ |\vec{p}|^6$$
$$p_{0i} \ p^{0i} \ = \ \alpha_1 \ (p_0)^2 \ |\vec{p}|^2, \ \ p_{0ij} \ p^{0ij} \ = \ \alpha_2 \ (p_0)^2 \ |\vec{p}|^4, \ \ p_{0ijk} \ p^{0ijk} \ = \ \alpha_3 \ (p_0)^2 \ |\vec{p}|^6 \tag{1.53}$$

where the α's and β's are numerical parameters. The mass-shell condition in C-space $P_A P^A = -\mathcal{M}^2$ becomes after inserting the conditions (1.53) and taking into account the chosen signature $(-, +, +, +)$

$$|\vec{p}|^2 \ \left(\ \frac{\pi^2}{|\vec{p}|^2} \ +1 \ + \beta_1 \ |\vec{p}|^2 \ + \beta_2 \ |\vec{p}|^4 \ \right) - (p_0)^2 \ \left(\ 1 \ + \alpha_1 \ |\vec{p}|^2 \ + \alpha_2 \ |\vec{p}|^4 \ + \ \alpha_3 \ |\vec{p}|^6 \ \right) \ = \ - \ \mathcal{M}^2 \tag{1.54}$$

One may notice that the terms inside the parenthesis in eq-(1.54) behave as if one had a *rainbow* metric as follows

$$g^{ij}(\pi^2,\ |\vec{p}|^2)\ p_i\ p_j\ +\ g^{00}(|\vec{p}|^2)\ p_0\ p_0\ =\ g^2(\pi^2,\ |\vec{p}|^2)\ |\vec{p}|^2\ -\ f^2(|\vec{p}|^2)\ E^2\ =\ -\ \mathcal{M}^2 \quad (1.55)$$

A rainbow metric [20] is a one-parameter family of metrics which depends on the energy (momentum) of the test particles moving in a given spacetime background, and forming a rainbow of metrics (rainbow geometry). Setting $\pi^2 = 0$ in eq-(1.55) one has then that the squared rainbow functions are given by

$$g^2(\pi^2 = 0,\ |\vec{p}|^2)\ \equiv\ 1\ +\beta_1\ |\vec{p}|^2\ +\beta_2\ |\vec{p}|^4,\quad \beta_1, \beta_2 > 0 \qquad (1.56a)$$

$$f^2(|\vec{p}|^2)\ \equiv\ 1\ +\alpha_1\ |\vec{p}|^2\ +\alpha_2\ |\vec{p}|^4\ +\ \alpha_3\ |\vec{p}|^6,\quad \alpha_1, \alpha_2, \alpha_3 > 0 \qquad (1.56b)$$

Given

$$g^{ij}\ =\ g^2(\pi^2 = 0,\ |\vec{p}|^2)\ \delta^{ij}\ =\ \Big(\ 1\ +\beta_1\ |\vec{p}|^2\ +\beta_2\ |\vec{p}|^4\ \Big)\ \delta^{ij} \qquad (1.57a)$$

$$g^{00}\ =\ -\ f^2(|\vec{p}|^2)\ \delta^{00}\ =\ -\ \Big(\ 1\ +\alpha_1\ |\vec{p}|^2\ +\alpha_2\ |\vec{p}|^4\ +\ \alpha_3\ |\vec{p}|^6\ \Big) \qquad (1.57b)$$

the *rainbow* metric is then *defined* as

$$ds^2\ =\ g_{\mu\nu}\ dx^\mu\ dx^\nu\ =$$

$$-\ \Big(\ 1\ +\alpha_1\ |\vec{p}|^2\ +\alpha_2\ |\vec{p}|^4\ +\ \alpha_3\ |\vec{p}|^6\ \Big)^{-1}\ (dt)^2\ +\ \Big(\ 1\ +\beta_1\ |\vec{p}|^2\ +\beta_2\ |\vec{p}|^4\ \Big)^{-1}\ (dx^i)^2 \qquad (1.58)$$

Another physical consequence is that the rainbow metric (1.58) when $\alpha_3 = 0;\ \alpha_1 = \beta_1; \alpha_2 = \beta_2$ yields *modifications* of the Weyl-Heisenberg algebra

$$[x^\mu,\ p^\nu]\ =\ i\ \hbar\ g^{\mu\nu}(|\vec{p}|^2) \qquad (1.59)$$

resulting from the momentum-dependent metric (1.58), and which in turn leads to the following uncertainty relations

$$\Delta x^\mu\ \Delta p^\nu\ \geq\ \frac{\hbar}{2}\ |\ <\Big(\ 1\ +\alpha_1\ |\vec{p}|^2\ +\alpha_2\ |\vec{p}|^4\ \Big)>\ \eta^{\mu\nu}\ | \qquad (1.60)$$

where $< .... >$ denote the QM expectation values $< \Psi|......|\Psi >$. See [21] for rigorous mathematical details.

From (1.60) one arrives at the minimal length stringy uncertainty relations [22]

$$\Delta x\ \Delta p_x\ \geq\ \frac{\hbar}{2}\ \Big(\ 1\ +\alpha_1\ (\Delta p_x)^2\Big)\ \Rightarrow\ \Delta x\ \geq\ \frac{\hbar}{2\Delta p_x}\ +\ (\frac{\hbar\alpha_1}{2})\ \Delta p_x \qquad (1.61)$$

Minimizing the expression in (1.61) and inserting the Planck scale L_P which was set to unity one has for the minimum position uncertainty a quantity of the order of the Planck scale

$$(\Delta x)_{min}\ =\ L_P\ \sqrt{\alpha_1},\quad \alpha_1 > 0 \qquad (1.62)$$

Higher order corrections to the stringy uncertainty relations in eq-(1.62) stem from the higher grade polymomentum variables in C-space appearing in eq-(1.61) and correspond, physically, to the membrane contributions to the modified uncertainty relations. Hence, the stringy and membrane corrections to the uncertainty relations in $D = 4$ are of the form (similar equations follow for the other spatial coordinates)

$$\Delta x \ \Delta p_x \ \geq \ \frac{\hbar}{2} \, [\ 1 \ + \alpha_1 \ (\Delta p_x)^2 \ + \ \alpha_2 \ (\Delta p_x)^4 \ ] \tag{1.63}$$

leading to

$$\Delta x \ \geq \ \frac{\hbar}{2} \, [\ \frac{1}{\Delta p_x} \ + \alpha_1 \ (\Delta p_x) \ + \ \alpha_2 \ (\Delta p_x)^3 \ ] \tag{1.64}$$

the extremization problem of (1.64) is more complicated but there is a local minimum when $\alpha_1 > 0, \alpha_2 > 0$. The value of Δp_x which yields the local minimum for Δx is

$$(\Delta p_x)_o \ = \ \left(\frac{- \ \alpha_1 \ + \ \sqrt{(\alpha_1)^2 + 12\alpha_2}}{6\alpha_2} \right)^{\frac{1}{2}}, \quad \alpha_1 > 0, \ \alpha_2 > 0 \tag{1.65}$$

If one sets the above value of $(\Delta p_x)_o$ and minimal length uncertainty to coincide with the Planck momentum and Planck scale, respectively, one can fix the numerical values of α_1, α_2. In higher dimensions than $D = 4$ one will capture the p-brane contributions beyond the membrane case due to the contributions of the higher grade polymomenta components. The dimensions (units) of the parameters in eqs-(1.63-1.65) are $[\alpha_1] = (L/\hbar)^2$, $[\alpha_2] = (L/\hbar)^4$.

Related to the minimal length uncertainty in eq-(1.62) one should mention that the theory of Scale Relativity proposed by Nottale [23] is based on a minimal observational length-scale, the Planck scale, as there is in Special Relativity a maximum speed, the speed of light, and deserves to be looked within the Clifford algebraic perspective. In future work we shall address the fractal nature of quantum spacetime [23] within the framework of quantum Clifford algebras and Scale Relativity. In the quantization program of gravity a key role must be played by quantum Clifford-Hopf algebras since the latter q-Clifford algebras naturally contain the κ-deformed Poincare algebras [89], [90], which are essential ingredients in the formulation of DSR within the context of Noncommutative spaces. The Minkowski spacetime quantum Clifford algebra structure associated with the conformal group and the Clifford-Hopf alternative κ-deformed quantum Poincare algebra was investigated [91].

1.5 Generalized Lorentz Transformations and Weyl-Heisenberg Algebra

We shall study next another *different* approach to the construction of generalized Lorentz transformations involving only polyvector components of *equal* grade. One may define a generalized Lorentz algebra in terms of anti-Hermitian operators $\mathcal{J}^{AB} = -\mathcal{J}^{BA}$ as [2]

$$[\ \mathcal{J}^{AB},\ \mathcal{J}^{CD}\ ]\ =\ -\ G^{AC}\ \mathcal{J}^{BD}\ +\ G^{AD}\ \mathcal{J}^{BC}\ -\ G^{BD}\ \mathcal{J}^{AC}\ +\ G^{BC}\ \mathcal{J}^{AD} \tag{1.66}$$

where $A, B, C, ...$ are polyvector-valued indices. One must emphasize that $\mathcal{J}^{AB} \neq [\Gamma^A, \Gamma^B]$, except in the case $\mathcal{J}^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$. To simplify matters, the generalized metric $G^{AB} = G^{BA}$ shall be chosen to be $G^{AB} = 0$ when the *grade* $A \neq$ *grade* B. And for the *same* grade metric components $g^{[a_1 a_2 ... a_k]\ [b_1 b_2 ... b_k]}$ of G^{AB}, the metric can decomposed into its irreducible factors as antisymmetrized sums of products of η^{ab} given by the following *determinant* [16]

$$G^{AB}\ \equiv\ det \left(\begin{array}{cccc} \eta^{a_1 b_1} & \ldots & \ldots & \eta^{a_1 b_k} \\ \eta^{a_2 b_1} & \ldots & \ldots & \eta^{a_2 b_k} \\ ------------ & & ---------------- & \\ \eta^{a_k b_1} & \ldots & \ldots & \eta^{a_k b_k} \end{array} \right)\ =\ G^{BA} \tag{1.67}$$

The spacetime signature is chosen to be $(-, +, +, ..., +)$.

One can verify next that a realization of the algebra (1.66) can be obtained in terms of polyvector-valued coordinates and momenta $\hat{X}^A, \hat{P}^B$ operators obeying the generalized Weyl-Heisenberg algebra

$$[\hat{X}^A,\ \hat{P}^B]\ =\ i\ (\hbar)^{(|A|+|B|)/2}\ G^{AB},\quad G^{AB} = G^{BA} \tag{1.68}$$

where $|A|, |B|,$ = grade of A, B, respectively.

The C-space polyvector-valued momentum is defined as

$$\mathbf{P}\ =\ \mathcal{M}\ \frac{d\mathbf{X}}{d\Sigma}\ =\ P^A\ \Gamma_A\ =\ \pi\ +\ p^\mu\ \gamma_\mu\ +\ p^{\mu\nu}\ \gamma_\mu \wedge \gamma_\nu\ +\ ...... \tag{1.69}$$

where $(d\Sigma)^2 =< d\mathbf{X}^\dagger\ d\mathbf{X} >$. Σ is the analog of "proper time" in C-space. To match physical units, powers of a suitable mass/length parameter must be introduced in eq-(1.69). Like the Planck mass and length. If $\mathbf{X}$ and $\mathbf{P}$ are taken to have length and momentum dimensions, respectively, then $\mathcal{M}$ has mass dimensions. By inspection one learns that the commutator of the *zero* grade components, the scalar parts of $\hat{X}^A$ and $\hat{P}^B$, does not involve $\hbar$ but a *dimensionless* parameter that can be given by the ratio of an ultraviolet L_P and infrared Hubble scale R_H as follows

$$[\hat{s},\ \hat{\pi}]\ =\ i\ \frac{L_P}{R_H}\ G^{**} \tag{1.70}$$

G^{**} is the scalar-scalar component of the generalized metric G^{AB}. The classical limit is attained when $L_P/R_H\ \rightarrow\ 0$ so that the above commutator vanishes. This ratio

[2]We choose anti-Hermitian operators in order to avoid having to introduce $\mathbf{i}$ factors in the right hand side of the commutators

L_P/R_H is also related to the observed vacuum energy density $\rho \sim (L_P/R_H)^2\ (M_P)^4 \sim 10^{-122}\ (Planck\ Mass)^4$.

Hence, if $\mathcal{J}^{AB} = 0$ when the grade $A \neq$ grade B, a Weyl-Heisenberg algebra allows to find a realization of the dimensionless anti-Hermitian generators $\mathcal{J}^{AB}$ in eq-(1.66) as follows

$$\mathcal{J}^{AB} = \frac{i}{\hbar^{(|A|+|B|)/2}} \left(\hat{X}^A\ \hat{P}^B - \hat{X}^B\ \hat{P}^A \right) = -\mathcal{J}^{BA}, \ \ \mathcal{J}^{AB} = 0\ if\ |A| \neq |B| \quad (1.71)$$

$\hat{X}^A$ and $\hat{P}^B$ are Hermitian operators.

To sum up, when $|A| = |B|$, $G^{AB} \neq 0, \mathcal{J}^{AB} \neq 0$; and $G^{AB} = 0, \mathcal{J}^{AB} = 0$ for $|A| \neq |B|$, a generalization of the Poincare algebra involving polyvector-valued indices is given by the commutators in eq-(1.66) and

$$[\ \mathcal{J}^{AB},\ \hat{P}^C\ ] = -\ G^{AC}\ \hat{P}^B + G^{BC}\ \hat{P}^A, \ \ [\hat{P}^A,\ \hat{P}^B] = 0, \ \ [\hat{X}^A,\ \hat{X}^B] = 0, \quad (1.72)$$

where $\hat{P}^A$ are the polymomentum operators and $\mathcal{J}^{AB}$ are the generalized Lorentz generators. The $[\mathcal{J}^{AB}, \mathcal{J}^{CD}], [\mathcal{J}^{AB}, \hat{P}^C], ...$ commutators obey the Jacobi identities.

A generalization of the Poincare algebra permits the construction of gauge theories of extended gravitational theories in *curved* C-spaces in term of the analogs of a vielbein E^A_M and spin connection Ω^{AB}_M. The generalized connection is $\mathcal{A}_M = E^A_M\ \mathcal{P}_\mathcal{A} + \Omega^{AB}_M\ \mathcal{J}_{AB}$. There is a nontrivial torsion as shown in [16].

A question still remains whether or not it is possible to construct the generators of the algebra displayed by eq-(1.66) in terms of a judicious *superposition* of Clifford algebra generators like

$$\mathcal{J}^{AB} = M^{AB}_{\ \ C}\ \Gamma^C \quad (1.73)$$

By inspection one learns that $\mathcal{J}^{AB} \neq [\Gamma^A, \Gamma^B]$, nor proportional to the commutators, except in the case $\mathcal{J}^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$. Therefore, the coefficients $M^{AB}_{\ \ C} \neq f^{AB}_{\ \ C}$ are not given by the structure constants. Inserting the ansatz of (1.73) into the commutators (1.66) leads to an algebraic set of equations involving $M^{AB}_{\ \ C}, f^{AB}_{\ \ C}, G^{AB}$ as the indices A, B, C run from 1 to 2^D. It is unknown (to our knowledge) if a solution for the coefficients $M^{AB}_{\ \ C}$ exists given the complexity of the (anti) commutator relations in any dimension provided in the Appendix. A computer algebra package would be required.

2 Generalized Gravity in Curved Clifford Spaces

2.1 The Differential Geometry of Curved C-spaces

In curved C-space [1], [7] one introduces the $\mathbf{X}$-dependent basis generators γ_M, γ^M defined in terms of the beins E^A_M, inverse beins E^M_A and the flat tangent space generators γ_A, γ^A as follows $\gamma_M = E^A_M(\mathbf{X})\gamma_A, \gamma^M = E^M_A(\mathbf{X})\gamma^A$. The curved C- space metric expression

$g_{MN} = E_M^A E_N^B \eta_{AB}$ also agrees with taking the scalar part of the Clifford geometric product $< \gamma_M \ \gamma_N >= g_{MN}$.

From now one we shall denote the curved C-space basis generators γ_M, γ^M by E_M, E^M, and the flat tangent space generators γ_A, γ^A by E_A, E^A. The indices $A, B, C, ...$ from the beginning of the alphabet represent the tangent space indices, while those from the middle of the alphabet $L, M, N, ...$ represent the base world indices. The covariant derivative of $E_M^A(\mathbf{X}), E_A^M(\mathbf{X})$ involves the generalized connection and spin connection and are defined as

$$\nabla_K E_M^A \ = \ \partial_K E_M^A \ - \ \Gamma_{KM}^L \ E_L^A \ + \ \omega_{KB}^A \ E_M^B \tag{2.1a}$$

$$\nabla_K E_A^M \ = \ \partial_K E_A^M \ + \ \Gamma_{KL}^M \ E_A^L \ - \ \omega_{KA}{}^B \ E_B^M \tag{2.1b}$$

If the nonmetricity is zero then $\nabla_K E_M^A = 0, \ \nabla_K E_A^M = 0$ in eqs-(2.1).

The coefficients (functions) $W_{LM}{}^N$ associated to the Clifford geometric product are defined by

$$E_A \ E_B \ = \ W_{AB}{}^C \ E_C, \ \ given \ \ E_L = E_L^A \ E_A, \ E_M \ = \ E_M^A \ E_A \ \Rightarrow$$

$$E_L \ E_M \ = \ W_{LM}{}^N \ E_N \Rightarrow W_{LM}{}^N \ = \ E_L^A \ E_M^B \ E_C^N \ W_{AB}{}^C \tag{2.2}$$

the Clifford algebra structure functions $f_{LM}{}^N, d_{LM}{}^N$ are defined by

$$[E_A, \ E_B] \ = \ f_{AB}{}^C \ E_C, \ \ [E_L, \ E_M] \ = \ f_{LM}{}^N \ E_N \Rightarrow f_{LM}{}^N \ = \ E_L^A \ E_M^B \ E_C^N \ f_{AB}{}^C \tag{2.3}$$

$$\{E_A, \ E_B\} \ = \ d_{AB}{}^C \ E_C, \ \ \{E_L, \ E_M\} \ = \ d_{LM}{}^N \ E_N \Rightarrow d_{LM}{}^N \ = \ E_L^A \ E_M^B \ E_C^N \ d_{AB}{}^C \tag{2.4}$$

Due to the antisymmetry property $\Omega_{KAB} = -\Omega_{KBA}$ of the generalized spin connection one has

$$\nabla_K(\eta_{AB}) \ = \ - \ \Omega_{KA}{}^C \ \eta_{CB} \ - \ \Omega_{KB}{}^C \ \eta_{AC} \ = \ - \ (\Omega_{KAB} \ + \ \Omega_{KBA} \ ) \ = \ 0 \tag{2.5}$$

as expected and such that

$$\nabla_K(g_{MN}) \ = \ \nabla_K(E_M^A \ E_N^B \ \eta_{AB}) \ = \ 0 \ \Rightarrow \ \nabla_K E_M^A \ = \ 0 \tag{2.6}$$

From

$$\nabla_K(E_M^A) \ = \ 0 \ \Rightarrow \ \partial_K(E_M^A) \ - \ \Gamma_{KM}{}^L \ E_L^A \ + \ \Omega_{KB}^A \ E_M^B \ = \ 0 \ \Rightarrow$$

$$\partial_K(E_M^A) \ = \ \Gamma_{KM}{}^L \ E_L^A \ - \ \Omega_{KB}^A \ E_M^B \tag{2.7}$$

one obtains the relationship between the connection and the spin connection. Having

$$\nabla_K(E_M^A) \ = \ 0 \ \Rightarrow \ \nabla_K(E_M) \ = \ \nabla_K(E_M^A \ E_A) \ = \ E_M^A \ \nabla_K E_A \ =$$

$$E_M^A \ (\ \partial_K E_A \ - \ \Omega_{KA}{}^B \ E_B \ ) \ = \ 0 \ \Rightarrow$$

$$\partial_K \; E_A \;=\; \Omega_{KA}^{\;\;\;B} \; E_B \tag{2.8}$$

Hence under parallel transport, $\partial_K E_A$, the tangent space basis E_A generators are rotated as displayed by eq-(2.8). More details of the role of the generalized spin connection in C-spaces can be found in [7].

The result $\nabla_K(E_M) = 0$ is also consistent with the zero nonmetricity condition

$$\nabla_K \; g_{MN} \;=\; \nabla_K \;<\; E_M \; E_N \;>\;=\;<\; \nabla_K(E_M) \; E_N \;>\;+\;<\; E_M \; \nabla_K(E_N) \;>\;=\; 0 \tag{2.9}$$

therefore, the Clifford algebra basis elements E_M in a *curved* C-space are covariantly constant with respect to a metric-compatible connection $\nabla_K g_{MN} = 0$.

Upon taking derivatives on both sides of the equalities in eqs-(2.2-2.4) and after using eqs-(2.7, 2.8) gives the covariantly constancy conditions of the structure functions

$$\nabla_K(f_{LMN}) = 0, \quad \nabla_K(d_{LMN}) = 0, \quad \nabla_K(W_{LMN}) = 0 \tag{2.10}$$

A careful analysis reveals that eq-(2.10) does *not* impose any *additional* constraints on the generalized connection and spin connection. This result is an improvement over our prior findings in [10] and is consistent with the fact that performing a derivative operation on both sides of an *equality* should not introduce additional constraints on the connection.

For simplicity we shall set the nonmetricity Q^L_{MN} to zero. In Appendix **B** we show that the *torsionless* Levi-Civita connection is given by

$${}^{(lc)}\Gamma^L{}_{MN} \;=\; \{^{\,L}_{MN}\} \;+\; \frac{1}{2}\, g^{LK} \;(\; f_{MKN} \;+\; f_{NKM} \;+\; f_{MNK} \;) \tag{2.11}$$

where

$$\{^{\,L}_{MN}\} \;=\; \frac{1}{2}\, g^{LK} \;(\; \partial_N \; g_{KM} \; + \partial_M \; g_{KN} \; - \partial_K \; g_{MN} \;) \tag{2.12}$$

and f_{MKN} are the Clifford algebra structure functions (coefficients). We should notice that the Levi-Civita connection in eq-(2.11) has a symmetric ${}^{(lc)}\Gamma^L{}_{(MN)}$ and antisymmetric ${}^{(lc)}\Gamma^L{}_{[MN]}$ piece. The symmetric piece is given by the first three terms in (2.11), while the antisymmetric piece is given by the last term in (2.11).

The Torsion is defined by

$$\mathbf{T} \;=\; \nabla_{\mathbf{X}} \, \mathbf{Y} \;-\; \nabla_{\mathbf{Y}} \, \mathbf{X} \;-\; [\mathbf{X}, \mathbf{Y}] \tag{2.13}$$

so that by inspection one can see that the LC connection (2.11) is torsionless

$${}^{(lc)}T_{MN}^{\;\;\;\;L} \;\equiv\; {}^{(lc)}\Gamma^L{}_{MN} \;-\; {}^{(lc)}\Gamma^L{}_{NM} - f_{MN}^{\;\;\;\;L} \;=\; 0 \tag{2.14}$$

The last term $-f_{MN}^{\;\;\;\;L}$ in the expression for the torsion (2.14) originates from the non-vanishing $[\mathbf{X}, \mathbf{Y}] \neq 0$ contribution and resulting from the fact that $[E_M, E_N] = f_{MN}^{\;\;\;\;L} \; E_L \neq 0$.

The Torsion can be introduced explicitly by the addition of the contorsion term $K^L{}_{MN}$

$$\Gamma^L{}_{MN} \;=\; {}^{(lc)}\Gamma^L{}_{MN} \;+\; K^L{}_{MN} \tag{2.15}$$

The contorsion tensor is defined in terms of the components of the torsion tensor as

$$K^L{}_{MN} \;=\; \frac{1}{2}\,(T_M{}^L{}_N \;+\; T_N{}^L{}_M \;+\; T^L{}_{MN}), \;\; T^L{}_{MN} \;=\; -\; T^L{}_{NM} \tag{2.16}$$

so that now the torsion is no longer zero $T^L{}_{MN} = \Gamma^L{}_{MN} - \Gamma^L_{NM} - f^L{}_{MN} \neq 0$.

After recurring to the result in eq-(2.7) $\partial_K(E^A_M) = \Gamma_{KM}{}^L\; E^A_L - \Omega^A_{KB}\; E^B_M$ and defining $T^L_{MN}\; E^A_L = T^A_{MN}$, one can verify that

$$T^A{}_{MN} \;=\; \partial_M\; E^A_N \;-\; \partial_N\; E^A_M \;+\; \Omega^A{}_{MB}\; E^B_N \;-\; \Omega^A{}_{NB}\; E^B_M \;-\; f_{MN}{}^L\; E^A_L \tag{2.17}$$

therefore, T^A_{MN} can be written in terms of the generalized spin connection and the generalized vielbeins . The expression (2.17) bears a resemblance with the Cartan structure equations for the torsion 2-form $\mathbf{T}^a = T^a_{\mu\nu}\; dx^\mu \wedge dx^\nu$ in ordinary spaces when it is written in terms of differential forms, exterior derivatives and exterior products

$$\mathbf{T}^a \;=\; \mathbf{d}\Theta^a \;+\; \omega^a{}_b \;\wedge\; \Theta^b, \;\; \Theta^a \equiv e^a_\nu\; dx^\nu, \;\; \omega^a{}_b \;=\; \omega^a{}_{b\;\mu}\; dx^\mu \tag{2.18}$$

The curvature is defined as

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\;\mathbf{Z} \;=\; [\nabla_{\mathbf{X}},\; \nabla_{\mathbf{Y}}]\;\mathbf{Z} \;-\; \nabla_{[\mathbf{X},\mathbf{Y}]}\;\mathbf{Z} \tag{2.19}$$

such that the explicit curvature components are given by

$$\mathbf{R}_{MNJ}{}^K \;=\; \partial_M\; \Gamma_{NJ}{}^K \;-\; \partial_N\; \Gamma_{MJ}{}^K \;-\; \Gamma_{MJ}{}^L\; \Gamma_{NL}{}^K \;+\; \Gamma_{NJ}{}^L\; \Gamma_{ML}{}^K \;-\; f_{MN}{}^L\; \Gamma_{LJ}{}^K \tag{2.20}$$

In Appendix **D** it is shown explicitly that the curvature (2.20) transforms homogeneously under coordinate transformations $X^M \rightarrow \tilde{X}^M(X^N)$ despite that the connection Γ^K_{MJ} transforms inhomogeneously.

The above curvature expression has a similarity to the *nonholonomic* coordinates description of the curvature tensor in ordinary spacetime, where one replaces the derivatives $\partial_\mu, \partial_\nu, ...$ with the derivative operators $\hat{e}_a = e^\mu_a \partial_\mu, \hat{e}_b = e^\nu_b \partial_\nu$ which are defined in terms of the inverse vielbeins; one replaces $\Gamma^\rho_{\mu\nu}$ with Γ^c_{ab} and instead of using the structure functions (coefficients) of the Clifford algebra one uses the nonholonomy coefficients defined by $[\hat{e}_a, \hat{e}_b] \;=\; C^c_{ab}\; \hat{e}_c$. To sum up, because the Clifford algebra structure functions (coefficients) are not zero $f_{MN}^{\;\;L} \neq 0$ one must *include* them into the definitions of the torsion and curvature. In the curvature case there are terms $f_{MN}{}^L \Gamma_{LJ}{}^K$ as displayed in eq-(2.20). While in the torsion case we must include the term $f_{MN}^{\;\;L}$ as shown in eq-(2.17).

The same-grade C-space metric components obeying $g_{MN} = g_{NM}$ are of the form

$$g_{\mathbf{00}}, \;\; g_{\mu\nu}, \;\; g_{\mu_1\mu_2\;\nu_1\nu_2}, \;\; \cdots, \;\; g_{\mu_1\mu_2...\mu_D\;\nu_1\nu_2...\nu_D} \tag{2.21}$$

In the most general case the metric *does not factorize* into antisymmetrized sums of products of the form

$$g_{[\mu_1\mu_2]\;[\nu_1\nu_2]}(x^\mu) \;\neq\; g_{\mu_1\nu_1}(x^\mu)\; g_{\mu_2\nu_2}(x^\mu) \;-\; g_{\mu_2\nu_1}(x^\mu)\; g_{\mu_1\nu_2}(x^\mu)$$

$$g_{[\mu_1\mu_2\ldots\mu_k]\ [\nu_1\nu_2\ldots\nu_k]}(x^\mu)\ \neq\ det\ G_{\mu_i\nu_j}\ =\ \epsilon^{j_1j_2\ldots j_k}\ g_{\mu_1\nu_{j_1}}\ g_{\mu_2\nu_{j_2}}\ \ldots\ g_{\mu_2\nu_{j_k}},\quad k=1,2,3,\ldots D \tag{2.22}$$

The determinant of $G_{\mu_i\nu_j}$ can be written as

$$det\left(\begin{array}{cccc} g_{\mu_1\nu_1}(x^\mu) & \ldots & \ldots & g_{\mu_1\nu_k}(x^\mu) \\ g_{\mu_2\nu_1}(x^\mu) & \ldots & \ldots & g_{\mu_2\nu_k}(x^\mu) \\ ------------ & & --------------- & \\ g_{\mu_k\nu_1}(x^\mu) & \ldots & \ldots & g_{\mu_k\nu_k}(x^\mu) \end{array}\right), \tag{2.23}$$

The metric component $g_{\mathbf{00}}$ involving the scalar "directions" in C-space of the Clifford polyvectors must also be included. It behaves like a Clifford scalar. The other component $g_{[\mu_1\mu_2\ldots\mu_D]\ [\nu_1\nu_2\ldots\nu_D]}$ involves the pseudo-scalar "directions". The latter scalar and pseudo-scalars might bear some connection to the dilaton and axion fields in Cosmology and particle physics.

The Bianchi identities when the torsion is zero are given by

$$\mathbf{R}_{MNJK}\ +\ \mathbf{R}_{NJMK}\ +\ \mathbf{R}_{JMNK}\ =\ 0 \tag{2.24}$$

$$\nabla_L(\mathbf{R}_{MNJK})\ +\ \nabla_M(\mathbf{R}_{NLJK})\ +\ \nabla_N(\mathbf{R}_{LMJK})\ =\ 0 \tag{2.25}$$

When the torsion is not zero there are nonvanishing terms in the right hand side of eqs-(2.24, 2.25) of the form $(\nabla+\mathbf{T})\mathbf{T}$, and $\mathbf{T}\times\mathbf{R}$, respectively, where $\mathbf{T}$ is the torsion and $\mathbf{R}$ is the curvature.

After multiplying the differential Bianchi identities (2.25), by $g^{MK}g^{NJ}$, and performing the contractions of polyvector-valued indices, one arrives at the vacuum field equations in C-space in the absence of torsion and nonmetricity

$$\mathbf{R}_{(MJ)}\ -\frac{1}{2}\ g_{MJ}\ \mathbf{R}\ =\ 0,\quad \mathbf{R}_{[MJ]}\ =\ 0 \tag{2.26}$$

where

$$\mathbf{R}_{(MJ)}\ \equiv\ \frac{1}{2}(\mathbf{R}_{MJ}+\mathbf{R}_{JM}),\quad \mathbf{R}_{[MJ]}\ \equiv\ \frac{1}{2}(\mathbf{R}_{MJ}-\mathbf{R}_{JM}) \tag{2.27}$$

Due to the fact that the Levi-Civita connection in eq-(2.11) has a symmetric ${}^{(lc)}\Gamma^L{}_{(MN)}$, and antisymmetric ${}^{(lc)}\Gamma^L{}_{[MN]}$ piece, $\mathbf{R}_{MJ}$ has a symmetric and anti-symmetric components. For this reason one must symmetrize the indices as displayed in the first expression of eq-(2.26). The on-shell value of the antisymmetric piece is $\mathbf{R}_{[MJ]}=0$.

One may include matter fields by introducing the C-space analog of the symmetric stress energy tensor $\mathbf{T}_{(MJ)}$ into the right-hand side of the first expression of eq-(2.26). While also introducing the antisymmetric piece of $\mathbf{T}_{MJ}$ into the right-hand side of the second expression of eq-(2.26) $\mathbf{R}_{[MJ]}\sim\mathbf{T}_{[MJ]}$.

The typical example of these sort of field equations, in ordinary spacetimes, are the field equations associated with the Einstein-Cartan-Dirac theory [24]. The nontrivial torsion $T^{\mu\nu\rho}$ tensor is generated (sourced) by the spin density tensor $S^{\mu\nu\rho}\sim\bar\Psi\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}\Psi$. In this case the torsion is non-propagating in the sense that it is an algebraic function given by fermion bilinear terms.

Up to numerical coefficients, the symmetric part of the stress energy tensor $\mathcal{T}_{(\mu\nu)}$ is of the form $\bar{\Psi}\gamma_{(\mu}\nabla_{\nu)}\Psi + g_{\mu\nu}S^{\alpha\beta\sigma}S_{\alpha\beta\sigma} + \cdots$, where $\nabla_\nu\Psi$ is defined in terms of the spin connection $\nabla_\nu\Psi = (\partial_\nu + \frac{1}{2}\omega_\nu^{ab}[\gamma_a, \gamma_b])\Psi$. Whereas the antisymmetric part $\mathcal{T}_{[\mu\nu]}$ of the stress energy tensor is of the form $\nabla_\alpha S^\alpha_{\mu\nu} + S^\beta_{\alpha\beta}S^\alpha_{\mu\nu} + \cdots$ [24].

To finalize this section we shall discuss the notion of poly-differential forms. In C-space one has now that

$$dx^{\mu\nu} \neq dx^\mu \wedge dx^\nu, \quad dx^{\mu\nu\rho} \neq dx^\mu \wedge dx^\nu \wedge dx^\rho, \ ..... \tag{2.28}$$

because the areal-coordinates $x^{\mu\nu}$, volume-coordinates $x^{\mu\nu\rho}$, ... associated with the world-sheet, world-volume, ... evolution of a string, membrane, ... are not related to the vector coordinates x^μ associated with the evolution of a point particle. For this reason the antisymmetry property of the poly-differential forms is given by $dX^M \wedge dX^N = - dX^N \wedge dX^M$. In particular one has the following combinations

$$dx^{\mu_1\mu_2\cdots\mu_{2m}} \wedge dx^{\rho_1\rho_2\cdots\rho_{2n}} = - dx^{\rho_1\rho_2\cdots\rho_{2n}} \wedge dx^{\mu_1\mu_2\cdots\mu_{2m}} \tag{2.29a}$$

$$dx^{\mu_1\mu_2\cdots\mu_{2m-1}} \wedge dx^{\rho_1\rho_2\cdots\rho_{2n-1}} = - dx^{\rho_1\rho_2\cdots\rho_{2n-1}} \wedge dx^{\mu_1\mu_2\cdots\mu_{2m-1}} \tag{2.29b}$$

$$dx^{\mu_1\mu_2\cdots\mu_{2m-1}} \wedge dx^{\rho_1\rho_2\cdots\rho_{2n}} = - dx^{\rho_1\rho_2\cdots\rho_{2n}} \wedge dx^{\mu_1\mu_2\cdots\mu_{2m-1}} \tag{2.29c}$$

$$dx^{\mu_1\mu_2\cdots\mu_{2m}} \wedge dx^{\rho_1\rho_2\cdots\rho_{2n-1}} = - dx^{\rho_1\rho_2\cdots\rho_{2n-1}} \wedge dx^{\mu_1\mu_2\cdots\mu_{2m}} \tag{2.29d}$$

and which $differs$ from the antisymmetry property of ordinary differential forms. Given an ordinary p-form $\mathbf{A}_p$ and an ordinary q-form $\mathbf{B}_q$ one has $\mathbf{A}_p \wedge \mathbf{B}_q = (-1)^{pq}\ \mathbf{B}_q \wedge \mathbf{A}_p$. The antisymmetry property displayed by the C-space poly-differential forms in eqs-(2.29) will ensure that the generalized curvature tensor is antisymmetric under the following exchange of polyvector-valued indices : $R_{MNJ}{}^K = - R_{NMJ}{}^K$.

The C-space poly-differential forms analogs of the Cartan-structure equations in ordinary spacetime are

$$\mathbf{T}^A = \mathbf{d}\Theta^A + \Omega^A{}_B \wedge \Theta^B, \quad \Theta^A \equiv E^A_M\ dX^M, \quad \Omega^A{}_B = \Omega^A{}_{B\ N}\ dX^N \tag{2.30a}$$

$$\mathbf{R}^{AB} = \mathbf{d}\Omega^{AB} + \Omega^A{}_C \wedge \Omega^{CB}, \quad \mathbf{R}^{AB} = \mathbf{R}^{AB}_{MN}\ dX^M \wedge dX^N \tag{2.30b}$$

where A, B are the tangent space indices and M, N are the base (world) indices.

The above equations are the starting point to formulate a gauge theory of extended gravity in C-spaces based on the analogs of a vielbein E^A_M and spin connection Ω^{AB}_M. The generalized connection is $\mathbf{A}_M = E^A_M\ P_A + \Omega^{AB}_M\ J_{AB}$. P_A is the translation generator and J_{AB} is the generalized Lorentz generator. The connection poly-differential one-form is $\mathbf{A}_M\ dX^M$ and the poly-differential curvature 2-form is $\mathbf{R} = (\mathbf{d} + \mathbf{A}) \wedge \mathbf{A}$. In component form, the curvature is $\mathbf{R}_{MN}\ dX^M \wedge dX^N = (\mathbf{R}^A_{MN}P_A + \mathbf{R}^{AB}_{MN}J_{AB})\ dX^M \wedge dX^N$. This gauge theory approach to C-space gravity is the C-space generalization of the Poincare gauge theory formulation of ordinary gravity [11].

3 The Generalized Gravitational Action and the Cosmological Constant

In this section we shall derive the field equations from a variational action principle instead from the differential Bianchi identities. Before embarking into this final section we shall work with the natural units $\hbar = c = G = L_{Planck} = 1$. Upon performing contractions of the curvature yields the analog of the Ricci tensor $\delta^N_K \, \mathbf{R}_{MNJ}{}^K = \mathbf{R}_{MJ}$ and the Ricci scalar $g^{MJ}\mathbf{R}_{MJ} = \mathbf{R}$. One may then construct an Einstein-Hilbert-Cartan like action based on the C-space curvature scalar R

$$\frac{1}{2\kappa^2} \int ds \prod dx^\mu \prod dx^{\mu_1\mu_2} \ \ldots \ dx^{\mu_1\mu_2\ldots\mu_D} \ \mu_m(g_{MJ}) \ \mathbf{R} \ \equiv$$

$$\frac{1}{2\kappa^2} \int [\mathbf{DX}] \ \mu_m(g_{MJ}) \ \mathbf{R} \tag{3.1}$$

where $\mu_m(g_{MJ})$ is a suitable integration measure and κ^2 is the gravitational coupling constant in the 2^D-dimensional C-space.

At this point it is important to remark that the analog of the Ricci tensor $\mathbf{R}_{MJ} \neq \mathbf{R}_{JM}$ is *no* longer *symmetric* in the indices because $\mathbf{R}_{MJ}$ (and $\mathbf{R}$) are defined now in terms of the non-symmetric connection $\Gamma^K_{MN} \neq \Gamma^K_{NM}$ as displayed in eq-(2.14). There is an antisymmetric piece in the connection given explicitly by the very last term of eq-(2.14). The curvature scalar becomes $\mathbf{R} = g^{MJ}\mathbf{R}_{MJ} = g^{MJ}\mathbf{R}_{(MJ)} + g^{MJ}\mathbf{R}_{[MJ]} = g^{MJ}\mathbf{R}_{(MJ)}$. Hence, it is the symmetric part of the Ricci tensor analog that appears in the vacuum field equations below. Torsion can also be added to the connection explicitly in terms of the contorsion terms as shown in eqs-(2.18,2.19).

In a given coordinate system (a generalized Lorentz frame) the mixed-grade components of the metric g_{MN}, g^{MN}, beins E^A_M, inverse beins E^M_A, can be set to zero in order to considerably simplify the calculations; i.e. namely due to the very large diffeomorphism symmetry in C-space, one may choose a frame ("diagonal gauge") such that the *mixed* grade components of the metric g_{MN}, beins E^A_M, inverse beins E^M_A are zero. In this case the C-space metric components can be chosen to be given by the determinant expressions in eq-(2.26).

The advantage of having $g_{MN} = 0$ if the grade of M is not the same as the grade of N is that the determinant of the C-space metric can be factorized as the product of determinants of matrices which are comprised of entries given themselves by determinants (2.26) . If an ordering prescription of indices is introduced, $\mu_1 < \mu_2 < \ldots.\mu_n$ and $\nu_1 < \nu_2 < \ldots.\nu_n$, the bivector-bivector components of the C-space metric in $D = 4$ dimensions $g_{\mu_1\mu_2 \ \nu_1\nu_2}$ can be arranged into an *ordered* square array of entries given by a 6×6 matrix, since the number of independent bivector components in $D = 4$ is $4 \times 3/2 = 6$. For instance, the entries of the square 6×6 matrix $g_{\mu_1\mu_2 \ \nu_1\nu_2}$ are given themselves by determinants : $g_{12 \ 12} = g_{11}g_{22} - g_{12}g_{21}$; $g_{13 \ 13} = g_{11}g_{33} - g_{13}g_{31}$, ... etc, and such that its determinant is given by the ordinary determinant of an square 6×6 matrix.

The trivector-trivector components of the C-space metric in $D = 4$ dimensions $g_{\mu_1\mu_2\mu_3 \ \nu_1\nu_2\nu_3}$ can be arranged into an *ordered* square array of entries given by a 4×4 matrix, since the number of independent trivector components in $D = 4$ is $4 \times 3 \times 2/2 \times 3 = 4$.

The entries of this square 4×4 matrix are given themselves by the determinants as shown in eq-(2.26). Following a similar procedure with the other C-space metric components, in this way one can write the measure of integration in $D = 4$ as the square root of the product of determinants

$$\mu_m(g_{MJ}) = \sqrt{|g| \ |det \ (g_{\mu\nu})| \ |det \ (g_{\mu_1\mu_2 \ \nu_1\nu_2})| \ |det \ (g_{\mu_1\mu_2\mu_3 \ \nu_1\nu_2\nu_3})| \ |det \ (g_{\mu_1\mu_2\mu_3\mu_4 \ \nu_1\nu_2\nu_3\nu_4})|} \tag{3.2}$$

where g is the scalar-scalar part of the C-space metric. The generalization to other dimensions is straightforward.

In the most general case one can have a C-space metric with non-vanishing mixed grade components such that the metric g_{MJ} components can be assembled into arrays of ordered *rectangular* matrices. The problem becomes that one cannot longer define a determinant of a rectangular matrix. One can also view the g_{MJ} as a hyper-matrix but the construction (if possible) of the hyper-determinant of the C-space metric (a hyper matrix) is a more difficult problem [13], [14].

Despite that in the most general case the measure $\mu_m(g_{MJ})$ is *not* given by eq-(3.2) one can still assume that $\mu_m(g_{MJ})$ is a suitable measure of integration obeying the condition

$$\delta\mu_m(g_{MJ}) = - \frac{1}{2} \ \mu_m(g_{MJ}) \ g_{MJ} \ \delta g^{MJ} \tag{3.3}$$

and which is similar to the variational behavior of the square root of an ordinary determinant of the spacetime metric $\sqrt{|det \ g_{\mu\nu}|}$.

Before continuing some important remarks are in order. It is known that the definition of an alternative measure substantially affects the discussion of the cosmological constant problem, as has been found in the study of Two Measures Theories [18]. For example a metric independent measure will not satisfy eq-(3.3) [18]. In the most general case the measure is not given by eq-(3.2) and this would modify the discussion of the cosmological constant problem. By using two choices for the measure as in the Two Measures Theory one improves the behavior concerning the vacuum energy density since the discussion of the cosmological constant problem depends crucially on what vacuum one takes [18].

An alternative measure of integration in four dimensions independent of the metric can be obtained, for example, in terms of four scalars $\varphi^a (a = 1, 2, 3, 4)$ as follows [18] $\Phi = \epsilon^{\mu\nu\rho\tau} \epsilon_{abcd} (\partial_\mu \varphi^a)(\partial_\nu \varphi^b)(\partial_\rho \varphi^c)(\partial_\tau \varphi^d)$. Such measure of integration can trigger a number of remarkable physically important phenomena [18] such as: (i) a new mechanism of dynamical generation of the cosmological constant; (ii) a new type of "quintessential inflation" scenario in cosmology; (iii) non-singular initial "emergent universe" phase of cosmological evolution preceding the inflationary phase; (iv) a new mechanism of dynamical spontaneous breakdown of supersymmetry in supergravity; (v) gravitational electrovacuum "bags".

The C-space extension of such measure in four dimensions would involve a Clifford-valued scalar field $\varphi^A \Gamma_A$ where the Clifford-valued index A in four dimensions spans $2^4 = 16$ components. Four of the components can be identified to the four scalars $\varphi^a (a = 1, 2, 3, 4)$. The C-space extension of the alternative measure Φ is now given by $\epsilon^{M_1 M_2 .... M_{16}} \epsilon_{A_1 A_2 .... A_{16}} (\partial_{M_1} \varphi^{A_1})(\partial_{M_2} \varphi^{A_2}).....(\partial_{M_{16}} \varphi^{A_{16}})$.

Therefore, to sum up, when the torsion is set to zero and the measure obeys (3.3), a variation of the action (3.1) leads to

$$\int [\mathbf{DX}] \ \mu_m(g_{MJ}) \ \left(\mathbf{R}_{(MJ)} \ - \frac{1}{2} \ g_{MJ} \ \mathbf{R} \right) \ \delta g^{MJ} \ = \ 0 \qquad (3.4)$$

after discarding the total derivative terms that do not contribute to the variation of the action when the variation of the fields vanishes at the boundaries. These total derivative terms stem from the variation of the Ricci tensor. In Appendix C it is shown after straightforward algebra that the variation of the Ricci tensor is given by

$$\delta \ (\mathbf{R}_{MJ}) \ = \ \nabla_M(\delta\Gamma^N_{NJ}) \ - \ \nabla_N(\delta\Gamma^N_{MJ}) \ + 2 \ \Gamma^N_{[ML]} \ \delta(\ \Gamma^L_{NJ} \ ) \ - \ \delta(\ f_{ML}{}^N \ \Gamma^L_{NJ} \ ) \qquad (3.5a)$$

when the torsion is zero one has $2\Gamma^N_{[ML]} = f_{ML}^N$ such that eq-(3.5a) becomes

$$\delta \ (\mathbf{R}_{MJ}) \ = \ \nabla_M(\delta\Gamma^N_{NJ}) \ - \ \nabla_N(\delta\Gamma^N_{MJ}) \ - \ \delta(\ f_{ML}{}^N \ ) \ \Gamma^L_{NJ} \qquad (3.5b)$$

Because Γ^L_{NJ} does *not* behave like a tensor the term $\delta(\ f_{ML}{}^N \ ) \ \Gamma^L_{NJ}$ in eq-(3.5b) is *spurious* unless one is forced to impose the variational condition $\delta(\ f_{ML}{}^N \ ) = 0$ on the structure functions, and whose physical interpretation is that one should *not* vary the Clifford algebraic structure functions in C-space. Therefore, when one sets $\delta(\ f_{ML}{}^N \ ) = 0$, the variation $\delta \ (\mathbf{R}_{MJ})$ in eq-(3.5b) becomes finally

$$\delta \ (\mathbf{R}_{MJ}) \ = \ \nabla_M(\delta\Gamma^N_{NJ}) \ - \ \nabla_N(\delta\Gamma^N_{MJ}) \qquad (3.5c)$$

and which is the analog of the Palatini identity in C-space.

If one does not wish to impose the condition $\delta(\ f_{ML}{}^N \ ) = 0$, by inspection one can see that another way of eliminating the spurious term $-\delta(\ f_{ML}{}^N \ ) \ \Gamma^L_{NJ}$ from the variation in eq-(3.5b) might be attained when the variation δ does *not* commute with the derivative operation ∂_M and such that an additional term of the following form must be added to the variation (3.5a, 3.5b)

$$[\ \delta, \ \partial_M \ ] \ \Gamma^L_{LJ} \ - \ [\ \delta, \ \partial_L \ ] \ \Gamma^L_{MJ} \qquad (3.5d)$$

if the above commutators $[\delta, \partial_M], [\delta, \partial_L]$ are defined such that eq-(3.5d) becomes $\delta(\ f_{ML}{}^N \ ) \ \Gamma^L_{NJ}$ one can then eliminate the presence of the spurious term $-\delta(\ f_{ML}{}^N \ ) \ \Gamma^L_{NJ}$ in eq-(3.5b) without having to impose the variational condition $\delta(\ f_{ML}{}^N \ ) = 0$ on the Clifford algebra structure functions. For a detailed analysis of the *noncommutativity* of the variation δ operation with the derivatives ∂ we refer to [17].

The variation (3.5c) contributes to a sum of total derivatives by noticing that one can pull the ∇ derivatives to the left of all the terms in the integrand because $\nabla_K g_{MJ} = 0$ and $\nabla \mu_m(g_{MJ}) = 0$ when the nonmetricity is zero. This yields finally $\int \nabla(\mu_m ......\delta\Gamma)$, which is a total derivative leading then to a boundary term that vanishes, either by imposing a zero variation at the boundaries and/or by having the fields vanish at infinity.

Finally, the vacuum field equations in C-space are given by

$$\mathbf{R}_{(MJ)} \ - \frac{1}{2} \ g_{MJ} \ \mathbf{R} \ = \ 0 \qquad (3.6)$$

One must supplement the above equations with the variation of the action with respect to the scalar-scalar component g of the C-space metric $\delta S/\delta g = 0$.

If there is torsion due to the presence of spinning matter in the Lagrangian one has *extra* terms

$$\frac{1}{2\kappa^2} \int [\mathbf{DX}] \ \mu(g_{MJ}) \ g^{MJ} \ T^L_{MN} \ \delta\Gamma^N_{LJ} \tag{3.7a}$$

in the variation of the action that are matched with the variation $-\delta S_{matter}$ of the matter terms, if and only if, the variation $\delta\Gamma^N_{LJ}$ is taken to be *independent* of the variation δg^{MN}. In this case the torsion obeys the relation

$$\frac{1}{2\kappa^2} \int [\mathbf{DX}] \ \mu(g_{MJ}) \ g^{MJ} \ T^L_{MN} \ = \ - \ \frac{\delta S_{matter}}{\delta\Gamma^N_{LJ}} \tag{3.7b}$$

One must also add the contribution of the *symmetric* part of the analog of the stress energy tensor $\kappa^2 \mathbf{T}_{(MJ)}$ to the right hand side of eq-(3.6), where $\mathbf{T}_{MJ}$ is defined by

$$\mathbf{T}_{MJ} \ \equiv \ - \ \frac{2}{\mu(g_{MJ})} \ \frac{\delta(\mu(g_{MJ}) \ \mathcal{L}_{matter})}{\delta g^{MJ}} \tag{3.7c}$$

We have arrived now at the most salient physical feature of the vacuum field equations. By inserting the torsionless connection expression in eq-(2.14) of the form $\Gamma^L_{MN} = \{^{\ L}_{MN}\} + f^L_{MN}$..... terms, and after using the covariantly constancy condition on the curved C-space Clifford algebra structure functions $\nabla_M f_{JKL} = 0$, one can decompose the Ricci tensor as $\mathbf{R}_{(MJ)} \sim R_{MJ} + f^{KL}_M f_{KLJ} + f^{KL}_J f_{KLM}$, and the Ricci scalar as $\mathbf{R} \sim R + f^{JKL} f_{JKL}$. $R_{MJ} = R_{JM}, R$ are the Ricci tensor and Ricci scalar analogs in C-space associated with the symmetric Christoffel connection $\{^{\ L}_{MN}\} = \{^{\ L}_{NM}\}$.

The physical significance of this curvature decomposition is that these extra terms involving the curved C-space Clifford algebra structure functions can be interpreted as an $effective$ stress energy tensor which can *mimic* the effects of "dark" matter/energy. To see how the cosmological constant Λ emerges, it is straightforward to infer that the contraction $f^{JKL} \ f_{JKL}$ involving the Clifford-algebra structure functions in curved C-space turns out to be *equal* to $f^{ABC} \ f_{ABC} \sim \Lambda_1 =$ constant, when f^{ABC}, f_{ABC} are the tangent space Clifford algebra structure *constants*. This finding is just a consequence of the definitions of f^{JKL} and f_{JKL} in terms of the beins E^A_J , and inverse beins E^J_A given by eqs-(2.2-2.4), and obeying $E^J_A \ E^A_M = \delta^J_M, ...$

Therefore, when the torsion is set to zero, the measure obeys (3.3), and after writing $\mathbf{R}_{(MJ)} = R_{(MJ)} + \Delta R_{(MJ)}$, and $\mathbf{R} = R + \Lambda_1$, the vacuum field equations in C-space can be *rewritten* as

$$R_{(MJ)} \ + \ \Delta R_{(MJ)} \ - \frac{1}{2} \ g_{MJ} \ R \ - \ \frac{1}{2} \ g_{MJ} \ \Lambda_1 \ = \ 0 \tag{3.8}$$

where $\Lambda_1 \equiv \Delta R \sim f^{JKL} \ f_{JKL} = f^{ABC} \ f_{ABC} = constant$. The other terms

$$\Delta R_{(MJ)} \ \sim \ f^{KL}_M f_{KLJ} \ + \ f^{KL}_J f_{KLM} \ \sim \ \Lambda_2 \ \kappa_{MJ} \tag{3.9}$$

are proportional to the curved space Clifford algebra Killing metric $\kappa_{MJ} = E^A_M E^D_J f^{BC}_A f_{BCD} = E^A_M E^D_J \kappa_{AD}$. If the Killing metric κ_{AD} coincides with η_{AD} then $\kappa_{MJ} = g_{MJ}$ and the combined effect of the two constants Λ_1, Λ_2 gives the sought-after cosmological constant term

$$\frac{1}{2} g_{MJ} \ (\ 2\Lambda_2 \ - \ \Lambda_1 \ ) \ \equiv \ \Lambda \ g_{MJ}, \ \ with \ \ 2\Lambda_2 \ - \ \Lambda_1 \ \equiv \ 2 \ \Lambda \tag{3.10}$$

If the Killing metric κ_{AD} does not coincide with η_{AD} then one will have for the $\Delta R_{(MJ)}$ terms the following $\Lambda_2 \ \kappa_{MJ}$ contribution which can be interpreted as (minus) an effective stress energy tensor $-\kappa^2 \ T_{MJ}$ term mimicking the effects of "dark" matter.

To conclude, one of the most salient physical feature of the extended gravitational theory in C-spaces is that one can generate an $effective$ stress energy tensor mimicking the effects of "dark" matter/energy. In particular the cosmological constant term. One could explicitly add a cosmological constant term, by hand, to the original action (3.1) but the main point of our above argument is that it is *not* necessary, to do so. The cosmological constant term is automatically *encoded* in the $f^{JKL} \ f_{JKL} = f^{ABC} \ f_{ABC}$ (= constant) term which naturally forms part of the C-space scalar curvature.

In ordinary Relativity, when the torsion is zero, one can construct the Einstein tensor by performing two successive contractions of the differential Bianchi identity [12]. It also leads to the conservation of the stress energy tensor in the right hand side. In C-space the differential Bianchi identities are satisfied when the torsion is zero. By performing two successive contractions of the differential Bianchi identities one arrives at the field equations

$$\nabla^M \ (\ \mathbf{R}_{(MJ)} \ - \frac{1}{2} \ g_{MJ} \ \mathbf{R} \ ) \ = \ 0 \ \Rightarrow \ \mathbf{R}_{(MJ)} \ - \frac{1}{2} \ g_{MJ} \ \mathbf{R} \ \ = \ \kappa^2 \ \mathbf{T}_{(MJ)}, \ \ \nabla^M(\mathbf{T}_{(MJ)}) = 0 \tag{3.11}$$

The advantage of recurring to the differential Bianchi identities in C-space to derive the field equations (3.11) is that it is *not* necessary to invoke an action and confront the subtleties in constructing a suitable measure of integration.

One may introduce a cosmological constant as an integration constant Λ' in the right hand side of eq-(3.11) giving the modified field equations

$$\mathbf{R}_{(MJ)} \ - \frac{1}{2} \ g_{MJ} \ \mathbf{R} \ = \ \Lambda' \ g_{MJ} \tag{3.12}$$

After decomposing the curvature terms of the left hand side of eq-(3.12) in the same form as in eq-(3.8), and bringing the term $\Lambda' \ g_{MJ}$ into the left hand side, one ends up with an $effective$ cosmological constant term of the form $(\Lambda - \Lambda')g_{MJ}$. Hence a *cancellation* of the effective cosmological constant is possible when $\Lambda - \Lambda' = 0$. This scenario for a plausible explanation of the extremely small value of the observed cosmological constant warrants further investigation.

Let us proceed with the vacuum field equations (3.6). To simplify matters we shall only consider the action (3.1) whose measure is given by (3.2) involving the C-space metric components and whose entries are given by the determinant expressions (E.1). Namely, the C-space metric is being decomposed into antisymmetrized sums of products of the ordinary metric components of spacetime. Besides the scalar-scalar component g of the C-space metric (not to be confused with the $|det\ g_{\mu\nu}|$), the other *independent* variables are now given by the ordinary metric components $g_{\mu\nu} = g_{\nu\mu}$, hence a variation of the action in C-space with respect to $g_{\mu\nu}$ leads to the generalized vacuum field equations that do *not* coincide with the Einstein vacuum field equations.

Hence, in the torsionless case, and in this simplified case, the vacuum field equations in $D = 4$ are obtained from the variation of the action with respect to $g^{\mu\nu}$ after using the chain rule of differentiation

$$\int [\mathbf{DX}]\ \mu_m(g_{MJ}) \left(\mathbf{R}_{(MJ)} - \frac{1}{2}\ g_{MJ}\ \mathbf{R} \right) \delta g^{MJ} =$$

$$\int [\mathbf{DX}]\ \mu_m(g_{MJ}) \left(\mathbf{R}_{(MJ)} - \frac{1}{2}\ g_{MJ}\ \mathbf{R} \right) \frac{\delta g^{MJ}}{\delta g^{\mu\nu}}\ \delta g^{\mu\nu} = 0 \tag{3.13}$$

The above variation in (3.13) yields the simplified version of the vacuum field equations

$$\mathbf{R}_{(\mu\nu)} - \frac{1}{2}\ g_{\mu\nu}\ \mathbf{R} + \left(\mathbf{R}_{(\hat{M}\hat{J})} - \frac{1}{2}\ g_{\hat{M}\hat{J}}\ \mathbf{R} \right) \frac{\delta g^{\hat{M}\hat{J}}}{\delta g^{\mu\nu}} = 0 \tag{3.14}$$

where the contributions of the polyvector-components of the C-space metric are denoted explicitly by the *hatted* indices. Clearly, the vacuum field equations (3.14) differ from the Einstein field equations in ordinary spacetime due to the extra terms stemming from Clifford algebraic structure and polyvector-valued contributions to the C-space metric. These extra terms, once again, can be interpreted as the contribution of (minus) an $effective$ stress energy tensor $-\kappa^2 \mathbf{T}^{eff}_{\mu\nu}$ which could mimic the effects of "dark" matter/energy. As a reminder, one must also include the equation associated with the scalar-scalar component g of the C-space metric $\delta S/\delta g = 0$. Such scalar-scalar C-space metric component might also have cosmological implications like the axion and dilaton.

There are still many challenges ahead to test the viability of the Extended Gravitational Theory in C-spaces. Other physical applications of C-space gravity were studied in [10] in relationship to higher curvature theories of gravity, like Lanczos-Lovelock-Cartan gravity (with torsion) [9] and to $f(R)$ extended theories of gravity [51]. Our finding that the presence of the cosmological constant, along with a plausible mechanism to explain its extremely small value and/or its cancellation, can be understood from a purely Clifford algebraic and geometric perspective, alone, is very appealing and deserves further investigation. The reader might have noticed a similarity of our expressions for the torsion and curvature to those which appear in a *nonholonomic* coordinate description of geometry, a la Finsler, for example. Recent cosmological applications of this *nonholonomic* approach to geometry, and related to the universe accelerated expansion, can be found in [15]. To finalize we include the important Appendices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}$ with the technical details of the calculations.

4 Generalized Metrics in C-spaces

4.1 (Anti) de Sitter Metrics in C-Spaces

The d-dim Anti de Sitter space AdS_d can be parametrized in terms of stereographic coordinates by embedding the d-dim hyperboloid (whose throat radius is $L/2$) in a $d+1$-dim pseudo-Euclidean flat space $R^{d-1,2}$ of signature $(-,+,+,\cdots,+,-)$ as follows

$$y^\mu \ = \ \frac{x^\mu}{(1 - x_\mu x^\mu / L^2)}, \quad \mu \ = \ 0, 1, 2, \cdots, d-1 \tag{4.1}$$

$$y^{d+1} \ = \ \frac{L}{2} \ \frac{(1 + x_\mu x^\mu/L^2)}{(1 - x_\mu x^\mu/L^2)}, \quad x_\mu x^\mu \ = \ - \ (x^0)^2 \ + \ (x^1)^2 \ + \ (x^2)^2 \ + \ \ldots \ + \ (x^{d-1})^2 \tag{4.2}$$

one can infer from eqs-(4.1,4.2) that

$$- \ (y^{d+1})^2 \ - \ (y^0)^2 \ + \ (y^1)^2 \ + \ (y^2)^2 \ + \ \ldots \ + \ (y^{d-1})^2 \ = \ - \ (\frac{L}{2})^2 \tag{4.3}$$

The d-dim de Sitter space dS_d can be parametrized by the stereographic coordinates by embedding the d-dim hyperboloid (whose throat radius is $L/2$) into a $d+1$-dim pseudo-Euclidean flat space $R^{d,1}$ of signature $(-,+,+,\cdots,+,+)$ as follows

$$y^\mu \ = \ \frac{x^\mu}{(1 + x_\mu x^\mu / L^2)}, \quad \mu \ = \ 0, 1, 2, \cdots, d-1 \tag{4.4}$$

$$y^{d+1} \ = \ \frac{L}{2} \ \frac{(1 - x_\mu x^\mu/L^2)}{(1 + x_\mu x^\mu/L^2)}, \quad x_\mu x^\mu \ = \ - \ (x^0)^2 \ + \ (x^1)^2 \ + \ (x^2)^2 \ + \ \ldots \ + \ (x^{d-1})^2 \tag{4.5}$$

obeying

$$(y^{d+1})^2 \ - \ (y^0)^2 \ + \ (y^1)^2 \ + \ (y^2)^2 \ + \ \ldots \ + \ (y^{d-1})^2 \ = \ (\frac{L}{2})^2 \tag{4.6}$$

The (Anti) de Sitter metric in stereographic coordinates become respectively

$$(d\tau)^2_{AdS} \ = \ \frac{(dx_\mu) \ (dx^\mu)}{(1 - x_\mu x^\mu/L^2)^2}, \quad (d\tau)^2_{dS} \ = \ \frac{(dx_\mu) \ (dx^\mu)}{(1 + x_\mu x^\mu/L^2)^2} \tag{4.7}$$

namely, the metric is conformally flat. It is well known (to the experts) that the scalar curvature of the d-dim Lorentzian spacetime corresponding to the conformally flat metric $g = e^{2\phi}\eta_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$, and written in terms of inertial coordinates, is given by the expression

$$R(g) \ = \ \Omega^{-2} \ [\ -2 \ (d-1) \ (\partial_\mu \partial^\mu ln \ \Omega) \ - \ (d-2) \ (d-1) \ (\partial_\mu ln \ \Omega) \ (\partial^\mu ln \ \Omega) \ ] \tag{4.8}$$

hence, given the conformal factors displayed above and plugging their values into eq-(4.8) one ends up, respectively, with

$$R_{AdS} \ = \ - \ \frac{d\ (d-1)}{(L/2)^2}, \quad R_{dS} \ = \ \frac{d\ (d-1)}{(L/2)^2} \tag{4.9}$$

Given this preamble we are going to exploit the conformally flat nature of (Anti) de Sitter spaces and show that the generalization of the d-dim Anti de Sitter space AdS_d metric to C-spaces is given

$$(d\Sigma)^2 \ = \ \frac{(dX_M)\ (dX^M)}{(1 - X_M X^M/L^2)^2} \tag{4.10}$$

the C-space conformal factor is

$$\Omega^2(X_M) \ = \ \frac{1}{(1 - X_M X^M/L^2)^2} \tag{4.11}$$

the infinitesimal displacement squared is

$$(dX_M)\ (dX^M) \ = \ (L_P)^2\ (ds)^2 \ + \ (dx_\mu)\ (dx^\mu) \ + \ (L_P)^{-2}\ (dx_{\mu\nu})\ (dx^{\mu\nu}) \ + \\ (L_P)^{-4}\ (dx_{\mu\nu\rho})\ (dx^{\mu\nu\rho}) \ + \ \ldots \tag{4.12}$$

The norm squared is

$$X_M X^M \ = \ (L_P)^2\ s^2 \ + \ x_\mu x^\mu \ + \ (L_P)^{-2}\ x_{\mu\nu}\ x^{\mu\nu} \ + \ (L_P)^{-4}\ x_{\mu\nu\rho}\ x^{\mu\nu\rho} \ + \ \ldots \tag{4.13}$$

The Clifford scalar s is chosen to be dimensionless. We choose $X_M X^M$ to have units of $(length)^2$ and for this reason suitable powers of the Planck scale L_P must appear in eqs-(4.10-4.12).

The bivectors, trivectors, ... infinitesimal displacements containing the temporal direction will appear with a negative sign due to the chosen Lorentzian signature

$$(dx_\mu)\ (dx^\mu) \ = \ - \ (dx^0)^2 \ + \ (dx^1)^2 \ + \ (dx^2)^2 \ + \ \ldots \ + \ (dx^{d-1})^2 \tag{4.14a}$$

$$(dx_{\mu\nu})\ (dx^{\mu\nu}) \ = \ -\ (dx^{01})^2 - (dx^{02})^2 - (dx^{03})^2 - \ \ldots \ + (dx^{12})^2 + (dx^{13})^2 + \ \ldots \tag{4.14b}$$

$$(dx_{\mu\nu\rho})\ (dx^{\mu\nu\rho}) \ = \ -\ (dx^{012})^2 - (dx^{013})^2 - (dx^{014})^2 - \ \ldots \ + (dx^{123})^2 + (dx^{124})^2 + \ \ldots \tag{4.14c}$$

etc. There is an ambiguity in choosing the sign in the Clifford scalar part $(ds)^2$ of eq-(4.10). We choose the + sign so the overall signature of the 2^d-dimensional C-space is *split* into an equal number of positive/negative signs.

Because the C-space corresponding to the Clifford algebra $Cl(d-1,1)$ is 2^d-dimensional one can show, after some straightforward and lengthy algebra is performed in the defining expressions for the connection and curvature in eqs-(2.11, 2.12, 2.20), that the generalization of the Anti de Sitter space scalar curvature to the 2^d-dimensional C-space, and evaluated for the symmetric Christoffel connection $\{\}$, is

$$\mathbf{R}(\{\}) \ = \ \Omega^{-2}\ \Big[\ -2\ (2^d - 1)\ (\partial_M \partial^M ln\ \Omega)\ \Big] \ -$$

$$\Omega^{-2} \left[(2^d - 2) (2^d - 1) (\partial_M ln \ \Omega) (\partial^M ln \ \Omega) \right] \quad (4.15)$$

where the expression for the C-space conformal factor $\Omega(X_M)$ is given by eq-(4.11). Hence, one arrives finally at

$$\mathbf{R} = - \frac{2^d (2^d - 1)}{(L/2)^2} \quad (4.16)$$

The generalization of the de Sitter space scalar curvature to the 2^d-dimensional C-space is derived from the C-space metric

$$(d\Sigma)^2 = \frac{(dX_M) (dX^M)}{(1 + X_M X^M / L^2)^2} \quad (4.17a)$$

leading to the (positive) value

$$\mathbf{R} = \frac{2^d (2^d - 1)}{(L/2)^2} \quad (4.17b)$$

The generalized vacuum field equations in C-space in the presence of a cosmological constant

$$\mathbf{R}_{MN}(\{\}) - \frac{1}{2} \mathbf{g}_{MN} \mathbf{R}(\{\}) + \Lambda \mathbf{g}_{MN} = 0 \quad (4.18)$$

are obeyed when the values for Λ associated with the C-space version of (Anti) de Sitter spacetimes are respectively given by

$$\Lambda = - \frac{(2^d - 1) (2^d - 2)}{2(L/2)^2}, \quad \Lambda = \frac{(2^d - 1) (2^d - 2)}{2(L/2)^2} \quad (4.19)$$

These results are consistent with a throat radius $\rho = L/2$ of the underlying (Anti) de Sitter spacetimes. The generalized Ricci tensors are respectively given by

$$\mathbf{R}_{MN} = - \frac{(2^d - 1)}{(L/2)^2} \mathbf{g}_{MN}, \quad \mathbf{R}_{MN} = \frac{(2^d - 1)}{(L/2)^2} \mathbf{g}_{MN} \quad (4.20)$$

The embedding of the 2^d-dimensional C-space "hyperboloid" into an *abstract* space of $2^d + 1$ dimensions in the Anti de Sitter version of C-space can be attained by writing

$$Y^M = \frac{X^M}{(1 - X_M X^M / L^2)}, \quad M = 1, 2, \cdots, 2^d \quad (4.21a)$$

$$Y^{M+1} = \frac{L}{2} \frac{(1 + X_M X^M / L^2)}{(1 - X_M X^M / L^2)}, \quad (4.21b)$$

whereas for the de Sitter version one has

$$Y^M = \frac{X^M}{(1 + X_M X^M / L^2)}, \quad M = 1, 2, \cdots, 2^d \quad (4.22a)$$

$$Y^{M+1} = \frac{L}{2} \frac{(1 - X_M X^M/L^2)}{(1 + X_M X^M/L^2)} \quad (4.22b)$$

and leading to the a generalization of eqs-(4.1-4.6). Note that $2^d + 1 \neq 2^{d+1}$, unless $d = 0$, however the abstract space of $2^d + 1$ dimensions is associated to the dimensions of the direct sum of the Clifford algebras $Cl(d-1, 1) \oplus Cl(0)$.

4.2 A different family of C-space metrics

Another C-space metric associated with the generalization of the d-dim Anti de Sitter space AdS_d to C-spaces is given by a "diagonal sum" of the Clifford scalar, vector, bivector, trivector, ... contributions

$$(d\Sigma)^2 = (L_P)^2 \frac{(ds)^2}{(1-s^2)^2} + \frac{(dx_\mu)\ (dx^\mu)}{(1 - x_\mu x^\mu/L^2)^2} + (L_P)^{-2} \frac{(dx_{\mu\nu})\ (dx^{\mu\nu})}{(1 - x_{\mu\nu} x^{\mu\nu}/L^4)^2} +$$
$$(L_P)^{-4} \frac{(dx_{\mu\nu\rho})\ (dx^{\mu\nu\rho})}{(1 - x_{\mu\nu\rho} x^{\mu\nu\rho}/L^6)^2} + \ \ldots \quad (4.23)$$

The above C-space metric is *not* the same as

$$(d\Sigma)^2 = \frac{(dX_M)\ (dX^M)}{(1 - X_M X^M/L^2)^2} \quad (4.24)$$

and for this reason the metric associated with the embedding (4.21) does *not* obey the field equations (2.27).

The above "diagonal sum" version in the de Sitter case is

$$(d\Sigma)^2 = (L_P)^2 \frac{(ds)^2}{(1+s^2)^2} + \frac{(dx_\mu)\ (dx^\mu)}{(1 + x_\mu x^\mu/L^2)^2} + (L_P)^{-2} \frac{(dx_{\mu\nu})\ (dx^{\mu\nu})}{(1 + x_{\mu\nu} x^{\mu\nu}/L^4)^2} +$$
$$(L_P)^{-6} \frac{(dx_{\mu\nu\rho})\ (dx^{\mu\nu\rho})}{(1 + x_{\mu\nu\rho} x^{\mu\nu\rho}/L^6)^2} + \ \ldots \quad (4.25)$$

The above C-space metric does *not* solve the field equations (2.26) and does not have the form

$$(d\Sigma)^2 = \frac{(dX_M)\ (dX^M)}{(1 + X_M X^M/L^2)^2} \quad (4.26)$$

4.3 Analog of Spherically Symmetric Metrics in C-spaces

To search for a generalization of static spherically symmetric metrics in C-spaces, let us focus on the Clifford algebra $Cl(3,1)$ associated with a four-dim Lorentzian spacetime and which is $2^4 = 16$ dimensional. The C-space metric defining the infinitesimal interval $(d\Sigma)^2$ has a *split* signature $(8,8)$ [1]. Let us examine what would be the analog of a "spherically" symmetric metric in C-space. The analog of the "spatial radial distance" squared in the 16-dim C-space is

$$|X|^2 \;=\; (L_P)^2\, s^2 \;+\; (x^1)^2 \;+\; (x^2)^2 \;+\; (x^3)^2 \;+$$
$$(L_P)^{-2} \;\Big(\; (x^{12})^2 \;+\; (x^{13})^2 \;+\; (x^{23})^2 \;\Big) \;+ (L_P)^{-4}\; (x^{123})^2 \tag{4.27}$$

from which one can infer that

$$d|X| \;=\; |X|^{-1} \;\Big[\; (L_P)^2\; sds \;+\; x^1 dx^1 \;+\; x^2 dx^2 \;+\; x^3 dx^3 \;\Big] \;+$$
$$|X|^{-1} \;\Big[\; (L_P)^{-2} \;\Big(\; x^{12} dx^{12} \;+\; x^{13} dx^{13} \;+\; x^{23} dx^{23} \;\Big) \;+ (L_P)^{-4}\; x^{123} dx^{123} \;\Big] \tag{4.28}$$

where $|X|$ is the square root of eq-(4.27).

The analog of the "temporal radial distance" squared in the 16-dim C-space is

$$|T|^2 \;=\; (x^0)^2 \;+ (L_P)^{-2} \;\Big(\; (x^{01})^2 \;+\; (x^{02})^2 \;+\; (x^{03})^2 \;\Big) \;+$$
$$(L_P)^{-4} \;\Big(\; (x^{012})^2 \;+\; (x^{013})^2 \;+\; (x^{023})^2 \;\Big) \;+ (L_P)^{-6}\; (x^{0123})^2 \tag{4.29}$$

from which one can infer the expression for the infinitesimal temporal displacement

$$d|T| \;=\; |T|^{-1} \;\Big[\; x^0 dx^0 \;+ (L_P)^{-2} \;\Big(\; x^{01} dx^{01} \;+\; x^{02} dx^{02} \;+\; x^{03} dx^{03} \;\Big) \;\Big] \;+$$
$$|T|^{-1} \;\Big[(L_P)^{-4} \;\Big(\; x^{012} dx^{012} \;+\; x^{013} dx^{013} \;+\; x^{023} dx^{023} \;\Big) \;+ (L_P)^{-6}\; x^{0123} dx^{0123} \;\Big] \tag{4.30}$$

where $|T|$ is the square root of eq-(4.29).

Hence, an ansatz for the analog of a spherically symmetric metric in the 16-dim C-space of split signature $(8,8)$ is

$$(d\Sigma)^2 \;=\; -\, f(|X|)\; (d|T|)^2 \;-\; |T|^2\; (d\chi_7)^2 \;+\; h(|X|)\; (d|X|)^2 \;+\; |X|^2\; (d\Omega_7)^2 \tag{4.31}$$

where $|X|^2 (d\Omega_7)^2$ is the C-space metric analog a 7-dim sphere determined by the spatial directions, and $|T|^2 (d\chi_7)^2$ is the C-space metric analog of a 7-dim sphere determined by the temporal directions. Ω_7, χ_7 are the respective solid angles of the 7-dim spheres. All the other terms in (4.31) are defined by eqs-(4.27-4.30). The real-valued functions $f(|X|), h(|X|)$ in (4.31) are determined by solving the very *complicated* C-space field equations (2.26). The flat C-space limit is attained when $f(|X|) = h(|X|) = 1$.

One should note that due to the presence of the terms $|T|^2 (d\chi_7)^2$, the C-space metric (4.31) is strictly speaking *not* static. One could extract a static slice in (4.31) by *freezing* the temporal solid angle degrees of freedom by setting $(d\chi_7)^2 = 0$, and leading to an interval

$$(d\Sigma)^2 \rightarrow - f(|X|)\ (d|T|)^2 + h(|X|)\ (d|X|)^2 + |X|^2\ (d\Omega_7)^2 \tag{4.32}$$

which resembles more closely a static spherically symmetric metric. Rigorously speaking, the Schwarzschild metric is only static in the region outside the horizon but it is not static in the interior region after performing the Kruskal-Szekeres coordinate transformations $r = r(u,v); t = t(u,v)$. The black hole singularity $r(u,v) = 0$ is spacelike.

The $4D$ (Anti) de Sitter-Schwarzschild metric in natural units $\hbar = c = G = 1$

$$(d\tau)^2 = - (1 - \frac{2M}{r} - \frac{\lambda}{3} r^2)\ (dt)^2 + (1 - \frac{2M}{r} - \frac{\lambda}{3} r^2)^{-1}\ (dr)^2 + r^2\ (d\Omega_2)^2 \tag{4.33}$$

is a solution of Einstein's field equations in $4D$ with a cosmological constant ($\lambda < 0$ in the AdS case). This metric is just a "slice" of the 16-dim C-space of split signature $(8,8)$ given by eq-(4.31). Guided by this metric (4.33) one could attempt to find the real-valued functions $f(|X|), h(|X|)$ in (4.31) which solve the C-space field equations (2.26).

We finalize this section by discussing the very restricted class of C-space metrics ($\mathbf{g}_{MN} = \mathbf{g}_{NM}$) that can be *decomposed* into products of ordinary metrics in spacetime. Firstly, one needs to have a C-space metric whose components have the *same* grade like

$$g_{\mathbf{00}},\quad g_{\mu\nu},\quad g_{\mu_1\mu_2\ \nu_1\nu_2},\ \ldots,\ g_{\mu_1\mu_2\ldots\mu_D\ \nu_1\nu_2\ldots\nu_D} \tag{4.33}$$

and which can be decomposed as

$$g_{[\mu_1\mu_2]\ [\nu_1\nu_2]}(x^\mu) = g_{\mu_1\nu_1}(x^\mu)\ g_{\mu_2\nu_2}(x^\mu) - g_{\mu_2\nu_1}(x^\mu)\ g_{\mu_1\nu_2}(x^\mu)$$

$$g_{[\mu_1\mu_2\ldots\mu_k]\ [\nu_1\nu_2\ldots\nu_k]}(x^\mu) = det\ G_{\mu_i\nu_j} = \epsilon^{j_1 j_2\ldots j_k}\ g_{\mu_1\nu_{j_1}}\ g_{\mu_2\nu_{j_2}}\ \ldots\ g_{\mu_2\nu_{j_k}}, \tag{4.34}$$

The determinant of $G_{\mu_i\nu_j}$ can be written as

$$det \left(\begin{array}{cccc} g_{\mu_1\nu_1}(x^\mu) & \ldots & \ldots & g_{\mu_1\nu_k}(x^\mu) \\ g_{\mu_2\nu_1}(x^\mu) & \ldots & \ldots & g_{\mu_2\nu_k}(x^\mu) \\ ------------ & & & --------------- \\ g_{\mu_k\nu_1}(x^\mu) & \ldots & \ldots & g_{\mu_k\nu_k}(x^\mu) \end{array} \right), \tag{4.35}$$

The metric component $g_{\mathbf{00}}$ involving the Clifford scalar "directions" $X_{\mathbf{0}} = s$ of the Clifford polyvectors in C-space must also be included. $X_{\mathbf{0}} = s$ must not be confused with the temporal coordinate x_0. $g_{\mathbf{00}}$ behaves like a Clifford scalar under coordinate transformations in C-space. The other component $g_{[\mu_1\mu_2\ldots\mu_D]\ [\nu_1\nu_2\ldots\nu_D]}$ involves the pseudo-scalar "direction". The latter scalar and pseudo-scalar components of the C-space metric might bear some connection to the dilaton and axion fields in Cosmology and particle physics. In the most general case the C-space metric *does not factorize* into antisymmetrized sums of products of ordinary metrics . We presented above examples of metrics in C-space which cannot be decomposed into antisymmetrized sums of products of ordinary metrics.

4.4 Areal Geometry and Strings

C-space metrics are an extension of *areal* metrics of the form $(d\tau)^2 = \frac{1}{4} h_{ijkl}(dx^i \wedge dx^j) \otimes (dx^k \wedge dx^l)$ which were studied long ago by Cartan. An areal metric generalization of the usual metric to Finsler geometry was developed by [28]. Such a generalized notion of area, and more generally the volume of m-dimensional submanifolds embedded in an n-dimensional space, have been considered under the terminology of "areal geometry". In these considerations, the metric and connection in general depend not only on $\mathbf{x}$ but also on the derivatives of $\mathbf{x}$ with respect to world-volume coordinates. Applications of the Kawaguchi Lagrangian formulation to string theory and p-branes can be found in [29]. The classification of area metrics and the construction of vacuum field equations were analyzed in [27]. Another family of equations for area metrics that reduce to the vacuum Einstein's equations in very special cases were studied in [26]. Static spherical symmetric solutions were found for the generalized Einstein equation in vacuum, including the Schwarzschild solution as a special case.

The Nambu-Goto action corresponding to the bosonic string is defined in terms of its worldsheet area. Motivated by the possibility that string theory admits backgrounds where the notion of length is not well defined but a definition of area is, propelled the authors [26] to study space-time geometries based on the generalization of length metrics to area metrics. In analogy with Riemannian geometry, they defined the analogues of connections, curvatures and Einstein tensor.

In Einstein's theory of gravity, the Bianchi identity provides a hint on how to define Einstein's equation such that the conservation of energy-momentum tensor is guaranteed. The situation is different for the gravitational theory of area metrics [26]. The conservation of energy-momentum is a result of the invariance of the theory under general coordinate transformations. In the theory of area metrics, the gauge symmetry is still merely general coordinate transformations but the number of degrees of freedom of the areal metric, connection and curvature are much larger than in the case of ordinary metrics. Therefore, the authors [26] argued that one should not try to define the generalized Einstein equation from the generalized Bianchi identity as one did in Einstein's theory.

However, a key difference that gravity in C-spaces has is that one has full diffeomorphism invariance under the polyvector-valued coordinate changes $X_M \to X'_M$, thus the generalized energy-momentum polytensor in C-space is conserved and consistent with the generalized C-space Bianchi identities, in the absence of torsion and nonmetricity, and which in turn, allows us to write down the generalization of Einstein equations in C-spaces [54]. A discussion of matter fields in C-spaces can be found in [1].

Another problem with the formulation of gravity of area metrics is that it does not seem to admit an action principle due to the fact that the tensor $\mathbf{R}_{ij}^{kl}{}_{mn}$ does *not* admit the definition of scalar curvature through the contraction of indices, if the only additional tensor available is the area metric [26]. A possibility is that the action principle for the area metric theory is available only in certain dimensions when the volume form can be used to do the trick to appropriately be able to contract indices [26]. Fortunately, in C-spaces this problem does *not* arise since all polyvector-valued indices are contracted with the C-space metric $\mathbf{g}_{MN} = \mathbf{g}_{NM}$, its inverse $\mathbf{g}^{MN} = \mathbf{g}^{NM}$, and δ^M_N which in general have polyvector valued indices M, N of the same and different grades : $\mathbf{g}_{[\mu_1\mu_2\cdots\mu_i]\ [\nu_1\nu_2\cdots\nu_j]}$, for example.

To finalize, we should point out that when the C-space metric components are of the same grade, and *admit* a decomposition as shown in eq-(4.35), it is plausible to have in the putative quantum gravitational theory cases where the expectation values of the areal metrics are not zero $< \hat{g}_{\mu\nu}\hat{g}_{\rho\sigma} > \neq 0$, despite that the expectation of the metric is $< \hat{g}_{\mu\nu} >= 0$ (Topological QFT's are characterized by physical correlations independent of the metric). This could be a very natural explanation as to why quantum gravitational effects could be essentially "stringy". If on average $< \hat{g}_{\mu\nu} >= 0$, one does not observe lengths but areas instead. Quantum gravitational effects are intrinsically manifested at the Planck-scale (there are quantum gravitational phenomena which have cosmological signatures at larger scales due to inflation, and/or compounding effects). Since the Planck scale L_P is an essential ingredient in the construction of the extended relativity in C-spaces [1], and Quantum Gravity, this suggests that C-space geometry is a natural arena to be explored. For this reason, we believe that more novel physical phenomena could be unraveled behind C-space gravity than we previously thought.

5 A Unified Description of Particles, Strings and Branes in Clifford Spaces

We will show next how the Extended Relativity Theory in C-spaces (Clifford spaces) allows a unified formulation of point particles, strings, membranes and p-branes, moving in ordinary target spacetime backgrounds, within the description of a single *polyparticle* moving in C-spaces. The degrees of freedom of the latter are provided by Clifford polyvector-valued coordinates (antisymmetric tensorial coordinates). A correspondence between the p-brane (p-loop) wave functional "Schrödinger-like" equations of Ansoldi-Aurilia-Spallucci and the polyparticle wave equation in C-spaces is found via the polyparticle/p-brane duality/correspondence. The crux of exploiting this correspondence is that it might provide another unexplored avenue to quantize p-branes (a notoriously difficult and unsolved problem) from the more straightforward quantization of the polyparticle in C-spaces, even in the presence of external interactions. We conclude this section with some comments about the *compositeness* nature of the polyvector-valued coordinate operators in terms of ordinary p-brane coordinates via the evaluation of n-ary commutators.

5.1 Branes in Clifford Spaces

An ordinary p-brane moving in a D-dim flat target background spacetime spans a $p+1$-dimensional world volume and the Nambu-Goto action written in terms of the Nambu-Poisson bracket is given by

$$S \ = \ T \int d^{p+1}\sigma \ \sqrt{ |det \ \partial_a X^\mu \ \partial_b X^\nu \ \eta_{\mu\nu}| } \ =$$

$$T \int d^{p+1}\sigma \ \sqrt{(\ \{X^{\mu_1}, X^{\mu_2}, X^{\mu_3}, \ldots, X^{\mu_{p+1}}\} \ )^2} \tag{5.1}$$

T is the p-brane tension whose units are $(mass)^{p+1}$. $X^\mu(\sigma^a)$ are the embedding functions of the $p+1$-dim world volume of the p-brane into the D-dim target spacetime background $(D \geq p+1)$. The world volume coordinates are $\sigma = (\sigma^1, \sigma^2, \cdots, \sigma^{p+1})$.

Let us generalize the action (5.1) to the C-space case when the world manifold and target space coordinates are both Clifford-valued. Given the 2^d polyvector-valued world manifold variables

$$\sigma^A \ = \ s, \quad \sigma^a, \quad \sigma^{a_1 a_2}, \quad \cdots, \quad \sigma^{a_1 a_2 \ldots a_d} \tag{5.2}$$

and the 2^D polyvector-valued target space coordinates

$$X^M \ = \ X, X^\mu, X^{\mu_1\mu_2}, \cdots, X^{\mu_1\mu_2\ldots\mu_D} \tag{5.3}$$

with $D \geq d$ one can write the analog of the expression for a Nambu-Poisson bracket in C-space as [1]

$$\epsilon^{A_1 A_2 \ldots A_{2d}} \ \frac{\partial X^{M_1}}{\partial \sigma^{A_1}} \ \frac{\partial X^{M_2}}{\partial \sigma^{A_2}} \ \ldots \ \frac{\partial X^{M_{2^d}}}{\partial \sigma^{A_{2^d}}} \tag{5.4}$$

In general each polyvector-valued coordinate component $X^M(\sigma^A)$ is a function of *all* the 2^d polyvector-valued world manifold variables σ^A. The expression (5.4) simplifies considerably in the very special case when

$$X \ = \ X(s), \ \ X^\mu \ = \ X^\mu(\sigma^a), \ \ X^{\mu_1\mu_2} \ = \ X^{\mu_1\mu_2}(\sigma^{a_1 a_2}), \ \ \ldots \ , X^{\mu_1\mu_2\cdots\mu_d} \ = \ X^{\mu_1\mu_2\cdots\mu_d}(\sigma^{a_1 a_2 \cdots a_d}),$$

$$X^{\mu_1\mu_2\cdots\mu_{d+1}} \ = \ X^{\mu_1\mu_2\cdots\mu_{d+2}} \ = \ \ldots \ = \ X^{\mu_1\mu_2\cdots\mu_D} \ = \ 0 \tag{5.5}$$

The above conditions of eq-(7) basically describe grade-preserving maps from the Clifford-valued world manifold to the target Clifford-valued C-space. It comprises of maps such that points are mapped to points; areas to areas; volumes to volumes ... and where one freezes to zero the polyvector valued target coordinates whose grade *exceeds* the dimension d (of the world manifold an whose associated C-space is 2^d-dim).

In this case the determinant-like expression (5.4) *factorizes* as follows

$$\frac{\partial X}{\partial s} \ \left(\epsilon^{a_1 a_2 \ \ldots \ a_d} \ \frac{\partial X^{\mu_1}}{\partial \sigma^{a_1}} \ \frac{\partial X^{\mu_2}}{\partial \sigma^{a_2}} \ \ldots \ \frac{\partial X^{\mu_d}}{\partial \sigma^{a_d}} \right) \ \left(\epsilon^{[a_1 b_1] \ [a_2 b_2] \ \ldots} \ \frac{\partial X^{\mu_1\nu_1}}{\partial \sigma^{a_1 b_1}} \ \frac{\partial X^{\mu_2\nu_2}}{\partial \sigma^{a_2 b_2}} \ \ldots \ \right)$$

$$\left(\epsilon^{[a_1 b_1 c_1] \ [a_2 b_2 c_2] \ \ldots} \ \frac{\partial X^{\mu_1\nu_1\rho_1}}{\partial \sigma^{a_1 b_1 c_1}} \ \frac{\partial X^{\mu_2\nu_2\rho_2}}{\partial \sigma^{a_2 b_2 c_2}} \ \ldots \right) \ \left(\epsilon^{[a_1 b_1 c_1 d_1] \ [a_2 b_2 c_2 d_2] \ \ldots} \ \frac{\partial X^{\mu_1\nu_1\rho_1\tau_1}}{\partial \sigma^{a_1 b_1 c_1 d_1}} \ \frac{\partial X^{\mu_2\nu_2\rho_2\tau_2}}{\partial \sigma^{a_2 b_2 c_2 d_2}} \ \ldots \right) \ \ldots \tag{5.6}$$

It is convenient to introduce an ordering of indices to avoid having to introduce factorial numerical factors. For the bivector coordinates $\sigma^{a_1b_1}, \sigma^{a_2b_2}, \cdots$ one requires to have $a_1 < b_1; a_2 < b_2; \cdots$. For the trivector coordinates $\sigma^{a_1b_1c_1}, \sigma^{a_2b_2c_2}, \cdots$ one requires to have $a_1 < b_1 < c_1; a_2 < b_2 < c_2; \cdots$, etc. Similar ordering prescription applies for the $X^{\mu_1\nu_1}, X^{\mu_1\nu_1\rho_1}, \cdots$ target C-space polyvector valued coordinates. The generalized version of the epsilon symbols ensures that no polyvector-valued indices are repeated.

A remark is in order. One could also view the induced metric $H_{AB} = \partial_A X^M \ \partial_B X^N \ G_{MN}$ on the 2^d-dim Clifford-valued world manifold as a hyper-matrix but the construction (if possible) of the hyper-determinant of the C-space metric (a hyper matrix) is a more difficult problem [13], [14]. For this reason we shall not pursue this road at the moment to build generalized p-brane actions in C-spaces based on Nambu-Goto actions associated with the square roots of hyper-determinants.

Let us proceed. For example, when $d = 2$ and $D \geq d$, eq-(5.6) is given by

$$\frac{\partial X}{\partial s} \left(\epsilon^{a_1a_2} \ \frac{\partial X^{\mu_1}}{\partial \sigma^{a_1}} \ \frac{\partial X^{\mu_2}}{\partial \sigma^{a_2}} \right) \ \frac{\partial X^{\mu_1\nu_1}}{\partial \sigma^{12}} \tag{5.7}$$

The term $\frac{\partial X}{\partial s}$ corresponds *effectively* to the motion of a point particle parametrized by the variable s and moving in one-dimension described by the coordinate X. The term $\epsilon^{a_1a_2} \ \frac{\partial X^{\mu_1}}{\partial \sigma^{a_1}} \ \frac{\partial X^{\mu_2}}{\partial \sigma^{a_2}}$ is the standard Poisson bracket $\{X^{\mu_1}, X^{\mu_2}\}$ with respect to the variables σ^1, σ^2 and associated with the motion of a string in a D-dimensional background. The term $\frac{\partial X^{\mu_1\nu_1}}{\partial \sigma^{12}}$ corresponds *effectively* to the motion of a point particle parametrized by the (bivector) variable σ^{12} and moving in a $\frac{D(D-1)}{2}$-dimensional background described by the (bivector) coordinates $X^{\mu_1\nu_1}$.

The analog of the Nambu-Goto action would now be

$$S \ = \ \kappa \int \ ds \ d\sigma^1 d\sigma^2 \ d\sigma^{12} \ \sqrt{ (\frac{\partial X}{\partial s})^2 } \ \sqrt{ (\ \{X^{\mu_1}, X^{\mu_2}\} \ )^2 } \ \sqrt{ (\ \frac{\partial X^{\mu_1\nu_1}}{\partial \sigma^{12}} \ )^2 } \ =$$

$$\kappa \int \ ds \ \sqrt{ (\frac{\partial X}{\partial s})^2 } \int \ d\sigma^1 d\sigma^2 \ \sqrt{ (\ \{X^{\mu_1}, X^{\mu_2}\} \ )^2 } \int \ d\sigma^{12} \ \sqrt{ (\ \frac{\partial X^{\mu_1\nu_1}}{\partial \sigma^{12}} \ )^2 } \ =$$

$$\kappa \int \ dX \int \ d\sigma^1 d\sigma^2 \ \sqrt{ (\ \{X^{\mu_1}, X^{\mu_2}\} \ )^2 } \int \ \sqrt{ (\ dX^{\mu_1\nu_1} \ )^2 } \tag{5.8}$$

Concluding, the action (5.8) *factorizes* and *collectively* describes a point particle moving in one-dim; a string moving in D-dim, and a point particle moving in $\frac{D(D-1)}{2}$-dim. Furthermore, one should notice that one has a product of terms instead of a summation of individual actions.

The conditions of eq-(5.5) in the more general case ($D \geq d > 2$) describe richer dynamics . Taking into account that a p-brane spans a $p+1$ dimensional world volume one has that the second term in eq-(5.6) describes the standard Nambu-Poisson bracket associated to a $d-1$-brane (spanning a d-dim world volume) moving in a D-dim target background. The third term describes effectively a $\frac{d(d-1)}{2} - 1$-brane moving in a $\frac{D(D-1)}{2}$-dim target background. The fourth term describes effectively a $\frac{d(d-1)(d-2)}{3!} - 1$-brane moving in a $\frac{D(D-1)(D-2)}{3!}$-dim target background; and so forth. The final term $(\partial X^{\mu_1\mu_2\cdots\mu_d}/\partial\sigma^{a_1a_2\cdots a_d})$ corresponds *effectively* to the motion of a point particle parametrized by the (highest grade polyvector) variable $\sigma^{123\cdots d}$ and moving in a $\frac{D(D-1)(D-2)\cdots(D-d)}{d!}$-dimensional background described by the (polyvector) coordinates $X^{\mu_1\mu_2\cdots\mu_d}$.

When the conditions of eq-(5.5) are imposed on the target C-space polyvector valued coordinates the number of degrees of freedom N (when $D \geq d$) is given by

$$N = \binom{D}{0} + \binom{D}{1} + \binom{D}{2} + \ldots + \binom{D}{d} \geq \binom{d}{0} + \binom{d}{1} + \binom{d}{2} + \ldots + \binom{d}{d} = 2^d \quad (5.9)$$

hence the number of *transverse* degrees of freedom is $N - 2^d \geq 0$. When $D = d$ one has $N = 2^D = 2^d$ and the number of *transverse* degrees of freedom is zero as expected. Therefore, by choosing $D > d$ one will have non-trivial dynamics since the number of transverse degrees of freedom are not zero.

To sum up, the action associated with the expression in eq-(5.6) is defined to be

$$S = \kappa \int ds\ d\sigma^1\ d\sigma^2\ \ldots\ d\sigma^{12}\ d\sigma^{13}\ \ldots\ d\sigma^{123}\ d\sigma^{124}\ \ldots\ d\sigma^{123....d}\ \sqrt{\Delta} \quad (5.10)$$

where Δ is the *square* of the expression given by eq-(5.6) and effectively describes a collective ensemble of points and p-branes, for certain specific values of p, and each moving in different target space dimensions as discussed above.

5.2 p-brane/polyparticle Duality

We will describe now how a polyparticle in C-space may have a correspondence with a *nested* hierarchy of point particles, strings, membranes and p-branes in ordinary space. Let us begin by recalling the infinitesimal interval displacement in C space

$$(d\Sigma)^2 = dX_M\ dX^M = (dX)^2 + L^{-2}\ dX_\mu\ dX^\mu + L^{-4}\ dX_{\mu_1\mu_2}\ dX^{\mu_1\mu_2} + \ldots + L^{-2D}\ dX_{\mu_1\mu_2.....\mu_D}\ dX^{\mu_1\mu_2.....\mu_D} \quad (5.11)$$

X is the Clifford-scalar part of the Clifford-valued coordinate X^M. The values of M range from $1, 2, \cdots, 2^D$. Σ and X are both taken to be dimensionless by introducing suitable powers of the length scale L (it can be chosen to be equal to the Planck scale) . The polyparticle dynamics is parametrized by the C-space proper time variable Σ such that the polyvector valued coordinates describing the motion of the polyparticle in C-space are determined by the 2^D functions $X^M = X^M(\Sigma)$.

When $n = p+1$, the p-brane/polyparticle duality/correspondence is defined as follows

$$L^{2n}\ \{X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_n}\}^2_{\sigma^a} \leftrightarrow (\frac{dX^{\mu_1\mu_2.....\mu_n}}{d\Sigma})^2 \quad (5.12a)$$

$$\frac{1}{L^n} \int d\Sigma \sqrt{ (\frac{dX^{\mu_1\mu_2.....\mu_n}}{d\Sigma})^2 } \leftrightarrow T_n \int d^n\sigma \sqrt{ \{X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_n}\}^2 }, \qquad (5.12b)$$

T_n is the $n-1$-brane tension and whose physical units are $(mass)^n = (length)^{-n}$. The values of n range from $1, 2, \cdots, D$. For $n = 1$ one has the point particle action parametrized by the time-like variable σ^1 of the world line in ordinary spacetime.

$$S = m \int d\sigma^1 \sqrt{ (\frac{dX^\mu}{d\sigma^1})^2 } \qquad (5.13)$$

For $n = 2$ one has the string action

$$S = T_2 \int d\sigma^1 d\sigma^2 \sqrt{(\{X^{\mu_1}, X^{\mu_2}\})^2}, \quad \{X^{\mu_1}, X^{\mu_2}\} = \epsilon^{a_1a_2} \frac{\partial X^{\mu_1}}{\partial\sigma^{a_1}} \frac{\partial X^{\mu_2}}{\partial\sigma^{a_2}}, \quad a = 1, 2 \qquad (5.14)$$

where σ^1, σ^2 are the temporal and spatial coordinates of the worldsheet, respectively; and so forth. For a p-brane whose world volume is $n = p+1$-dim one writes the p-brane action given by the right hand side of eq-(14b) in terms of the Nambu-Poisson bracket

$$\{X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_n}\} = \epsilon^{a_1a_2 \cdots a_{p+1}} \frac{\partial X^{\mu_1}}{\partial\sigma^{a_1}} \frac{\partial X^{\mu_2}}{\partial\sigma^{a_2}} \ldots \frac{\partial X^{\mu_{p+1}}}{\partial\sigma^{a_{p+1}}}, \quad a = 1, 2, 3, \cdots, p+1 \qquad (5.15)$$

One should note that we are using the epsilon symbol in defining all the above brackets. One could have used the epsilon symbol only in the highest grade case, corresponding to the case when $p + 1 = D$, and defined the lower grade brackets in terms of an auxiliary number of antisymmetric tensor fields $\omega^{a_1a_2.....}$ of different ranks if one performed the derivatives with respect to *all* the σ^a variables $\sigma^1, \sigma^2, \cdots, \sigma^D$. For instance

$$\{X^{\mu_1}, X^{\mu_2}\} = \omega^{a_1a_2} \frac{\partial X^{\mu_1}}{\partial\sigma^{a_1}} \frac{\partial X^{\mu_2}}{\partial\sigma^{a_2}}, \quad a = 1, 2, 3, \cdots, D \qquad (5.16a)$$

$$\{X^{\mu_1}, X^{\mu_2}, X^{\mu_3}\} = \omega^{a_1a_2a_3} \frac{\partial X^{\mu_1}}{\partial\sigma^{a_1}} \frac{\partial X^{\mu_2}}{\partial\sigma^{a_2}} \frac{\partial X^{\mu_3}}{\partial\sigma^{a_3}}, \quad a = 1, 2, 3, \cdots, D \qquad (5.16b)$$

etc, ... However this *multisymplectic* approach will complicate matters since one must satisfy the (generalized) Jacobi identities (fundamental identities) which will constrain the functional form of the auxiliary number of antisymmetric tensor fields $\omega^{a_1a_2.....}$. For this reason, to simplify matters we define the brackets solely in terms of epsilon symbols as shown above in eqs-(5.14,5.15). In this way we have a *nested* hierarchy of point particles, strings, membranes and p-branes in ordinary spacetime. The sequence of variables is nested as follows $\sigma^1 \subset (\sigma^1, \sigma^2) \subset (\sigma^1, \sigma^2, \sigma^3) \subset \cdots \subset (\sigma^1, \sigma^2, \cdots, \sigma^D)$.

A realization of the duality conditions in eq-(5.12a) can be simply realized when both sides of eq-(5.12a) are equal to a constant. Since the right hand side of eq-(5.12a) depends on Σ and the left hand side depends on σ^a an equality is possible when both sides are equal to a constant. In particular, the simplest choice to attain this equality is when

$$X^1 \;=\; c_1\,\sigma^1,\;\; X^2 \;=\; c_2\,\sigma^2,\;\; X^3 \;=\; c_3\,\sigma^3,\;\; X^n \;=\; c_n\,\sigma^n,\;\; \ldots\;, X^D \;=\; c_D\,\sigma^D \quad (5.17)$$

where $c_1, c_2, ...$ are constants and such that $\{X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_n}\} = 0$ *except* for the specific value

$$L^{2n}\,\{X^1, X^2, \ldots, X^n\}^2 \;=\; (c_1 c_2 ......c_n)^2\, L^{2n} \;=\; (\frac{dX^{\mu_1\mu_2.....\mu_n}}{d\Sigma})^2 \;=\; |\mathbf{V}|^2 \;=\; constant \quad (5.18)$$

On the other hand, from the equations of motion associated with the free polyparticle action in C-space [1], after taking into account that $(\frac{dX^M}{d\Sigma})^2 = 1$ and that Σ is chosen to be dimensionless,

$$S \;=\; \int d\Sigma \;=\; \int d\Sigma\,\sqrt{(\frac{dX^M}{d\Sigma})^2} \;\Rightarrow\; \frac{d^2 X^M}{d\Sigma^2} \;=\; 0,\;\; X^M \equiv X, X^\mu, X^{\mu_1\mu_2}, \cdots, X^{\mu_1\mu_2....\mu_D} \quad (5.19)$$

one concludes that the components $X^{\mu_1\mu_2.....\mu_n}(\Sigma)$ grow/decrease *linearly* with Σ so that the n-volume velocity components $V^{\mu_1\mu_2....\mu_n} = \frac{dX^{\mu_1\mu_2.....\mu_n}}{d\Sigma}$ are *constant*. This is indeed consistent with the results found in eq-(5.18) where the *magnitude* of the n-volume velocity $\mathbf{V}$ is *constant*. If the n-volume velocity components are constant then the magnitude of the n-volume velocity is also constant. The converse is not true. A typical example is ordinary circular motion. In [35] it was shown that when the areal velocities are *constant* the Nambu and Schild string actions lead to equivalent equations of motion. Similar conclusions hold for p-brane actions.

Proceeding with eqs-(5.17,5.18) one learns, if one equates both sides of eq-(5.12b) and sets $T_n = L^{-n}$, that the (dimensionless) polyparticle's proper time in C-space obeys

$$\Sigma \;=\; \frac{1}{L^n}\int d^n\sigma \;=\; \frac{\Omega_n}{L^n},\;\; n = 1, 2, \ldots, D \quad (5.20a)$$

and also

$$|\mathbf{V}|\;\Sigma \;=\; V_n \;=\; \int d^n\sigma\,\sqrt{\{X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_n}\}^2}\,,\;\; n = 1, 2, \ldots, D \quad (5.20b)$$

Because the magnitudes of the n-volume velocities $|\mathbf{V}|$ are *constant*, from (22b) one learns that the scale sizes of the evolving world lines, world sheets, world volumes, ... are linearly proportional to the polyparticle's proper time Σ in C-space. This is not unlike to cosmological models where the size of the cosmos is taken as a "dilational" clock. From eq-(22a) one may also infer directly that Σ only remains invariant under length, area, volume, hyper-volume preserving diffeomorphisms of the σ^a coordinates. We have chosen a very simple case in eqs-(5.17). More complicated cases warrant further investigation.

To study further the polyparticle/p-brane correspondence let us begin by writing the wave equation associated with a free polyparticle in C-space in natural units $\hbar = c = 1$ and when $L = 1$

$$i\,\frac{\partial\Psi(X^M, \Sigma)}{\partial\Sigma} \;=\; -\,\frac{\partial^2\Psi(X^M, \Sigma)}{\partial X^M\,\partial X_M},\;\; X^M \equiv X, X^\mu, X^{\mu_1\mu_2}, \cdots, X^{\mu_1\mu_2....\mu_D} \quad (5.21)$$

A solution of (23) consistent with the dispersion relations $P_M P^M = M^2$ is

$$\Psi(X^M, \Sigma) = Exp\ [\ i\ (P_M\ X^M - M^2\ \Sigma)\ ] \qquad (5.22a)$$

inserting the solution (5.22a) into (5.21) yields $(P_M P^M - M^2)\Psi = 0$, and which in turn, after replacing $P_M \to -i\partial/\partial X^M$ leads to the analog of the Klein-Gordon equation in C-space for the Clifford-scalar-field $\Phi(X^M)$

$$\left(\frac{\partial^2}{\partial X^M\ \partial X_M} + M^2 \right) \Phi(X^M) = 0, \quad \text{when} \quad \Psi(X^M, \Sigma) = \Phi(X^M)\ Exp\ [\ -i\ M^2\ \Sigma\ ] \qquad (5.22b)$$

It is important to emphasize that the decomposition of $\Psi(X^M, \Sigma)$ into a product of separate wave functions

$$\Psi(X, \Sigma)\ \Psi(X^\mu, \Sigma)\ \Psi(X^{\mu\nu}, \Sigma)\ \ldots =$$

$$Exp\ [\ i\ (P\ X - P^2\ \Sigma)\ ]\ Exp\ [\ i\ (P_\mu\ X^\mu - (P_\mu)^2\ \Sigma)\ ]\ Exp\ [\ i\ (P_{\mu\nu}\ X^{\mu\nu} - (P_{\mu\nu})^2\ \Sigma)\ ]\ \ldots \qquad (5.22c)$$

does not solve eq-(5.21). The reason is that the C-space invariant, and Σ-independent quantity, is given by the *net* sum of the terms $P^2 + (P^\mu)^2 + (P^{\mu\nu})^2 + \cdots = M^2$. Hence each Lorentz-invariant term $P^2, (P^\mu)^2, (P^{\mu\nu})^2, \cdots$ by $itself$ is not Σ-independent, nor invariant under the generalized C-space version of the Lorentz transformations.

Choosing instead an ansatz solution to eq-(5.21) given by the following "diagonal" sum

$$\Psi(X^M, \Sigma) = \Psi_0(X, \Sigma) + \Psi_1(X^\mu, \Sigma) + \Psi_2(X^{\mu_1\mu_2}, \Sigma) + \ldots + \Psi_D(X^{\mu_1\mu_2 \ldots \mu_D}, \Sigma) \quad (5.23)$$

and inserting it into eq-(5.21) leads to the family of decoupled equations

$$i\ \frac{\partial \Psi(X^{\mu_1\mu_2 \ldots \mu_{p+1}}, \Sigma)}{\partial \Sigma} = -\ \frac{\partial^2 \Psi(X^{\mu_1\mu_2 \ldots \mu_{p+1}}, \Sigma)}{\partial X^{\mu_1\mu_2 \ldots \mu_{p+1}}\ \partial X_{\mu_1\mu_2 \ldots \mu_{p+1}}}, \quad p = -1, 0, 1, 2, \cdots, D-1. \quad (5.24)$$

where $p = -1$ corresponds to the scalar part X of the polyvector X^M. In the string, M-theory literature $p = -1$ corresponds to brane-instantons [30].

We are going to compare the wave equations (5.24) with the p-brane (p-loop) wave functional "Schrödinger-like" equation obtained by [35] and which are based on the Schild p-brane actions that are invariant under length, area, volume, hyper-volume preserving diffeomorphisms of the σ^a coordinates. As a result the corresponding Hamiltonian density is *not* zero, whereas the Hamiltonian density associated with the fully reparametrization invariant Nambu-Goto p-brane action is zero. The p-brane (p-loop) wave functional "Schrödinger-like" equation [35] is given by

$$-\ \frac{1}{2(p+1)!m^{p+1}} \left(\oint_{C_p} d^p\mathbf{s}\sqrt{(\mathbf{Y}')^2} \right)^{-1} \oint_{C_p} d^p\mathbf{s}\sqrt{(\mathbf{Y}')^2}\ \frac{\delta^2 \Psi[C_p; V_{p+1}]}{\delta Y^{\mu_0\mu_1\mu_2 \ldots \mu_p}(\mathbf{s})\ \delta Y_{\mu_0\mu_1\mu_2 \ldots \mu_p}(\mathbf{s})} =$$

$$i \, \frac{\partial \Psi[C_p; V_{p+1}]}{\partial V_{p+1}} \tag{5.25}$$

where $\mathbf{s} \equiv s^1, s^2, \ldots, s^p$ are the intrinsic spatial coordinates of the spatial p-loop C_p of topology S^p (a closed p-brane) that is moving in a D-dimensional target spacetime background and sweeping a $p+1$-dimensional timelike world *hypertube* Ω_{p+1} whose $p+1$-dimensional proper volume is V_{p+1}. The proper volume V_{p+1} acts now as a clock/temporal variable and from eq-(5.20) one can see its relation to the polyparticle's proper time Σ in C-space.

The $p+1$-vectors $Y^{\mu_0\mu_1\mu_2\ldots\mu_p}[C_p(\mathbf{s})]$ are the *holographic* [35] coordinates associated with the spatial C_p loop located at the *boundary* of the time-like world *hypertube* after it has swept a V_{p+1} volume. The C_p-loop encloses a $p+1$-dim region and whose projections onto the coordinate planes define the values of the holographic coordinates $Y^{\mu_0\mu_1\mu_2\ldots\mu_p}[C_p(\mathbf{s})]$.

The measure of integration of the p-dimensional loop C_p of topology S^p is given in terms of the square root of

$$(\mathbf{Y}')^2 \equiv \{Y^{\mu_1}, Y^{\mu_2}, \ldots, Y^{\mu_p}\}^2, \;\; \{Y^{\mu_1}, Y^{\mu_2}, \ldots, Y^{\mu_p}\} \equiv \epsilon^{i_1 i_2 \ldots i_p} \frac{\partial Y^{\mu_1}}{\partial s^{i_1}} \frac{\partial Y^{\mu_2}}{\partial s^{i_2}} \cdots \frac{\partial Y^{\mu_p}}{\partial s^{i_p}} \tag{5.26}$$

where $Y^\mu(\mathbf{s})$ are the ordinary coordinates in spacetime of the points of the p-loop C_p.

The terms inside the integrand in the left-hand side of the loop wave equation (5.25) explicitly depend on the spatial loop coordinates $\mathbf{s}$ at each point of the p-loop C_p, whereas the terms in the right-hand side only depend on the *shape* of the p-loop. For this reason one must integrate the left-hand side along *all* the points of the p-loop. This integration amounts effectively to taking the loop-average of $\frac{\delta^2 \Psi[C_p; V_{p+1}]}{\delta Y^{\mu_0\mu_1\mu_2\ldots\mu_p}(\mathbf{s}) \; \delta Y_{\mu_0\mu_1\mu_2\ldots\mu_p}(\mathbf{s})}$. In the particular case when the non-zero modes contribution averages to zero, one is solely left with the zero-modes contribution

$$< \frac{\delta^2 \Psi[C_p; V_{p+1}]}{\delta Y^{\mu_0\mu_1\mu_2\ldots\mu_p}(\mathbf{s}) \; \delta Y_{\mu_0\mu_1\mu_2\ldots\mu_p}(\mathbf{s})} >_{average} \; = \frac{\partial^2 \Psi[Y_{(0)}^{\mu_0\mu_1\mu_2\ldots\mu_p}; V_{p+1}]}{\partial Y_{(0)}^{\mu_0\mu_1\mu_2\ldots\mu_p} \; \partial Y_{\mu_0\mu_1\mu_2\ldots\mu_p;(0)}} \tag{5.27a}$$

such that the wave equation associated with the latter zero-modes denoted by $Y_{(0)}^{\mu_0\mu_1\mu_2\ldots\mu_p}$ becomes

$$-\frac{1}{2(p+1)! m^{p+1}} \frac{\partial^2 \Psi[Y_{(0)}^{\mu_0\mu_1\mu_2\ldots\mu_p}; V_{p+1}]}{\partial Y_{(0)}^{\mu_0\mu_1\mu_2\ldots\mu_p} \; \partial Y_{\mu_0\mu_1\mu_2\ldots\mu_p;(0)}} = i \, \frac{\partial \Psi[Y_{(0)}^{\mu_0\mu_1\mu_2\ldots\mu_p}; V_{p+1}]}{\partial V_{p+1}} \tag{5.27b}$$

and bears now an *identical expression* (up to numerical factors) to the functional form of eq-(5.24) obtained from the wave equation of a free polyparticle in C-space. This also requires using the explicit correspondence $\Sigma \sim V_{p+1}$ derived in eq-(5.20b), when $n = p+1$, so that $\frac{\partial}{\partial \Sigma} \leftrightarrow \frac{\partial}{\partial V_{p+1}}$ and matching the *zero* modes coordinates (quenched-like degrees of freedom) $Y_{(0)}^{\mu_0\mu_1\mu_2\ldots\mu_p}$ to the Clifford polyvectors $X^{\mu_1\mu_2\cdots\mu_{p+1}}$.

Instead of recurring to the p-brane (p-loop) wave functional "Schrödinger-like" equation [35] the polyparticle/p-brane duality/correspondence at the quantum level should be given in terms of the functional (path) integrals for the partition functions

$$\int [DX^{\mu_1\mu_2.....\mu_n}(\Sigma)]\, e^{iS[X^{\mu_1\mu_2.....\mu_n}(\Sigma)]} \leftrightarrow \int [DX^\mu(\sigma^a)]\, e^{iS[X^\mu(\sigma^a)]},\ \ a = 1,2,3,\cdots, n = p{+}1 \tag{5.28}$$

the crux of exploiting the correspondence (5.28) is that it may provide another avenue to quantize p-branes (a notoriously difficult and unsolved problem) from the more straightforward quantization of the polyparticle in C-space. For instance, a preliminary relation between the quantum membrane propagator and Clifford-polyvectors was found in [36].

We conclude by adding some further comments. So far we have only discussed the physics of a free polyparticle and free p-branes. It is warranted to introduce interactions. The Hamiltonian density for p-branes in the presence of an external potential is now given by

$$\mathcal{H} \ = \ \frac{m^{(p+1)}}{2(p+1)!}\ \{X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_{p+1}}\}^2 \ + \ V(X^{\mu_1}, X^{\mu_2}, \cdots, X^{\mu_{p+1}}) \tag{5.29}$$

where $X^\mu \ = \ X^\mu(\sigma^a),\ a = 1,2,\cdots,p+1$. Above, we simply added a potential term to the kinetic terms of the p-brane Schild Hamiltonian density.

The polyparticle version of eq-(31) is (after reintroducing the length scale parameter L)

$$\mathcal{H} \ = \ \frac{m^{(p+1)}}{2L^{2(p+1)}(p+1)!}\ (\frac{dX^{\mu_1\mu_2.....\mu_{p+1}}}{d\Sigma})^2 \ + \ V(X^{\mu_1\mu_2\cdots\mu_{p+1}}) \tag{5.30}$$

One may choose for potential (density) V the *polyparticle* analog of the relativistic point-particle harmonic oscillator and whose wave-functions in both configuration and Bargmann-Fock like space were found using group-theoretical methods by [37]. These wave-functions are provided by generalized Hermite polynomials. Similarly, one may explore the polyparticle generalization in C-space of the Born-Dirac oscillator [38] which is now characterized by an equation of the form

$$\left(i\, \gamma^M\, \frac{\partial}{\partial X^M} \ + i\, \lambda\, \gamma^M\, X_M \ - \ \mathcal{M} \right)\ \Psi(X^M, \Sigma) \ = \ i\, \frac{\partial \Psi}{\partial \Sigma}, \tag{5.31}$$

with $X^M \equiv X, X^\mu, X^{\mu_1\mu_2}, \cdots, X^{\mu_1\mu_2\cdots\mu_D}$ and exhibiting an $X^M \leftrightarrow P^M$ Born's reciprocity symmetry. As usual, powers of a suitable length (inverse mass) must be inserted in (5.31) to match physical units. The quantization of the Born-Dirac *polyparticle* oscillator in C-spaces (construction of the explicit quantum states, spectrum, ...) may facilitate the quantization program of p-branes living in ordinary spacetimes, and experiencing an external interaction, via the p-brane/polyparticle duality/correspondence proposed in this work. The energy-angular momentum spectrum of the Born-Dirac point-particle oscillator behaves as $J \sim E^2$ [38] which resembles the Regge behavior $J \sim \alpha' m^2$ of the string (α' is the inverse string tension).

Another Schrödinger-like wave equations worth mentioning are those based on the De Donder-Weyl quantization approach to gravity [39] where a parameter of inverse spatial volume dimensions, Clifford-valued wave functions and Clifford-Dirac operators are essential. To finalize, and related to the quantization approach of p-branes via the quantum polyparticle, it is worth mentioning that the n-ary commutators of the quantum operators $\hat{X}^\mu$ can be expressed in terms of Clifford polyvector-valued coordinate operators as follows

$$[\ \hat{X}^{\mu_1},\ \hat{X}^{\mu_2}\ ] \ \sim\ \hat{X}^{\mu_1\mu_2},\ \ [\ \hat{X}^{\mu_1},\ \hat{X}^{\mu_2},\ \hat{X}^{\mu_3}\ ] \ \sim\ \hat{X}^{\mu_1\mu_2\mu_3},\ \ \ldots,$$
$$[\ \hat{X}^{\mu_1},\ \hat{X}^{\mu_2}, \cdots,\ \hat{X}^{\mu_{p+1}}\ ] \ \sim\ \hat{X}^{\mu_1\mu_2\cdots\mu_{p+1}} \tag{5.32}$$

when the coordinate algebra is isomorphic to the Clifford algebra [40]. The essence of (5.32) is the compositeness nature of the polyvector-valued coordinate operators in terms of the ordinary p-brane coordinate operators.

6 Clifford Gravity Cosmology and Dark Energy

We begin by explaining the relationship between Clifford-algebra-valued Gauge Field Theories and Conformal Gravity. By fixing some of the gauge symmetries and imposing some constraints one recovers ordinary gravity. Let us show how the conformal algebra in four dimensions admits a Clifford algebra realization; i.e. the generators of the conformal algebra can be expressed in terms of the Clifford algebra basis generators. The conformal algebra in four dimensions $so(4,2)$ is isomorphic to $su(2,2)$.

Let $\eta_{ab} = (-,+,+,+)$ be the Minkowski spacetime (flat) metric in $D = 3+1$-dimenisons. The epsilon tensors are defined as $\epsilon_{0123} = -\epsilon^{0123} = 1$, The real Clifford $Cl(3,1,R)$ algebra associated with the tangent space of a $4D$ spacetime $\mathcal{M}$ is defined by the anticommutators

$$\{\ \Gamma_a,\ \Gamma_b\ \}\ \equiv\ \Gamma_a\ \Gamma_b\ +\ \Gamma_b\ \Gamma_a\ =\ \ =\ 2\ \eta_{ab} \tag{6.1a}$$

such that

$$[\Gamma_a,\Gamma_b] = 2\Gamma_{ab},\ \ \Gamma_5\ =\ -\,i\ \Gamma_0\ \Gamma_1\ \Gamma_2\ \Gamma_3,\ \ (\Gamma_5)^2\ =\ 1;\ \ \ \{\Gamma_5,\Gamma_a\}\ =\ 0; \tag{6.1b}$$

$$\Gamma_{abcd}\ =\ \epsilon_{abcd}\ \Gamma_5;\quad \Gamma_{ab} = \frac{1}{2}\left(\Gamma_a\Gamma_b - \Gamma_b\Gamma_a\right). \tag{6.2a}$$

$$\Gamma_{abc}\ =\ \epsilon_{abcd}\ \Gamma_5\ \Gamma^d;\quad \Gamma_{abcd} = \epsilon_{abcd}\ \Gamma_5. \tag{6.2b}$$

$$\Gamma_a\ \Gamma_b\ =\ \Gamma_{ab} + \eta_{ab},\quad \Gamma_{ab}\ \Gamma_5\ =\ \frac{1}{2}\epsilon_{abcd}\ \Gamma^{cd}, \tag{6.2c}$$

$$\Gamma_{ab}\ \Gamma_c\ =\ \eta_{bc}\ \Gamma_a - \eta_{ac}\ \Gamma_b + \epsilon_{abcd}\ \Gamma_5\ \Gamma^d \tag{6.2d}$$

$$\Gamma_c\ \Gamma_{ab}\ =\ \eta_{ac}\ \Gamma_b - \eta_{bc}\ \Gamma_a + \epsilon_{abcd}\ \Gamma_5\ \Gamma^d \tag{6.2e}$$

$$\Gamma_a\ \Gamma_b\ \Gamma_c\ =\ \eta_{ab}\ \Gamma_c + \eta_{bc}\ \Gamma_a - \eta_{ac}\Gamma_b + \epsilon_{abcd}\ \Gamma_5\ \Gamma^d \tag{6.2f}$$

$$\Gamma^{ab}\ \Gamma_{cd}\ =\ \epsilon^{ab}{}_{cd}\ \Gamma_5\ -4\delta^{[a}_{[c}\ \Gamma^{b]}{}_{d]}\ -2\delta^{ab}_{cd}. \tag{6.2g}$$

$$\delta^{ab}_{cd} = \frac{1}{2} (\delta^a_c \ \delta^b_d \ - \ \delta^a_d \ \delta^b_c \ ). \qquad (6.2.h)$$

the generators $\Gamma_{ab}, \Gamma_{abc}, \Gamma_{abcd}$ are defined as usual by a signed-permutation sum of the anti-symmetrizated products of the gammas. A representation of the $Cl(3,1)$ algebra exists where the generators

$$\mathbf{1}; \ \Gamma_1, \ \Gamma_2, \ \Gamma_3, \ \Gamma_4 = -i\Gamma_0; \ and \ \Gamma_5 \qquad (6.3)$$

are Hermitian; while the generators $\Gamma_a \Gamma_5$ and Γ_{ab} for $a, b = 1, 2, 3, 4$ are anti-Hermitian. Using eqs-(6.1-6.3) allows to write the $Cl(3,1)$ algebra-valued one-form as

$$\mathbf{A} = \left(a_\mu \ \mathbf{1} + \ b_\mu \ \Gamma_5 \ + \ e^a_\mu \ \Gamma_a \ + \ f^a_\mu \ \Gamma_a \ \Gamma_5 \ + \ \frac{1}{4} \omega^{ab}_\mu \ \Gamma_{ab} \right) \ dx^\mu. \qquad (6.4)$$

The physical significance of the field components $a_\mu, b_\mu, e^a_\mu, f^a_\mu, \omega^{ab}_\mu$ in eq-(6.4) will be explained below.

The Clifford-valued gauge field A_μ transforms according to $A'_\mu \ = \ U^{-1} \ A_\mu \ U + U^{-1} \partial_\mu U$ under Clifford-valued gauge transformations. The Clifford-valued field strength is $F = dA + [A, A]$ so that F transforms covariantly $F' = U^{-1} \ F \ U$. Decomposing the field strength in terms of the Clifford algebra generators gives

$$F_{\mu\nu} \ = \ F^1_{\mu\nu} \ \mathbf{1} \ + \ F^5_{\mu\nu} \ \Gamma_5 \ + \ F^a_{\mu\nu} \ \Gamma_a \ + \ F^{a5}_{\mu\nu} \ \Gamma_a \ \Gamma_5 \ + \ \frac{1}{4} F^{ab}_{\mu\nu} \ \Gamma_{ab}. \qquad (6.5)$$

the Clifford-algebra-valued 2-form field strength is $F \ = \ \frac{1}{2} \ F_{\mu\nu} \ dx^\mu \wedge dx^\nu$ and $F_{\mu\nu} \ = \ \partial_\mu A_\nu \ - \ \partial_\nu A_\mu \ + \ [\ A_\mu, \ A_\nu]$ where $\partial_\mu A_\nu = \frac{\partial A_\nu}{\partial x^\mu}$. The field-strength components are given by

$$F^1_{\mu\nu} \ = \partial_\mu a_\nu - \partial_\nu a_\mu \qquad (6.6a)$$

$$F^5_{\mu\nu} \ = \ \partial_\mu b_\nu - \partial_\nu b_\mu + 2 e^a_\mu f_{\nu a} - 2 e^a_\nu f_{\mu a} \qquad (6.6b)$$

$$F^a_{\mu\nu} \ = \ \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^{ab}_\mu e_{\nu b} - \omega^{ab}_\nu e_{\mu b} + 2 f^a_\mu b_\nu - 2 f^a_\nu b_\mu \qquad (6.6c)$$

$$F^{a5}_{\mu\nu} \ = \ \partial_\mu f^a_\nu - \partial_\nu f^a_\mu + \omega^{ab}_\mu f_{\nu b} - \omega^{ab}_\nu f_{\mu b} + 2 e^a_\mu b_\nu - 2 e^a_\nu b_\mu \qquad (6.6d)$$

$$F^{ab}_{\mu\nu} \ = \ \partial_\mu \omega^{ab}_\nu + \omega^{ac}_\mu \omega_{\nu c}{}^{\ b} + 4 \left(e^a_\mu e^b_\nu - f^a_\mu f^b_\nu \right) - \mu \longleftrightarrow \nu. \qquad (6.6e)$$

At this stage we may provide the relation among the $Cl(3,1)$ algebra generators and the the conformal algebra $so(4,2) \sim su(2,2)$ in $4D$. It is well known to the experts that the operators of the Conformal algebra can be written in terms of the Clifford algebra generators as

$$P_a \ = \ \frac{1}{2} \Gamma_a \ (1 \ - \ \Gamma_5); \quad K_a \ = \ \frac{1}{2} \Gamma_a \ (1 \ + \ \Gamma_5); \quad D \ = \ - \ \frac{1}{2} \ \Gamma_5, \quad L_{ab} \ = \ \frac{1}{2} \ \Gamma_{ab}. \ (6.7)$$

P_a ($a = 1, 2, 3, 4$) are the translation generators; K_a are the conformal boosts; D is the dilation generator and L_{ab} are the Lorentz generators. The total number of generators is respectively $4 + 4 + 1 + 6 = 15$. From the above realization of the conformal algebra generators (6.7), the explicit evaluation of the commutators yields

$$[P_a,\ D]\ =\ P_a; \quad [K_a,\ D]\ =\ -\,K_a; \quad [P_a,\ K_b]\ =\ -\,2g_{ab}\,D\ +\ 2\ L_{ab}$$

$$[P_a,\ P_b]\ =\ 0; \quad [K_a, K_b]\ =\ 0; .... \tag{6.8}$$

which is consistent with the $su(2,2) \sim so(4,2)$ commutation relations. We should notice that the K_a, P_a generators in (6.7) are both comprised of Hermitian Γ_a *and* anti-Hermitian $\pm\Gamma_a\Gamma_5$ generators, respectively. The dilation D operator is Hermitian, while the Lorentz generator L_{ab} is anti-Hermitian. The fact that Hermitian and anti-Hermitian generators are required is consistent with the fact that $U(2,2)$ is a pseudo-unitary group as we shall see bellow.

Having established this one can infer that the real-valued tetrad V_μ^a field (associated with translations) and its real-valued partner $\tilde{V}_\mu^a$ (associated with conformal boosts) can be defined in terms of the real-valued gauge fields e_μ^a, f_μ^a as follows

$$e_\mu^a\,\Gamma_a\ +\ f_\mu^a\,\Gamma_a\Gamma_5\ =\ V_\mu^a\ P_a\ +\ \tilde{V}_\mu^a\ K_a \tag{6.9}$$

From eq-(6.7) one learns that eq-(2.9) leads to

$$e_\mu^a\ -\ f_\mu^a\ =\ V_\mu^a; \quad e_\mu^a\ +\ f_\mu^a\ =\ \tilde{V}_\mu^a\ \Rightarrow$$

$$e_\mu^a\ =\ \frac{1}{2}\,(V_\mu^a + \tilde{V}_\mu^a), \quad f_\mu^a\ =\ \frac{1}{2}(\tilde{V}_\mu^a - V_\mu^a). \tag{6.10}$$

The components of the torsion and conformal-boost curvature of conformal gravity are given respectively by the linear combinations of eqs-(6.6c, 6.6d)

$$F_{\mu\nu}^a\ -\ F_{\mu\nu}^{a5}\ =\ \widetilde{F}_{\mu\nu}^a[P]; \quad F_{\mu\nu}^a\ +\ F_{\mu\nu}^{a5}\ =\ \widetilde{F}_{\mu\nu}^a[K]\ \Rightarrow$$

$$F_{\mu\nu}^a\ \Gamma_a\ +\ F_{\mu\nu}^{a5}\ \Gamma_a\ \Gamma_5\ =\ \widetilde{F}_{\mu\nu}^a[P]\ P_a\ +\ \widetilde{F}_{\mu\nu}^a[K]\ K_a. \tag{6.11a}$$

Inserting the expressions for e_μ^a, f_μ^a in terms of the vielbein V_μ^a and $\tilde{V}_\mu^a$ given by (6.10), yields the standard expressions for the Torsion and conformal-boost curvature, respectively

$$\widetilde{F}_{\mu\nu}^a[P]\ =\ \partial_{[\mu}\ V_{\nu]}^a\ +\ \omega_{[\mu}^{ab}\ V_{\nu]b}\ -\ V_{[\mu}^a\ b_{\nu]}, \tag{6.11b}$$

$$\widetilde{F}_{\mu\nu}^a[K]\ =\ \partial_{[\mu}\ \tilde{V}_{\nu]}^a\ +\ \omega_{[\mu}^{ab}\ \tilde{V}_{\nu]b}\ +2\ \tilde{V}_{[\mu}^a\ b_{\nu]}, \tag{6.11c}$$

The Lorentz curvature in eq-(6.6e) can be recast in the standard form as

$$F_{\mu\nu}^{ab} = \mathcal{R}_{\mu\nu}^{ab}\ =\ \partial_{[\mu}\ \omega_{\nu]}^{ab}\ +\ \omega_{[\mu}^{ac}\ \omega_{\nu]c}^{b}\ +\ 2(\ V_{[\mu}^a\ \tilde{V}_{\nu]}^b\ +\ \tilde{V}_{[\mu}^a\ V_{\nu]}^b\ ). \tag{6.11d}$$

The components of the curvature corresponding to the Weyl dilation generator given by $F_{\mu\nu}^5$ in eq-(6.6b) can be rewritten as

$$F_{\mu\nu}^5\ =\ \partial_{[\mu}\ b_{\nu]}\ +\ \frac{1}{2}\ (\ V_{[\mu}^a\ \tilde{V}_{\nu]a}\ -\ \tilde{V}_{[\mu}^a\ V_{\nu]a}\ ). \tag{6.11e}$$

and the Maxwell curvature is given by $F^1_{\mu\nu}$ in eq-(6.6a). A re-scaling of the vielbein V^a_μ/l and $\tilde V^a_\mu/l$ by a length scale parameter l is necessary in order to endow the curvatures and torsion in eqs-(6.11) with the proper dimensions of $length^{-2}, length^{-1}$, respectively.

To sum up, the real-valued tetrad gauge field V^a_μ (that gauges the translations P_a) and the real-valued conformal boosts gauge field $\tilde V^a_\mu$ (that gauges the conformal boosts K_a) of conformal gravity are given, respectively, by the linear combination of the gauge fields $e^a_\mu \mp f^a_\mu$ associated with the $\Gamma_a,\ \Gamma_a\ \Gamma_5$ generators of the Clifford algebra $Cl(3,1)$ of the tangent space of spacetime $\mathcal{M}^4$ after performing a Wick rotation $-i\ \Gamma_0 = \Gamma_4$.

Gauge invariant actions involving Yang-Mills terms of the form $\int Tr(F\wedge^* F)$ and theta terms of the form $\int Tr(F\wedge F)$ are straightforwardly constructed. For example, a $SO(4,2)$ gauge-invariant action for conformal gravity is [45]

$$S\ =\ \int\ d^4x\ \epsilon_{abcd}\ \epsilon^{\mu\nu\rho\sigma}\ \mathcal{R}^{ab}_{\mu\nu}\ \mathcal{R}^{cd}_{\rho\sigma} \tag{6.12}$$

where the components of the Lorentz curvature 2-form $\mathcal{R}^{ab}_{\mu\nu} dx^\mu \wedge dx^\nu$ are given by eq-(6.11c) after re-scaling the vielbein V^a_μ/l and $\tilde V^a_\mu/l$ by a length scale parameter l in order to endow the curvature with the proper dimensions of $length^{-2}$.

The conformal boost symmetry can be fixed by choosing the gauge $b_\mu = 0$ because under infinitesimal conformal boosts transformations the field b_μ transforms as $\delta b_\mu = -2\ \xi^a\ e_{a\mu} = -2\ \xi_\mu$; i.e the parameter ξ_μ has the same number of degrees of feedom as b_μ. After further fixing the dilational gauge symmetry, setting the torsion to zero (which constrains the spin connection $\omega^{ab}_\mu(V^a_\mu)$ to be of the Levi-Civita form given by a function of the vielbein V^a_μ), and eliminating the $\tilde V^a_\mu$ field algebraically via its (non-propagating) equations of motion, the expression in eq-(6.12) leads to the de Sitter group $SO(4,1)$ invariant Macdowell-Mansouri-Chamseddine-West action (MMCW) [44] (suppressing spacetime indices for convenience)

$$S\ =\ \int\ (\ R^{ab}(\omega)\ -\ \frac{1}{l^2}\ V^a\wedge V^b\ )\ \wedge\ (\ R^{cd}(\omega)\ -\ \frac{1}{l^2}\ V^c\wedge V^d\ )\ \epsilon_{abcd}. \tag{6.13}$$

The action (6.13) is comprised of 3 terms. One term is the topological invariant Gauss-Bonnet term $\int R^{ab}(\omega)\wedge R^{cd}(\omega)\epsilon_{abcd}$. The standard Einstein-Hilbert gravitational action term is given by $-\frac{1}{l^2}\int R^{ab}(\omega)\wedge V^c\wedge V^d\epsilon_{abcd}$, and the cosmological constant term $\frac{1}{l^4}\int V^a\wedge V^b\wedge V^c\wedge V^d\epsilon_{abcd}$. l is the de Sitter space's throat size; i.e. l^2 is proportional to the square of the Planck scale (the Newtonian coupling constant).

The familiar Einstein-Hilbert gravitational action can also be obtained from a coupling of gravity to a scalar field like it occurs in a Brans-Dicke-Jordan theory of gravity

$$S\ =\ \frac{1}{2}\int\ d^4x\ \sqrt{g}\ \phi\ D^c_\mu\ D^\mu_c\ \phi\ =$$

$$\frac{1}{2}\int\ d^4x\ \sqrt{g}\ \phi\ \left(\ \frac{1}{\sqrt{g}}\partial_\nu(\sqrt{g}\ g^{\mu\nu}\ D^c_\mu\phi)\ +\ b^\mu\ (D^c_\mu\phi)\ +\ \frac{1}{6}\ R\ \phi\ \right). \tag{6.14a}$$

where the conformally covariant derivative acting on a scalar field ϕ of Weyl weight one is

$$D^c_\mu \phi \ = \ (\ \partial_\mu \ - \ b_\mu \ ) \phi \tag{6.14b}$$

Fixing the conformal boosts symmetry by setting $b_\mu = 0$ and the dilational symmetry by setting $\phi = constant$ leads to the Einstein-Hilbert action for ordinary gravity.

We proceed next with the cosmological applications by introducing the Clifford-valued scalar field (a hyper-complex valued scalar) defined as

$$\mathbf{\Phi} \ = \ \Phi^A \ \Gamma_A \ = \ \phi \ \mathbf{1} \ + \ \phi^a \ \gamma_a \ + \frac{1}{2!} \ \phi^{ab} \ \gamma_{ab} \ + \frac{1}{3!} \ \phi^{abc} \ \gamma_{abc} \ + \frac{1}{4!} \ \phi^{abcd} \ \gamma_{abcd} \tag{6.15}$$

Now we can propose the most general action as an *extension* of the MMCW action displayed in eq-(6.13) and given by

$$\mathbf{S} \ = \ \int d^4x \ \epsilon^{\mu\nu\rho\sigma} \ < \ \mathbf{F}_{\mu\nu} \ \mathbf{F}_{\rho\sigma} \ \mathbf{\Phi} \ > \ = \ \int d^4x \ \epsilon^{\mu\nu\rho\sigma} \ < \ F^A_{\mu\nu} \ F^B_{\rho\sigma} \ \Phi^C \ \Gamma_A \ \Gamma_B \ \Gamma_C > \tag{6.16}$$

The bracket operation $< ..... >$ denotes extracting the Clifford scalar part of the geometric product of Clifford-valued quantities. It is the analog of taking the trace of a matrix product. The most general action can be decomposed into several pieces $\mathbf{S} = S_1 + S_2 + S_3 + S_4 + S_5$. Defining $\phi^{abcd} \ = \ \epsilon^{abcd} \ \phi^5 = \epsilon^{abcd} \ \varphi$ we have

$$S_5 \ = \ \int \ d^4x \ \epsilon^{\mu\nu\rho\sigma} \ < \ F^A_{\mu\nu} \ F^B_{\rho\sigma} \ \phi^{abcd} \ \Gamma_A \ \Gamma_B \ \gamma_{abcd} > \ =$$

$$\int \ d^4x \ \epsilon_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ \varphi \ \left(\ a_{51} F^{ab}_{\mu\nu} \ F^{cd}_{\rho\sigma} \ + a_{52} \ F^a_{\mu\nu} \ F^{bcd}_{\rho\sigma} \ + a_{53} \ F_{\mu\nu} \ F^{abcd}_{\rho\sigma} \ \right) \ +$$

$$\int \ d^4x \ \epsilon_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ \varphi \ \left(\ a_{54} F^{ab}_{\mu\nu e} \ F^{ecd}_{\rho\sigma} \ + a_{55} F^a_{\mu\nu e} \ F^{ebcd}_{\rho\sigma} \ + a_{56} F^{ab}_{\mu\nu ef} \ F^{efcd}_{\rho\sigma} \right) \tag{6.17}$$

One can rewrite (6.17) in differential form notation as

$$S_5 \ = \ \int \ \epsilon_{abcd} \ \varphi \ \left(\ a_{51} \ F^{ab} \wedge F^{cd} \ + a_{52} \ F^a \wedge F^{bcd} \ + a_{53} \ F \wedge F^{abcd} \ \right) +$$

$$\int \ \epsilon_{abcd} \ \varphi \ \left(\ a_{54} \ F^{ab}{}_e \wedge F^{ecd} \ + a_{55} \ F^a{}_e \wedge F^{ebcd} \ + a_{56} \ F^{ab}{}_{ef} \wedge F^{efcd} \right) \tag{6.18}$$

One can recognize that the MMCW action (6.13) is contained in one *piece* of S_5 and given by

$$S_{MMCW} \ \subset \ \int \ d^4x \ \epsilon_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ \varphi \ \left(\ F^{ab}_{\mu\nu} \ F^{cd}_{\rho\sigma} \ \right) \tag{6.19}$$

when $\varphi = 1$ as described by eqs-(6.6e, 6.11). One should notice that when the scalar field φ is *not* constant the expression

$$\int \ d^4x \ \sqrt{g} \ \varphi \ (\ R_{\mu\nu\rho\sigma} \ R_{\mu\nu\rho\sigma} \ - 4 \ R_{\mu\nu} \ R^{\mu\nu} \ + \ R^2 \ ) \tag{6.20}$$

is no longer equal to the Gauss-Bonnet topological invariant due to the key $\varphi(x)$ factor and such terms will now contribute to the equations of motion.

The term $\epsilon_{abcd} F^a \wedge F^{bcd}$ in (6.18) can be rewritten as $F^a \wedge \tilde{F}_a$, while the term $\epsilon_{abcd} F \wedge F^{abcd} = F \wedge \tilde{F}$, etc.... The components $F^{bcd} = F^{bcd}_{\mu\nu} dx^\mu \wedge dx^\nu$, $F^{abcd} = F^{abcd}_{\mu\nu} dx^\mu \wedge dx^\nu$, etc. ... are all given by eqs-(6.4,6.5,6.6) after taking into account the relations among the Clifford algebra generators (gamma matrices) in eqs-(6.1, 6.2). The other terms in the action are

$$S_1 \ = \ \int \ d^4x \ \epsilon^{\mu\nu\rho\sigma} \ < \ F^A_{\mu\nu} \ F^B_{\rho\sigma} \ \phi \ \Gamma_A \ \Gamma_B \ \mathbf{1} \ > \ =$$

$$\int \ d^4x \ \epsilon^{\mu\nu\rho\sigma} \ \phi \ \Big(\ a_{11} \ F_{\mu\nu} \ F_{\rho\sigma} \ + a_{12} \ F^a_{\mu\nu} \ F_{a \ \rho\sigma} \ + a_{13} \ F^{ab}_{\mu\nu} \ F_{ab \ \rho\sigma} \ \Big) \ +$$

$$\int \ d^4x \ \epsilon^{\mu\nu\rho\sigma} \ \phi \ \Big(\ a_{14} \ F^{abc}_{\mu\nu} \ F_{abc \ \rho\sigma} \ + a_{15} \ F^{abcd}_{\mu\nu} \ F_{abcd \ \rho\sigma} \ \Big) \tag{6.21}$$

One can rewrite (6.21) in differential form notation as

$$S_1 \ = \ \int \ \phi \ \Big(\ a_{11} \ F \ \wedge \ F \ + a_{12} \ F^a \ \wedge \ F_a \ + a_{13} \ F^{ab} \ \wedge \ F_{ab} \ \Big) \ +$$

$$\int \ \phi \ \Big(\ a_{14} \ F^{abc} \ \wedge F_{abc} \ + a_{15} \ F^{abcd} \ \wedge \ F_{abcd} \ \Big) \tag{6.22}$$

$$S_3 = \ \int \ d^4x \ \epsilon^{\mu\nu\rho\sigma} \ < \ F^A_{\mu\nu} \ F^B_{\rho\sigma} \ \phi^{ab} \ \Gamma_A \ \Gamma_B \ \gamma_{ab} \ > \ =$$

$$\int \ \phi_{ab} \ \Big(\ a_{31} \ F^a \wedge F^b \ + a_{32} \ F^{ab} \wedge F \ + a_{33} \ F^a_c \wedge F^{cb} \ \Big) \ +$$

$$\int \ \phi_{ab} \ \Big(\ a_{34} \ F^a_{cd} \wedge F^{cdb} \ + a_{35} \ F^a_{cde} \wedge F^{cdeb} \ \Big) \tag{6.23}$$

$$S_2 = \ \int \ d^4x \ \epsilon^{\mu\nu\rho\sigma} \ < \ F^A_{\mu\nu} \ F^B_{\rho\sigma} \ \phi^a \ \Gamma_A \ \Gamma_B \ \gamma_a \ > \ =$$

$$\int \ \phi_a \ \Big(\ a_{21} \ F^a \wedge F \ + a_{22} \ F^a_{\ b} \ \wedge \ F^b \ + a_{23} \ F^a_{bc} \wedge F^{bc} \ + a_{24} \ F^a_{bcd} \wedge F^{bcd} \ \Big) \tag{6.24}$$

$$S_4 = \ \int \ d^4x \ \epsilon^{\mu\nu\rho\sigma} \ < \ F^A_{\mu\nu} \ F^B_{\rho\sigma} \ \phi^{abc} \ \Gamma_A \ \Gamma_B \ \gamma_{abc} \ > \ =$$

$$\int \ \phi_{abc} \ \Big(\ a_{41} \ F^{abc} \wedge F \ + a_{42} \ F^{ab} \wedge F^c \ + a_{43} \ F^{abc}{}_d \wedge F^d \ \Big) \ +$$

$$\int \ \phi_{abc} \ \Big(\ a_{44} \ F^{ab}{}_d \wedge F^{dc} \ + a_{45} \ F^{ab}{}_{de} \wedge F^{dec} \ \Big) \tag{6.25}$$

the way to obtain the numerical coefficients a_{ij} is explained in the Appendix.

It is essential to introduce dynamics for the *dimensionless* Clifford-valued scalar field $\mathbf{\Phi}$ otherwise a variation of the action (6.16) with respect to the $\mathbf{\Phi}$ field will trivially constraint the action to zero since in this case $\mathbf{\Phi}$ will act as a Lagrange multiplier. The scalar field contribution to the action for the signature $(-,+,+,+)$ is

$$S[\mathbf{\Phi}] \ = \ \int \ d^4x \ \sqrt{g} \ < \ - \ \frac{1}{2l^2} \ (D_\mu \ \mathbf{\Phi}^\dagger) \ (D^\mu \ \mathbf{\Phi}) \ - \frac{1}{l^4} \ V(\mathbf{\Phi}) \ > \tag{6.26a}$$

The dagger operation $\mathbf{\Phi}^\dagger$ denotes the reversal operation and is obtained by reversing the order of the Clifford generators. For example, $(\gamma_a \wedge \gamma_b)^\dagger = \gamma_b \wedge \gamma_a$, $(\gamma_a \wedge \gamma_b \wedge \gamma_c)^\dagger = \gamma_c \wedge \gamma_b \wedge \gamma_a$, etc ..... so that

$$< \ (D_\mu \ \mathbf{\Phi}^\dagger) \ (D^\mu \ \mathbf{\Phi}) \ > \ = \ (D_\mu \phi) \ (D^\mu \phi) \ + \ (D_\mu \phi_a) \ (D^\mu \phi^a) \ + \ (D_\mu \phi_{ab}) \ (D^\mu \phi^{ab}) \ +$$

$$(D_\mu \phi_{abc}) \ (D^\mu \phi^{abc}) \ + \ (D_\mu \phi_{abcd}) \ (D^\mu \phi^{abcd}) \quad (6.26b)$$

where we have omitted combinatorial numerical factors for convenience.

The potential, for example, may be given by a polynomial $V(\mathbf{\Phi}) = \sum_{n=0} \ a_n \ \mathbf{\Phi}^n$ or a more complicated function. Upon taking the Clifford scalar part of the potential one has $< V(\mathbf{\Phi}) >= \mathcal{V}(\phi, \phi^a, \phi^{ab}, \phi^{abc}, \phi^{abcd})$ which is a complicated (polynomial, for example) expression given in terms of the 16 scalars. For simplicity we shall choose the analog of a quartic Higgs-like potential given by

$$\mathcal{V} \ = \ \frac{1}{l^4} \ \lambda \ (\ |\Phi^A \ \Phi_A| \ - \ \mathbf{v}^2)^2$$

$$\Phi^A \ \Phi_A \ = \ \phi^2 \ + \ \phi^a \phi_a \ + \frac{1}{2!} \ \phi^{ab} \phi_{ab} \ + \frac{1}{3!} \ \phi^{abc} \phi_{abc} \ + \frac{1}{4!} \ \phi^{abcd} \phi_{abcd} \quad (6.27)$$

the reason one must take the absolute value in $|\Phi^A \Phi_A|$ is because the Clifford scalar norm $\Phi^A \Phi_A$ is *not* positive definite since the 16-dimensional quadratic form has a split $(8, 8)$ signature [3] when the tangent space metric η_{ab} is Minkowskian $diag(-1, +1, +1, +1)$.

The gauge covariant derivative acting on the Clifford-valued scalar $\mathbf{\Phi}$ is defined as

$$(D_\mu \Phi^A) \ \Gamma_A \ = \ (\partial_\mu \ \Phi^A) \ \Gamma_A \ + \ [\ A_\mu^B \ \Gamma_B, \ \Phi^C \ \Gamma_C \ ] \ \Rightarrow$$

$$D_\mu \Phi^A \ = \ (\partial_\mu \ \Phi^A) \ + \ A_\mu^B \ \Phi^C \ < \ [\ \Gamma_B, \ \Gamma_C \ ] \ \Gamma^A \ > \ = \ (\partial_\mu \ \Phi^A) \ + \ A_\mu^B \ \Phi^C \ f_{BC}^{\ \ A} \quad (6.28a)$$

where we have written the commutator Clifford algebra as $[\Gamma_B, \Gamma_C] = f_{BC}^{\ \ A} \Gamma_A$ and whose structure constants are displayed in the Appendix. Under infinitesimal $Cl(3, 1)$ gauge transformations the Clifford-valued scalar $\mathbf{\Phi}$ field transforms as

$$\delta \Phi^C \ = \ f^C_{\ AB} \ \xi^A \ \Phi^B, \ \ \xi \ = \ \xi^A \ \Gamma_A \ = \ \tilde{\xi} \ \mathbf{1} \ + \ \xi^a \ \gamma_a \ + \frac{1}{2} \ \xi^{ab} \ \gamma_{ab} \ +$$

$$\frac{1}{3!} \ \xi^{abc} \ \gamma_{abc} \ + \frac{1}{4!} \ \xi^{abcd} \ \gamma_{abcd} \quad (6.28b)$$

and the gauge covariant derivative transforms as well $\delta(D_\mu \Phi^C) = f^C_{\ AB} \ \xi^A \ D_\mu \Phi^B$.

To sum up, the action $\mathbf{S} + S[\mathbf{\Phi}]$ given by eqs-(6.16-6.26) is comprised of (i) φ times the MMCW Lagrangian (6.13) that contains the Einstein-Hilbert and cosmological constant terms. (ii) Extra terms quadratic in the curvature and torsion. (iii) A coupling of curvature and torsion terms. (iv) kinetic and potential terms for a multiplet of 16 spacetime scalar fields $\phi, \phi^a, \phi^{ab}, \phi^{abc}, \phi^{abcd}$ that from the tangent space point of view behave as a scalar, vector, antisymmetric tensors of rank two and three and a pseudo-scalar field, respectively. (v) Non-minimal couplings of the scalars and curvature and torsion terms. (vi) terms involving the field strengths associated with conformal boosts, a dilational (Weyl gauge field) and a $U(1)$ Maxwell-like generator as displayed by eqs-(6.6, 6.11). A review of conformal (super) gravity can be found in [45].

Our action displayed by eqs-(6.16-6.26) is a more complex generalization of the $f(R)$ modified gravity models involving powers of the curvatures [51]. It is also a more general extension of the cosmological models based on Brans-Dicke-Jordan gravity [50] and non-minimally coupled Einstein-Electroweak theory [48]. It contains many more terms than a $U(2,2) = SU(2,2) \times U(1)$ gauge theory (conformal gravity and Maxwell theory) combined with the kinetic and potential terms of a multiplet of 16 scalar fields (corresponding to a 4×4 matrix-valued scalar in the 16-dimensional adjoint representation of $U(2,2)$).

Solving the equations of motion of the action $\mathbf{S} + S[\mathbf{\Phi}]$ after performing a variation with respect to all the fields is a very cumbersome project that requires a Clifford computer algebra package and which is beyond the scope of this work. Fixing and/or breaking some of the gauge symmetries will simplify things. Let us *truncate* the action given in eqs-(6.16,6.26) by *freezing* all the components of $\mathbf{\Phi}$ to zero except φ so that the following Higgs-like potential $\mathcal{V}$

$$\mathcal{V} = \frac{1}{l^4} \lambda (\varphi^2 - \mathbf{v}^2)^2, \quad \lambda > 0 \tag{6.29}$$

is minimized to zero when $\varphi_o = \mathbf{v}$. Focusing solely on the terms in eq-(6.19) and the Higgs potential in eq-(6.26a), we have (i) φ times the { Gauss-Bonnet terms, the Einstein-Hilbert action, and the cosmological constant }; and (ii) the *effective* potential energy density given by the scalar potential minus the running cosmological "constant" term

$$U_{eff} = \frac{1}{l^4} \lambda (\varphi^2 - \mathbf{v}^2)^2 - \frac{\varphi}{l^4} \tag{6.30}$$

Let us define the reduced Planck mass by $M_P^2 = (1/8\pi L_P^2)$ and equate the Planck energy density $\frac{1}{4} M_P^4$ to the value of U_{eff} when $\varphi = 0$ in eq-(6.30)

$$U_{eff}(\varphi = 0) = \frac{1}{l^4} (\lambda \mathbf{v}^4) = \frac{1}{4} M_P^4 = \frac{1}{(16\pi)^2 L_P^4} \tag{6.31}$$

By equating the value of the effective potential energy density at $\varphi = \varphi_*$ to the present-day observed vacuum energy density one has

$$U_{eff}(\varphi_*) = \frac{1}{l^4} \lambda (\varphi_*^2 - \mathbf{v}^2)^2 - \frac{\varphi_*}{l^4} = \rho_{obs} \sim \frac{1}{L_P^2 R_H^2} =$$

$$(\frac{L_P}{R_H})^2 \frac{1}{L_P^4} \sim 10^{-120} M_P^4 \tag{6.32}$$

where L_P and R_H are the Planck and Hubble scale, respectively. The ratio $(\frac{L_P}{R_H})^2$ is chosen to be of the order of 10^{-120}. Matching the present-day value of the Newtonian coupling constant with the running coupling appearing in the Einstein-Hilbert term in eq-(2.19), when $\varphi = \varphi_*$, gives

$$\frac{\varphi_*}{l^2} = \frac{1}{16\pi G} = \frac{1}{2} \frac{1}{8\pi L_P^2} = \frac{1}{2} M_P^2 \tag{6.33}$$

It is interesting to note that negative values of φ furnish a negative coupling G that would correspond to a repulsive gravitational regime. For the time being we shall focus in the case where $\varphi \geq 0$.

Finally, from eqs-(6.30, 6.31, 6.33) one arrives at the following numerical results for the $l, \mathbf{v}, \lambda$ parameters of the Higgs-like potential (6.29)

$$l \simeq R_H, \quad \mathbf{v} \simeq \frac{1}{16\pi} (\frac{R_H}{L_P})^2, \quad \lambda \simeq (16\pi)^2 (\frac{L_P}{R_H})^4 \tag{6.34}$$

and $\varphi_* \simeq \mathbf{v}$.

From the plot of the graph U_{eff}/ρ_{obs} versus φ one learns that $\varphi_* < \varphi_o = \mathbf{v}$ but its value is *very* close to $\mathbf{v}$. Since the throat size of the present de Sitter accelerating universe $l = R_H$ agrees with the value for l obtained in eq-(6.34) this is sign of consistency. The value of $\varphi_* + \epsilon$ is the *crossover* point when the effective potential energy density (6.30) switches from *positive* to *negative* values as φ increases (assuming it increases with the flow of time). Anti de Sitter spacetime has a constant negative energy density and positive pressure (attractive force) ; whereas de Sitter spacetime has a constant positive energy density and negative pressure (repulsive force). In our most simplified scenario, the universe has not entered yet the phase of negative energy density where its expansion will deccelerate, until the point φ_{**} , when it will crossover again into a positive energy density epoch of perpetual accelerated expansion.

Our results obtained above are compatible with a very rapid de Sitter inflationary phase in the very early universe because of the very large initial value of the (positive) energy density. An extensive and recent review (with a vast number of references) about cosmological inflation and its realization in quantum field theory and in string theory can be found [46]. Furthermore, our results are also consistent with the present-day de Sitter accelerating universe with a very small value of the vacuum energy density (6.32) due to the very large value of the Hubble scale. More recently, the authors [49] have argued that the so-called cosmological constant fine-tuning problem (why the cosmological constant observed today is so much smaller than the Planck scale or why the universe is accelerating at present) can be solved with the help of Higgs inflation by simply assuming a variable cosmological "constant" during the inflation epoch. This is compatible with our findings.

To sum up, in our simplified scenario all the parameters $l, \mathbf{v}, \lambda$ of the Higgs-like scalar potential (6.29) are given in terms of the *two* fundamental scales, L_P, R_H (a lower and upper scale) by eq-(6.34) which allows us to reproduce the extremely small observed vacuum energy density (6.32) and the current value of the Newtonian gravitational coupling (6.33). Nottale [23] in his development of the Scale Relativity Theory has proposed a resolution of the cosmological constant problem based also on these two fundamental scales L_P, R_H.

The fact that a running Newtonian coupling in eq-(6.33) leads to $G = \frac{l^2}{16\pi\varphi} \rightarrow \infty$ when $\varphi \rightarrow 0$, at the Big Bang singularity for example, does not mean that the Einstein-Hilbert action necessarily collapses to zero, because one may have $R = \infty$ at the singularity such that the ratio $R/16\pi G$ might still be well defined. In order to study the behavior of the scalar φ as a function of x^μ, one has to determine the spacetime dynamics of $\varphi(x^\mu)$ which is obtained by performing a variation of the truncated action with respect to φ, and yielding a very complex equation of the form

$$\frac{1}{l^2}\ D_\mu D^\mu \varphi\ - \frac{1}{l^4}\ \frac{\partial V(\varphi)}{\partial \varphi}\ +\ \epsilon_{abcd}\ \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}\ (a_{51}\ F^{ab}_{\mu\nu}\ F^{cd}_{\nu\sigma}\ +\ ......)\ +$$

$$\frac{1}{l^2}\ (A^a_\mu A^\mu_a\ +\ A^{abc}_\mu A^\mu_{abc})\ \varphi\ =\ 0 \tag{6.35}$$

the last terms in (6.35) stem from the contribution $[\mathbf{A}_\mu, \mathbf{\Phi}]^2$ to the $(D_\mu \Phi^A)(D^\mu \Phi_A)$ terms in the truncated action.

One cannot solve eq-(6.35) without performing a variation of the action with respect to the remaining gauge fields. In the most general case, one has to study the full space-time dynamics of *all* the gauge fields involved in the non-truncated action, with the key contribution of the kinetic and potential terms $(D_\mu \Phi_A)(D^\mu \Phi^A)$, $\mathcal{V}(\Phi^A)$ for *all* the scalars, to see whether or not there is a dynamical evolution of the 16 scalar fields that is consistent with the extremely small value of the vacuum energy density observed today, and associated with a de Sitter accelerated phase of expansion. The throat size of the de Sitter solution is $l = R_H$.

Fermionic matter terms and gauge fields of the Standard and GUT Models should be taken into account in the most general theory. A de Sitter, Anti de Sitter and Minkowski vacuum spacetime solution is also consistent with a breaking of the $SU(2,2) \sim SO(4,2)$ conformal symmetry down to the de Sitter $SO(4,1)$, Anti de Sitter $SO(3,2)$ and Minkowski $SO(4)$ one. Recently, the authors [47] studied the problem of obtaining de Sitter and inflationary vacua from dimensional reduction of double field theory (DFT) on non-geometric string backgrounds. They also considered a new class of effective potentials that admit Minkowski and de Sitter minima.

Before embarking into the study of the full action comprised of eqs-(6.16-6.26), one can start instead with the simpler Clifford-gravity inspired action

$$S\ =\ \int d^4x\ \sqrt{g}\ \Big(\ \varphi\ [\ R_{\mu\nu\rho\sigma}\ R_{\mu\nu\rho\sigma}\ - 4\ R_{\mu\nu}\ R^{\mu\nu}\ +\ R^2\ ]\ - \frac{\varphi}{l^2}\ R\ + \frac{\varphi}{l^4}\ \Big)\ -$$

$$\int d^4x\ \sqrt{g}\ (\ \frac{1}{2l^2}\ (\partial_\mu \varphi)\ (\partial^\mu \varphi)\ +\ \frac{1}{l^4}\ V(\varphi)\ ) \tag{6.36}$$

as a testing ground for cosmological scenarios. An even simpler action was the Weyl invariant action investigated in [52] where the source of dark energy was identified with a dilaton-like scalar field θ of dimensions $length^{-1}$ that is required to implement Weyl (scale) invariance of the action

$$S\ =\ \frac{1}{16\pi}\ \int d^4x\ \sqrt{g}\ \Big(\ -\ \theta^2\ \mathcal{R}_{Weyl}\ - \frac{1}{2}\ g^{\mu\nu}\ (D_\mu\theta)\ (D_\nu\theta)\ -\ V(\theta)\ \Big) \tag{6.37a}$$

under the Weyl scalings

$$\theta' = e^{-\Omega}\ \theta;\ \ g'_{\mu\nu} = e^{2\Omega} g_{\mu\nu},\ \ \mathcal{R}'_{Weyl} = e^{-2\Omega}\mathcal{R}_{Weyl},\ \ V(\theta') = e^{-4\Omega} V(\theta)$$

$$D_\mu\theta = \partial_\mu\theta - A_\mu\theta \ \rightarrow \ (D_\mu\theta)' = e^{-\Omega} D_\mu\theta, \quad A'_\mu = A_\mu - \partial_\mu\Omega, ... \tag{6.37b}$$

the Weyl symmetry naturally selects a quartic potential $V \sim \theta^4$. It was shown in [52] how the action was related to a Brans-Dicke-Jordan model whose ω parameter had its critical value $\omega = -3/2$ and leading to the observed constant vacuum energy density when the scalar field θ was scaled to a constant such that $(\theta_o)^2 = 1/G$. Closely related results have been obtained recently by [53], where dark energy is due to the existence of a Dirac scalar field in a conformal theory of gravitation. In this cosmological model, dark energy (described by an effective cosmological constant) is a function of a Dirac scalar field and such that there is an exponential decrease of the value of the scalar field (from the inflation stage) down to a constant limiting value at large times.

To conclude, we believe that Clifford-gravity-based cosmology is a promising avenue to understand the origins of the very small presently observed value of the vacuum energy density, and the 16 scalar fields corresponding to the Clifford-valued scalar $\mathbf{\Phi}$ in four-dimensions could be plausible dark energy/matter candidates.

7 Moyal Deformations of Clifford Gauge Theories of Gravity

In this section, a Moyal deformation of a Clifford $Cl(3,1)$ Gauge Theory of (Conformal) Gravity is performed for canonical noncommutativity (constant $\Theta^{\mu\nu}$ parameters). In the very special case when one imposes certain constraints on the fields, there are *no* first order contributions in the $\Theta^{\mu\nu}$ parameters to the Moyal deformations of Clifford gauge theories of gravity. However, when one does *not* impose constraints on the fields, there are first order contributions in $\Theta^{\mu\nu}$ to the Moyal deformations in variance with the previous results obtained by other authors and based on different gauge groups. Despite that the generators of $U(2,2), SO(4,2), SO(2,3)$ can be expressed in terms of the Clifford algebra generators this does *not* imply that these algebras are isomorphic to the Clifford algebra. Therefore one should not expect identical results to those obtained by other authors. In particular, there are Moyal deformations of the Einstein-Hilbert gravitational action with a cosmological constant to first order in $\Theta^{\mu\nu}$. Finally, we provide a mechanism which furnishes a plausible cancellation of the huge vacuum energy density.

Let us begin with the associative and noncommutative Moyal star product when the (inverse) symplectic form $\Omega^{\mu\nu} = -\Omega^{\mu\nu}$ does *not* have an X-dependence. It is defined as

$$(A_1 \ * \ A_2)(Z) \ = \ exp\left(\frac{1}{2} \ \Omega^{\mu\nu} \ \partial_{X^\mu} \ \partial_{Y^\nu} \right) \ A_1(X) \ A_2(Y)|_{X=Y=Z} \ =$$

$$\sum_{n=0}^{\infty} \ \frac{(\frac{1}{2})^n}{n!} \ \Omega^{\mu_1\nu_1} \ \Omega^{\mu_2\nu_2} \ .... \ \Omega^{\mu_n\nu_n} \ (\partial^n_{\mu_1\mu_2......\mu_n} \ A_1) \ (\partial^n_{\nu_1\nu_2......\nu_n} \ A_2) \tag{7.1}$$

$$\partial^n_{\mu_1\mu_2......\mu_n} \ A_1(Z) \ \equiv \ \partial_{\mu_1} \ \partial_{\mu_2} \ ...... \ \partial_{\mu_n} \ A_1(Z). \tag{7.2a}$$

$$\partial^n_{\nu_1\nu_2......\nu_n} \ A_2(Z) \ \equiv \ \partial_{\nu_1} \ \partial_{\nu_2} \ ...... \ \partial_{\nu_n} \ A_2(Z). \tag{7.2b}$$

For simplicity we shall take the very special case of canonical noncommutativity $[X^\mu, X^\nu]_* = i\Theta^{\mu\nu} = \Omega^{\mu\nu} = constants$, such that the star product is the standard Moyal one. If the fields and their derivatives vanishing fast enough at infinity, one has the *cyclicity* property of the integral

$$\int \ A * B \ = \ \int \ A \ B \ + \ total \ derivative \ = \ \int \ A \ B \ = \ \int \ B \ * \ A \tag{7.3}$$

$$\int \ A \ * \ B \ * \ C \ = \ \int \ A \ (B \ * \ C) \ + \ total \ derivative \ = \ \int \ A \ (B \ * \ C) \ =$$

$$\int \ (B \ * \ C) \ A \ = \ \int \ (B \ * \ C) \ * \ A \ + \ total \ derivative \ = \ \int \ B \ * C \ * \ A \tag{7.4}$$

therefore, when the star product is associative and the fields and their derivatives vanishing fast enough at infinity (or there are no boundaries) one has

$$\int \ A \ * \ B \ * C \ = \ \int \ B \ * C \ * \ A \ = \ \int \ C \ * A \ * \ B \ . \tag{7.5}$$

The relations (7.3-7.5) are essential in order to construct invariant actions under star gauge transformations of the form $\delta F_{\mu\nu} = i[\xi, F_{\mu\nu}]_*$. The invariance of the actions is due to the associativity property of the star products and the cyclicity property of the integrals and of the Clifford scalar part of the geometric product of the Clifford generators. Taking the scalar part is the analog of the trace of a matrix product.

One should notice, for example, that when one has a Lie-algebraic type of noncommutativity, the $\Theta's$ are now X-dependent $[X^\mu, X^\nu]_* = i\Theta^{\mu\nu}(X) = if^{\mu\nu}_\rho X^\rho$ so that the cyclicity property *no longer holds* since the star product is X-dependent. For a detailed study of how to remedy this problem see [64].

Due to the noncommutativity of the spacetime coordinates, the components of the Clifford-algebra valued field strength are now *modified* as follows

$$\mathcal{F}_{\mu\nu} \ = \ \mathcal{F}^C_{\mu\nu} \ \Gamma_C \ = \ (\ \partial_\mu \ \mathcal{A}^C_\nu \ - \ \partial_\nu \ \mathcal{A}^C_\mu \ ) \ \Gamma_C \ -$$

$$\frac{i}{2} \ (\ \mathcal{A}^A_\mu * \mathcal{A}^B_\nu \ - \ \mathcal{A}^B_\nu * \mathcal{A}^A_\mu \ ) \ \{ \ \Gamma_A, \ \Gamma_B \ \} \ - \ \frac{i}{2} \ (\ \mathcal{A}^A_\mu * \mathcal{A}^B_\nu \ + \ \mathcal{A}^B_\nu * \mathcal{A}^A_\mu \ ) \ [\ \Gamma_A, \ \Gamma_B \ ]. \tag{7.6}$$

The commutators $[\ \Gamma_A, \ \Gamma_B \ ]$ and anti-commutators $\{ \ \Gamma_A, \ \Gamma_B \ \}$ in eq-(7.6), where A, B are polyvector-valued indices, can be obtained from all the relations provided in the Appendix. Notice that both the standard commutators *and* anticommutators of the gammas appear in eq-(7.6) and which now define the Clifford-algebra valued field strength in noncommutative spacetimes; i.e. if the products of fields were to commute one would have had only the Lie algebra commutator $\mathcal{A}^A_M \mathcal{A}^J_B \ [\Gamma_A, \Gamma_B]$ pieces without the anti-commutator $\{\Gamma_A, \Gamma_B\}$ contributions in the r.h.s of eq-(7.6).

We should remark that one is *not* deforming the Clifford algebra involving $[\ \Gamma_A, \ \Gamma_B \ ]$ and $\{ \ \Gamma_A, \ \Gamma_B \ \}$ in eq-(7.6) but it is the "point" product algebra $\mathcal{A}^A_M * \mathcal{A}^B_N$ of the fields which is being deformed. (Quantum) q-Clifford algebras have been studied extensively by [68].

The symmetrized star product in terms of $\Theta^{\mu\nu} = constants$ is

$$\mathcal{A}^A_\mu \ *_s \ \mathcal{A}^B_\nu \ \equiv \ \frac{1}{2} \left(\mathcal{A}^A_\mu * \mathcal{A}^B_\nu + \mathcal{A}^B_\nu * \mathcal{A}^A_\mu \right) \ = \ \mathcal{A}^A_\mu \ \mathcal{A}^B_\nu \ + \frac{i^2}{2!} \Theta^{\alpha\beta} \ \Theta^{\kappa\lambda} \ (\partial_\alpha \ \partial_\kappa \ \mathcal{A}^A_\mu) \ (\partial_\beta \ \partial_\lambda \ \mathcal{A}^B_\nu) \ + \ ...... \tag{7.7}$$

the antisymmetrized (Moyal bracket) star product is

$$\mathcal{A}^A_\mu *_a \mathcal{A}^B_\nu \ \equiv \ \frac{1}{2} \left(\mathcal{A}^A_\mu * \mathcal{A}^B_\nu - \mathcal{A}^B_\nu * \mathcal{A}^A_\mu \right) \ = \ i \ \Theta^{\alpha\beta} \ (\partial_\alpha \ \mathcal{A}^A_\mu) \ (\partial_\beta \ \mathcal{A}^B_\nu) \ + \ ..... \tag{7.8}$$

Early works on Moyal deformations of gravity can be found in [60],[57],[55]. Examples of an X-dependent $\Theta^{\mu\nu}(x)$ occurs in κ-deformed Minkowski spacetimes [19]. An extension of the Seiberg – Witten (SW) map for X-dependent $\Theta^{\mu\nu}(x)$ was provided by [64], [58], [59], [65], [66], among others, relating the non-Abelian noncommutative gauge fields based on *noncommutative* coordinates and the non-Abelian gauge fields based on *commutative* coordinates. It is then when one may construct the proper expressions for the $deformed$ field strengths, associated with the *noncommutative* coordinates, in terms of the undeformed field strengths. Since the former involve the universal enveloping algebra that is $infinite$ dimensional one must find a criteria to reduce the number of the degrees of freedom to a finite one; this is attained via the Seiberg-Witten map.

The main advantage of recurring to a Clifford algebraic formulation described in this work, is that both the commutator and anticommutator algebra in eq-(7.6) *closes* and this will simplify the laborious and cumbersome Seiberg-Witten procedure, involving the universal enveloping algebra. One may now proceed to perform the Moyal deformations of the field strengths and the action in a straightforward fashion.

The Moyal deformation of the terms S_5 encoding the MMCW gravitational action with a cosmological constant is given by

$$S_{(5)*} \ = \ \int \ d^4x \ \epsilon^{\mu\nu\rho\sigma} \ < \ \mathcal{F}^A_{\mu\nu} \ * \mathcal{F}^B_{\rho\sigma} \ * \phi^{abcd} \ \Gamma_A \ \Gamma_B \ \gamma_{abcd} > \ =$$

$$\int \ d^4x \ \epsilon_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ \varphi \ * \Big(\ a_{51} \mathcal{F}^{ab}_{\mu\nu} \ * \mathcal{F}^{cd}_{\rho\sigma} \ + a_{52} \ \mathcal{F}^a_{\mu\nu} \ * \mathcal{F}^{bcd}_{\rho\sigma} \ + a_{53} \ \mathcal{F}_{\mu\nu} \ * \mathcal{F}^{abcd}_{\rho\sigma} \ \Big) \ +$$

$$\int \ d^4x \ \epsilon_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ \varphi \ * \Big(\ a_{54} \mathcal{F}^{ab}_{\mu\nu e} \ * \mathcal{F}^{ecd}_{\rho\sigma} \ + a_{55} \mathcal{F}^a_{\mu\nu e} \ * \mathcal{F}^{ebcd}_{\rho\sigma} \ + a_{56} \mathcal{F}^{ab}_{\mu\nu ef} \ * \mathcal{F}^{efcd}_{\rho\sigma} \Big) \tag{7.9}$$

Before studying the Moyal deformations given by the action (7.9) one needs to establish the dictionary among the different Clifford $Cl(3,1)$ gauge field components and the fields of conformal gravity. From eqs-(6.4-6.6) one can infer the following correspondence

$$\mathcal{A}^{ab}_\mu \leftrightarrow \omega^{ab}_\mu, \ \ \mathcal{A}^a_\mu \leftrightarrow e^a_\mu, \ \ \mathcal{A}^{abc}_\mu \leftrightarrow f^a_\mu, \ \ \mathcal{A}^{abcd}_\mu \leftrightarrow b_\mu, \ \ \mathcal{A}_\mu \leftrightarrow a_\mu \tag{7.10}$$

Let us look at the $first$ order Θ-corrections to the components of $F^{ab}_{\mu\nu}$ given by eq-(6.6e) upon using eq-(7.6) and the equations in the Appendix

$${}^{(1)}\mathcal{F}^{ab}_{\mu\nu} \ = \ \mathcal{F}^{ab}_{\mu\nu} \ + \ \ \Theta^{\alpha\beta} \ \partial_\alpha \mathcal{A}^{abe}_\mu \ \partial_\beta \mathcal{A}_{\nu e} \ - \ \ \Theta^{\alpha\beta} \ \partial_\alpha \mathcal{A}^{abef}_\mu \ \partial_\beta \mathcal{A}_{\nu ef} \tag{7.11}$$

Repeating this procedure with the other field strength components in eqs-(6.6a-6.6d) yields the *first* order Θ-corrections

$$^{(1)}\mathcal{F}_{\mu\nu} \;=\; \mathcal{F}_{\mu\nu} \;+ 2\;\Theta^{\alpha\beta}\;\partial_\alpha \mathcal{A}^e_\mu\;\partial_\beta \mathcal{A}_{\nu e}\;-$$

$$2\;\Theta^{\alpha\beta}\;\partial_\alpha \mathcal{A}^{ef}_\mu\;\partial_\beta \mathcal{A}_{\nu ef} - 2\;\Theta^{\alpha\beta}\;\partial_\alpha \mathcal{A}^{efg}_\mu\;\partial_\beta \mathcal{A}_{\nu efg}\;+$$

$$2\;\Theta^{\alpha\beta}\;\partial_\alpha \mathcal{A}^{efgh}_\mu\;\partial_\beta \mathcal{A}_{\nu efgh} \tag{7.12}$$

$$^{(1)}\mathcal{F}^a_{\mu\nu} \;=\; \mathcal{F}^a_{\mu\nu} \;- 2\;\Theta^{\alpha\beta}\;\partial_\alpha \mathcal{A}^{aef}_\mu\;\partial_\beta \mathcal{A}_{\nu ef} \tag{7.13}$$

$$^{(1)}\mathcal{F}^{abc}_{\mu\nu} \;=\; \mathcal{F}^{abc}_{\mu\nu} \;+ 2\;\Theta^{\alpha\beta}\;\partial_\alpha \mathcal{A}^{ab}_\mu\;\partial_\beta \mathcal{A}^c_\nu\;-\frac{1}{2}\;\Theta^{\alpha\beta}\;\partial_\alpha \mathcal{A}^{abef}_\mu\;\partial_\beta \mathcal{A}^c_{\nu ef} \tag{7.14}$$

$$^{(1)}\mathcal{F}^{abcd}_{\mu\nu} \;=\; \mathcal{F}^{abcd}_{\mu\nu} + 2\;\Theta^{\alpha\beta}\;\partial_\alpha \mathcal{A}^{ab}_\mu\;\partial_\beta \mathcal{A}^{cd}_\nu\;+\frac{1}{2}\;\Theta^{\alpha\beta}\;\partial_\alpha \mathcal{A}^{abe}_\mu\;\partial_\beta \mathcal{A}^{cd}_{\nu e}\;-$$

$$\frac{1}{4}\;\Theta^{\alpha\beta}\;\partial_\alpha \mathcal{A}^{abef}_\mu\;\partial_\beta \mathcal{A}^{cd}_{\nu ef} \tag{7.15}$$

We have indicated in the previous equations (7.11-7.15) that one has a first order correction by attaching explicitly a superscript $^{(1)}$ to the field strength expressions in the left hand side. The expressions for the components of $\mathcal{F}^A_{\mu\nu}$ in the right hand side are obtained explicitly from eqs-(6.6a-6.6e) by *replacing* the *commutative* gauge fields A^A_μ for the *noncommutative* ones $\mathcal{A}^A_\mu$.

Having written the above expressions (7.11-7.15) for the noncommutative field strengths in terms of the noncommutative gauge fields $\mathcal{A}^A_\mu$ it remains to write the latter noncommutative fields in terms of the commutative fields A^A_μ via the Seiberg-Witten map procedure. A lengthy procedure (see [61], [62]) yields the following expression for the noncommutative field strengths $\mathcal{F}_{\mu\nu}$ in terms of the commutative fields, after ommiting the Clifford-valued internal indices for simplicity since $\mathcal{F}_{\mu\nu} \equiv \mathcal{F}^A_{\mu\nu}\Gamma_A$, $\mathbf{F}_{\mu\nu} \equiv F^A_{\mu\nu}\Gamma_A$, $\mathbf{A}_\mu \equiv A^A_\mu\Gamma_A$,

$$\mathcal{F}_{\mu\nu} \;=\; \mathbf{F}_{\mu\nu} \;+\; \frac{1}{2}\Theta^{\alpha\beta}\{\mathbf{F}_{\mu\alpha},\;\mathbf{F}_{\nu\beta}\} \;-\; \frac{1}{4}\Theta^{\alpha\beta}\{\mathbf{A}_\alpha,\;(\partial_\beta + \mathcal{D}_\beta)\mathbf{F}_{\mu\nu}\} + \cdots \tag{7.16}$$

where the covariant derivative is defined in the adjoint representation

$$\mathcal{D}_\sigma \mathbf{F}_{\mu\nu} \;=\; \partial_\sigma \mathbf{F}_{\mu\nu} \;- i\;[\mathbf{A}_\sigma,\;\mathbf{F}_{\mu\nu}]. \tag{7.17}$$

Similarily, the Seiberg-Witten map allows to express the noncommutative scalar fields components present in the Clifford-valued field $\hat{\mathbf{\Phi}}$ in terms of the commutative scalar fields components present in the Clifford-valued field $\mathbf{\Phi}$

$$\hat{\mathbf{\Phi}} \;=\; \mathbf{\Phi} \;-\frac{1}{4}\;\Theta^{\alpha\beta}\;\{A_\alpha,\;(\partial_\beta + D_\beta)\mathbf{\Phi}\}\;+\;\ldots \tag{7.18}$$

see [61] for the case of a $SO(2,3)$-valued scalar field.

All that rests now is to evaluate the individual components of $\mathcal{F}_{\mu\nu} \equiv \mathcal{F}^A_{\mu\nu}\Gamma_A$ in the left hand side of (7.16) after performing the geometric products of the Clifford algebra generators appearing in the right hand side of (7.16) due to the decomposition of $\mathbf{F}_{\mu\nu} \equiv F^A_{\mu\nu}\Gamma_A$, $\mathbf{A}_\mu \equiv A^A_\mu\Gamma_A$. A similar procedure is performed in eq-(7.18).

We shall focus for now on the contribution up to first order in the Θ-terms to the Clifford bivector components $\mathcal{F}^{ab}_{\mu\nu}\gamma_{ab}$

$$ {}^{(1)}\mathcal{F}^{ab}_{\mu\nu} = F^{ab}_{\mu\nu} + \frac{1}{2}\,\Theta^{\alpha\beta}\,\Big(F^{abc}_{\mu\alpha}\,F_{\nu\beta c} - F^{abcd}_{\mu\alpha}\,F_{\nu\beta cd}\Big) + $$
$$ \frac{1}{2}\,\Theta^{\alpha\beta}\,\Big(F_{\mu\alpha c}\,F^{cab}_{\nu\beta} - F_{\mu\alpha cd}\,F^{cdab}_{\nu\beta}\Big) + \ldots \tag{7.19} $$

The extra terms in (7.19) are of the form $\Theta(A\partial F + AAF)$. For example

$$ -\frac{1}{4}\,\Theta^{\alpha\beta}\,\Big(A^{abc}_\alpha\,\partial_\beta F_{\mu\nu c} - A^{abcd}_\alpha\,\partial_\beta F_{\mu\nu cd}\Big) - \frac{1}{4}\,\Theta^{\alpha\beta}\,A^{abc}_\alpha\,A_{\beta cd}\,F^d_{\mu\nu} \tag{7.20} $$

A similar procedure yields the expression for the noncommutative scalar field $\hat{\phi}^{abcd} = \epsilon^{abcd}\hat{\varphi}$ in terms of the commutative scalar and gauge fields.

The higher order corrections in Θ are obtained from the higher order terms in the definition of the Moyal star products and in those terms generated by the Seiberg-Witten map. Comparing our results, based on the Moyal deformations provided by eq-(7.9), with the results of others we should emphasize that the authors [62] had for their starting $U(2,2)$ invariant Lagrangian only the two terms (omitting numerical factors)

$$ L = \epsilon_{abcd}\,\Big(F^{ab}\wedge F^{cd} + F\wedge F^{abcd}\Big) \tag{7.21} $$

instead of the *six* terms present in eq-(6.18). Secondly, they imposed by hand several *constraints* on the fields such that $F_{\mu\nu} = F^a_{\mu\nu} = F^{abc}_{\mu\nu} = F^{abcd}_{\mu\nu} = 0$. And thirdly, they set $\varphi = constant$.

Whereas the authors [61] used the Seiberg-Witten map procedure to construct a model of noncommutative gravity based on the gauge theory of $SO(2,3)$ defined over a noncommutative spacetime characterized by $\Theta^{\mu\nu}$ = constants. The starting Lagrangian in [61] was chosen to be

$$ L = \epsilon_{abcd}\,\varphi\,F^{ab}\wedge F^{cd} \tag{7.22} $$

They found a cancellation of the Θ-terms to first order and which agrees with the results obtained by the authors [62] (for the group $U(2,2)$) when one has a canonical noncommutativity. It appears that the cancellation of the first order terms in $\Theta^{\mu\nu}$ might be model-independent.

Let us examine carefully the Moyal deformation of the eq-(6.16) after one inserts the explicit expressions for the *noncommutative* fields inside the integral

$$ \int d^4x\;\epsilon^{\mu\nu\rho\sigma}\;<\;\hat{\mathbf{\Phi}}\,*\,\mathcal{F}_{\mu\nu}\,*\mathcal{F}_{\rho\sigma}\;> \tag{7.23} $$

the Θ-terms up to $first$ order in the integrand will be

$$\mathbf{\Phi}\ (\mathcal{F}_{\mu\nu}\ *\mathcal{F}_{\rho\sigma})^{(1)}\ +\ \hat{\mathbf{\Phi}}^{(1)}\ (F_{\mu\nu}\ F_{\rho\sigma})\ +$$
$$\frac{i}{2}\ \Theta^{\alpha\beta}\ \partial_\alpha\Phi\ \partial_\beta(F_{\mu\nu}\ F_{\rho\sigma}) \tag{7.24}$$

The last term is a total derivative after an integration by parts due to the condition $\Theta^{\alpha\beta}\partial_\alpha\partial_\beta(...) = 0$. Hence the last term decouples (it can be dropped if the fields vanish fast enough at infinity or there are no boundaries). This is to be expected if one does not wish to introduce imaginary terms to the Moyal deformed action. The hats represent the noncommuative scalars and $\hat{\mathbf{\Phi}}^{(1)}$ is the first order contribution in Θ to the noncommutative scalar field. $\mathbf{\Phi}$ is the Clifford-valued scalar field with *commutative* components.

The first two terms of eq-(7.24) gives

$$-\frac{\Theta^{\alpha\beta}}{4}\ <\ \{\mathbf{A}_\alpha,\ (\partial_\beta + D_\beta)\ \mathbf{F}_{\mu\nu}\}\ \mathbf{F}_{\rho\sigma}\ \mathbf{\Phi}\ >\ \epsilon^{\mu\nu\rho\sigma}\ -$$
$$-\frac{\Theta^{\alpha\beta}}{4}\ <\ \mathbf{F}_{\mu\nu}\ \mathbf{F}_{\rho\sigma}\ \{\mathbf{A}_\alpha,\ (\partial_\beta + D_\beta)\ \}\ \mathbf{\Phi}\ >\ \epsilon^{\mu\nu\rho\sigma}\ +$$
$$\frac{\Theta^{\alpha\beta}}{2}\ <\ \{\mathbf{F}_{\alpha\mu},\ \mathbf{F}_{\beta\nu}\}\ \mathbf{F}_{\rho\sigma}\ \mathbf{\Phi}\ >\ \epsilon^{\mu\nu\rho\sigma}\ +\ \ldots \tag{7.25}$$

The terms that one must extract the Clifford scalar part $< ... >$ are of the form

$$\Theta^{\alpha\beta}\ <\ \{\mathbf{F}_{\alpha\mu},\ \mathbf{F}_{\beta\nu}\}\ \mathbf{F}_{\rho\sigma}\ \mathbf{\Phi}\ >\ \epsilon^{\mu\nu\rho\sigma} \tag{7.26}$$

$$\Theta^{\alpha\beta}\ <\ \mathbf{F}_{\mu\nu}\ \ \{\ \mathbf{F}_{\alpha\rho},\ \mathbf{F}_{\beta\sigma}\ \}\ \mathbf{\Phi}\ >\ \epsilon^{\mu\nu\rho\sigma} \tag{7.27}$$

$$\Theta^{\alpha\beta}\ <\ \mathbf{F}_{\mu\nu}\ \mathbf{F}_{\rho\sigma}\ \{\ \mathbf{F}_{\alpha\beta},\ \mathbf{\Phi}\ \}\ >\ \epsilon^{\mu\nu\rho\sigma} \tag{7.28}$$

$$\Theta^{\alpha\beta}\ <\ \mathbf{F}_{\mu\nu}\ \mathbf{F}_{\rho\sigma}\ \{\ \mathbf{A}_\alpha,\ (\partial_\beta + D_\beta)\ \mathbf{\Phi}\ \}\ \ >\ \epsilon^{\mu\nu\rho\sigma} \tag{7.29}$$

$$\Theta^{\alpha\beta}\ <\ \mathbf{F}_{\mu\nu}\ \{\mathbf{A}_\alpha,\ (\partial_\beta + D_\beta)\ \mathbf{F}_{\rho\sigma}\}\ \mathbf{\Phi}\ >\ \epsilon^{\mu\nu\rho\sigma}\ +$$
$$\Theta^{\alpha\beta}\ <\ \{\mathbf{A}_\alpha,\ (\partial_\beta + D_\beta)\ \mathbf{F}_{\mu\nu}\}\ \mathbf{F}_{\rho\sigma}\ \mathbf{\Phi}\ >\ \epsilon^{\mu\nu\rho\sigma} \tag{7.30}$$

$$\frac{i}{2}\ \Theta^{\alpha\beta}\ <\ (\partial_\alpha\mathbf{F}_{\mu\nu})\ (\partial_\beta\mathbf{F}_{\rho\sigma})\ \mathbf{\Phi}\ > \tag{7.31}$$

To simplify the calculations let us *truncate* all the components of the field $\Phi = \Phi^A\Gamma_A$ to zero *except* $\Phi^{mnpq} \neq 0$, and all the components of $A^A_\mu\Gamma_A$ to zero *except* $A^{ab}_\mu \neq 0$. In this case one will have in explicit components form for the term in eq-(7.28) the following

$$\Theta^{\alpha\beta}\ <\ F^{ab}_{\mu\nu}\,\gamma_{ab}\ F^{cd}_{\rho\sigma}\,\gamma_{cd}\ \{F^{rs}_{\alpha\beta}\,\gamma_{rs},\ \phi_{mnpq}\ \gamma^{mnpq}\}\ >\ \epsilon^{\mu\nu\rho\sigma} \tag{7.32}$$

Recurring to the expressions displayed in the Appendix allow us to extract the Clifford scalar part $< ... >$ of the geometric products of the Clifford $Cl(3,1)$ algebra generators in eq-(7.32). After some straightforward but lengthy algebra it yields (up to a numerical factor)

$$\Theta^{\alpha\beta} \ \eta_{ac} \ F^{ap}_{\mu\nu} \ F^{cq}_{\rho\sigma} \ F^{mn}_{\alpha\beta} \ \phi_{mnpq} \ \epsilon^{\mu\nu\rho\sigma} \ = \ 0 \tag{7.33}$$

The reason this last expression eq-(7.33) is *vanishing* is due to the contraction structure of the tangent space indices and the *antisymmetry* of *all* the terms of eq-(7.33) under the exchange of indices with the *exception* of the (flat) tangent space metric $\eta_{ac} = \eta_{ca}$.

Following the same procedure with eq-(7.27) and using the same symmetry (antisymmetry) argument in the contraction of indices gives for the Clifford scalar part

$$\Theta^{\alpha\beta} \ \eta_{ac} \ F^{ap}_{\alpha\rho} \ F^{cq}_{\beta\sigma} \ F^{mn}_{\mu\nu} \ \phi_{mnpq} \ \epsilon^{\mu\nu\rho\sigma} \ = \ 0 \tag{7.34}$$

identical vanishing results occur with eq-(7.29)

$$\Theta^{\alpha\beta} \ \eta_{ac} \ F^{ap}_{\mu\nu} \ F^{cq}_{\rho\sigma} \ A^{mn}_{\alpha} \ (\partial_\beta + D_\beta)\phi_{mnpq} \ \epsilon^{\mu\nu\rho\sigma} \ = \ 0 \tag{7.35}$$

and with eq-(7.26).

The explicitly gauge noncovariant eq-(7.30) yields

$$\Theta^{\alpha\beta} \ \eta_{ac} \ \phi_{mnpq} \ F^{am}_{\mu\nu} \ A^{cn}_{\alpha} \ (\partial_\beta + D_\beta) F^{pq}_{\rho\sigma} \ \epsilon^{\mu\nu\rho\sigma} \ -$$

$$\Theta^{\alpha\beta} \ \eta_{ca} \ \phi_{mnpq} \ F^{cq}_{\rho\sigma} \ A^{an}_{\alpha} \ (\partial_\beta + D_\beta) F^{mp}_{\mu\nu} \ \epsilon^{\mu\nu\rho\sigma} \ = \ 0 \tag{7.36}$$

A way to see why eq-(7.36) is zero can be obtained by relabeling the indices $\mu\nu \leftrightarrow \rho\sigma, q \leftrightarrow m, a \leftrightarrow c$ in the second line of eq-(7.36) so that it becomes identical to the first line and leading to an exact cancellation due to the key minus sign in eq-(7.36) and antisymmetry $F^{pq}_{\rho\sigma} = -F^{qp}_{\rho\sigma}$.

Finally we examine eq-(7.31) giving

$$\frac{i}{2} \ \Theta^{\alpha\beta} \ (\partial_\alpha F^{mn}_{\mu\nu}) \ (\partial_\beta F^{pq}_{\rho\sigma}) \ \phi_{mnpq} \ \epsilon^{\mu\nu\rho\sigma} = \ 0 \tag{7.37}$$

The reason eq-(7.37) is zero is due to an overall antisymmetry. Relabeling the indices in eq-(7.37) $\mu\nu \leftrightarrow \rho\sigma, \alpha \leftrightarrow \beta, mn \leftrightarrow pq$ and due to the antisymmetry of $\Theta^{\alpha\beta} = -\Theta^{\beta\alpha}$ it leads to

$$\frac{i}{2} \ \Theta^{\alpha\beta} \ (\partial_\alpha F^{mn}_{\mu\nu}) \ (\partial_\beta F^{pq}_{\rho\sigma}) \ \phi_{mnpq} \ \epsilon^{\mu\nu\rho\sigma} = \ - \ \frac{i}{2} \ \Theta^{\alpha\beta} \ (\partial_\beta F^{pq}_{\rho\sigma}) \ (\partial_\alpha F^{mn}_{\mu\nu}) \ \ \phi_{pqmn} \ \epsilon^{\rho\sigma\mu\nu} =$$

$$- \ \frac{i}{2} \ \Theta^{\alpha\beta} \ (\partial_\alpha F^{mn}_{\mu\nu}) \ (\partial_\beta F^{pq}_{\rho\sigma}) \ \phi_{mnpq} \ \epsilon^{\mu\nu\rho\sigma} \tag{7.38}$$

therefore, if $X = -X \Rightarrow X = 0$.

Therefore, the Clifford scalar part of the $first$ order contributions in the $\Theta^{\alpha\beta}$ terms of the Moyal-deformed action is *vanishing* when one *truncates* all the components of $\Phi = \Phi^A \Gamma_A$ to zero *except* $\Phi^{mnpq} \neq 0$, and all the components of $A^A_\mu \Gamma_A$ to zero *except*

$A^{ab}_\mu \neq 0$. If one does *not* impose such truncation, one will have to consider the Moyal deformations of all other expressions in eqs-(6.21-6.25). It is unlikely that there is a cancellation of the Θ-terms up to first order in this most general case.

For example, let us examine the first order contribution in $\Theta^{\alpha\beta}$ of

$$\int \ < \ \left(F_{\mu\nu} * F^{abcd}_{\rho\sigma} * \phi_{abcd} \right)^{(1)} \ > \ \epsilon^{\mu\nu\rho\sigma} \tag{7.39}$$

One of the terms is

$$\frac{i}{2} \ \Theta^{\alpha\beta} \ (\partial_\alpha F_{\mu\nu}) \ (\partial_\beta F^{abcd}_{\rho\sigma}) \ \phi_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ \neq 0 \tag{7.40}$$

which is clearly nonvanishing and furnishes an imaginary contribution to the Moyal deformed action. The other imaginary contribution can be dropped because it yields a total derivative term

$$\int \ \frac{i}{2} \ \Theta^{\alpha\beta} \ \partial_\alpha (F_{\mu\nu} \ F^{abcd}_{\rho\sigma}) \ \partial_\beta \phi_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ =$$

$$\int \ \frac{i}{2} \ \Theta^{\alpha\beta} \ \partial_\alpha \left(\ F_{\mu\nu} \ F^{abcd}_{\rho\sigma} \ \partial_\beta \phi_{abcd} \ \right) \ \epsilon^{\mu\nu\rho\sigma} \tag{7.41}$$

after an integration by parts.

One may cancel the contribution in eq-(7.40) by adding to eq-(7.39) the term

$$\int \ < \ \left(F^{abcd}_{\rho\sigma} * F_{\mu\nu} * \phi_{abcd} \right)^{(1)} \ > \ \epsilon^{\mu\nu\rho\sigma} \tag{7.42}$$

which amounts to a trivial *symmetrization* of the ordering in the products of the field strengths. Not surprisingly, due to this trivial *symmetrization*, there is cancellation due to the antisymmetry of $\Theta^{\alpha\beta}$.

Eq-(7.40) is gauge covariant because $\partial_\alpha F_{\mu\nu} = D_\alpha F_{\mu\nu}$ and $\partial_\beta F^{abcd}_{\rho\sigma} = D_\beta F^{abcd}_{\rho\sigma}$ after writing $F^{abcd}_{\rho\sigma} = \epsilon^{abcd} G_{\rho\sigma}$. Because there are a lot of gauge noncovariant terms in the expansion in powers of Θ, the authors [63] used the method of composite fields which enables to write the final results in a manifestly gauge covariant way. Therefore, the final results are manifestly gauge covariant as they should be.

There are many other terms in eq-(7.39) whose contribution is nonvanishing and real to first order in Θ, for example

$$\Theta^{\alpha\beta} \ F^{rs}_{\alpha\rho} \ F_{\beta\sigma rs} \ F^{abcd}_{\mu\nu} \ \phi_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ \neq \ 0 \tag{7.43}$$

$$\Theta^{\alpha\beta} \ F_{\mu\nu} \ F^{ab}_{\alpha\rho} \ F^{cd}_{\beta\sigma} \ \phi_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ \neq \ 0 \tag{7.44}$$

due to the fact that now $F_{\mu\nu}$ and $F^{abcd}_{\mu\nu}$ are no longer zero. In particular, the terms of eq-(7.44) clearly form part of the deformed action $S_{(5)*}$ in eq-(7.9) and encoding the Moyal deformations of the MMCW gravitational action with a cosmological constant given by eq-(6.13) to first order in $\Theta^{\mu\nu}$. By setting $\phi_{abcd} = \epsilon_{abcd}\varphi$ and recurring to the decomposition of $F^{ab}_{\alpha\rho}, F^{cd}_{\beta\sigma}$ provided in eqs-(6.11d, 6.13) one will have that eq-(3.44) yields the following Θ corrections to the vacuum energy density (in the modified action)

$$\frac{\varphi}{l^4}\ \Theta^{\alpha\beta}\ F_{\mu\nu}\ V^a_\alpha\ V^b_\rho\ V^c_\beta\ V^d_\sigma\ \epsilon_{abcd}\ \epsilon^{\mu\nu\rho\sigma} \tag{7.45}$$

where V^a_α is the vielbein field. If one identifies $\frac{\varphi}{l^2} \sim \frac{1}{G} = \frac{1}{L_P^2}$ and $\frac{\varphi}{l^4} = \rho_{vacuum}$ one can cancel the enormous ρ_{vacuum} energy density (when $\varphi = 1$) if the terms in eq-(3.45) are of the same order of magnitude, which implies that

$$\frac{\varphi}{l^4}\ \left(V^a_\mu\ V^b_\nu\ V^c_\rho\ V^d_\sigma\ +\ \Theta^{\alpha\beta}\ F_{\mu\nu}\ V^a_\alpha\ V^b_\rho\ V^c_\beta\ V^d_\sigma \right)\ \epsilon_{abcd}\ \epsilon^{\mu\nu\rho\sigma}\ =\ 0 \tag{7.46}$$

Setting the magnitude of the constant $\Theta^{\alpha\beta}$ parameters to be of the order of the Planck scale squared L_P^2 will fix the values of $F_{\mu\nu}$ in eq-(7-46) that furnish a cancellation of the huge vacuum energy density. Hence, the second terms in eq-(7.46) provide in general the x-dependent corrections to the vacuum energy density (cosmological constant). This result should be contrasted with those in [61].

One should notice that despite the generators of $U(2,2), SO(4,2), SO(2,3)$ can be expressed in terms of the Clifford algebra generators this does *not* imply that these algebras are *isomorphic* to the Clifford algebra. Hence one should *not* expect identical results as those obtained by other authors.

To sum up, when one does *not* impose constraints on the fields, there are first order contributions in the $\Theta^{\mu\nu}$ (constants) parameters in the Moyal deformations of a Clifford gauge theory formulation of gravity in variance with the previous results obtained by other authors and based on different gauge groups. This could provide a plausible cancellation mechanism of the huge vacuum energy density $1/L_P^4$. The $first$ order contributions in the $\Theta^{\alpha\beta}$ terms of the Moyal-deformed action is *vanishing* in the special case when one $truncates$ all the components of $\Phi = \Phi^A \Gamma_A$ to zero *except* $\Phi^{mnpq} \neq 0$, and all the components of $A^A_\mu \Gamma_A$ to zero *except* $A^{ab}_\mu \neq 0$.

Similarily, one obtains the Moyal deformations of the action $S[\mathbf{\Phi}]$ corresponding to the Clifford-valued scalar field $\mathbf{\Phi}$. Firstly, there is a modification of the gauge covariant derivative term (6.28a) due to the noncommutativity of the pointwise product of fields. Both commutators and anticommutators will appear in the Moyal deformations of eq-(6.28a) as they did in eq-(7.6). This will lead to corrections in powers of Θ of the gauge covariant derivative terms. Secondly, one performs the Moyal star products among all the terms present in the Clifford-valued scalar field action as it was done in eq-(7.9) after recurring to eq-(7.18).

8 N-ary Algebras and Clifford Spaces

In this section Polyvector-valued gauge field theories in noncommutative Clifford spaces are presented. They are based on noncommutative (but associative) star products that require the use of the Baker-Campbell-Hausdorff formula. Using these star products allows the construction of actions for noncommutative p-branes (branes moving in noncommutative spaces). Noncommutative Clifford-space gravity as a poly-vector-valued gauge theory of twisted diffeomorphisms in Clifford-spaces would require quantum Hopf algebraic deformations of Clifford algebras. We proceed with the study of n-ary algebras and find an important relationship among the **n**-ary commutators of noncommuting spacetime coordinates $[X^1, X^2, ..., X^n]$ with the poly-vector valued coordinates $X^{123...n}$ in noncommutative Clifford spaces given by $[X^1, X^2, ..., X^n] = n! \ X^{123...n}$. The large N limit of **n**-ary commutators of n hyper-matrices $\mathbf{X}_{i_1 i_2 .... i_n}$ leads to Eguchi-Schild p-brane actions for $p+1 = n$. Finally, a noncomutative n-ary $\bullet$ product of n functions is constructed which is a generalization of the binary star product $*$ of two functions, and is associated with the deformation quantization of n-ary structures and deformations of the Nambu-Poisson brackets.

The study of n-ary algebras, *ternary* algebras, in particular, have recently resurfaced with great intensity in the study of $M2$-brane duality where M theory on $AdS_4 \times S^7$ is dual to a superconformal field theory in three dimensions, with the supergroup $OSp(8|4)$, after Bagger-Lambert-Gustavsson (BLG) [79] constructed a Chern-Simons gauge theory in three dimensions with maximal supersymmety $\mathcal{N} = 8$. However, their construction only works for the $SO(4)$ gauge group and it does not provide the desired dual to M-theory on $AdS_4 \times S^7$ [80]. The authors [81] later have shown that the dual gauge theory is actually an $\mathcal{N} = 6$ superconformal Chern-Simons theory in three-dimensions and is associated to M-theory on $AdS_4 \times S^7/Z_k$, with N units of flux. The $M5$-brane duality is based on M theory on $AdS_7 \times S^4$ being dual to a six dimensional superconformal field theory whose super group is $OSp(6, 2|4)$. Recently it was shown by [82] how the $M5$ brane can be obtained from a mass deformed BLG theory which is realized by a Nambu bracket and such that a maximally supersymmetric Lagrangian for the fluctuation fields exists corresponding to a single $M5$ brane on $\mathbf{R^{1,2}} \times \mathbf{S^3}$.

N-ary algebras have been known for some time [75] since Nambu introduced his bracket (a Jacobian) in the study of branes and the generalizations of Hamiltonian mechanics based on Poisson brackets. In this section we shall show how poly-vector valued coordinates admit a very natural interpretation in terms of $n-$ary commutators.

The ternary commutator for noncommuting coordinates is defined as

$$[X^1, X^2, X^3] \ = \ X^1 \ [X^2, X^3] \ + \ X^2 \ [X^3, X^1] \ + \ X^3 \ [X^1, X^2] \ =$$

$$\frac{1}{2} \ \{ \ X^1, \ [X^2, X^3] \ \} \ + \ \frac{1}{2} \ [\ X^1, \ [X^2, X^3] \ ] \ + \ cyclic \ permutations \tag{8.1}$$

Due to the Jacobi identities, the terms

$$\frac{1}{2} \ [\ X^1, \ [X^2, X^3] \ ] \ + \ cyclic \ permutations \ = \ 0. \tag{8.2}$$

so that the ternary commutators become

$$[X^1, X^2, X^3] \ = \ \frac{1}{2} \ \{ \ X^1, \ [X^2, X^3] \ \} \ + \ cyclic \ permutations. \tag{8.3}$$

The second step is to write down the *noncommutative* algebra associated with the noncommuting poly-vector-valued coordinates in $D = 4$ and which can be obtained from the Clifford algebra displayed in Appendix **A** by performing the following replacements (and relabeling indices)

$$\gamma^{\mu} \leftrightarrow X^{\mu}, \quad \gamma^{\mu_1\mu_2} \leftrightarrow X^{\mu_1\mu_2}, \quad .... \gamma^{\mu_1\mu_2.....\mu_n} \leftrightarrow X^{\mu_1\mu_2....\mu_n}. \tag{8.4}$$

When the spacetime metric components $g_{\mu\nu}$ are *constant*, from the replacements (8.4) and the Clifford algebra (after one relabels indices), one can then construct the following *noncommutative* algebra among the poly-vector-valued coordinates in $D = 4$, and *obeying* the Jacobi identities, given by the relations

$$[\ X^{\mu_1},\ X^{\mu_2}\ ]_* \ = \ X^{\mu_1}\ *\ X^{\mu_2}\ -\ X^{\mu_2}\ *\ X^{\mu_1}\ =\ 2\ X^{\mu_1\mu_2}. \tag{8.5}$$

In most of the remaining commutators a suitable length scale parameter must be introduced in order to match units. We shall set this length scale (let us say the Planck scale) to *unity*. Also, by choosing the C-space coordinates to behave like anti-Hermitian operators we avoid the need to introduce i factors in the right hand side.

$$[\ X^{\mu_1\mu_2},\ X^{\nu}\ ]_* \ = \ 4\ (\ g^{\mu_2\nu}\ X^{\mu_1}\ -\ g^{\mu_1\nu}\ X^{\mu_2}\ ). \tag{8.6}$$

$$[\ X^{\mu_1\mu_2\mu_3},\ X^{\nu}\ ]_* \ = \ 2\ X^{\mu_1\mu_2\mu_3\nu}, \quad [\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu}\ ]_* \ = \ -\,8\ g^{\mu_1\nu}\ X^{\mu_2\mu_3\mu_4}\ \pm ...... \tag{8.7}$$

$$[\ X^{\mu_1\mu_2},\ X^{\nu_1\nu_2}\ ]_* \ = \ -\,8\ g^{\mu_1\nu_1}\ X^{\mu_2\nu_2}\ +\ 8\ g^{\mu_1\nu_2}\ X^{\mu_2\nu_1}\ + \\ 8\ g^{\mu_2\nu_1}\ X^{\mu_1\nu_2}\ -\ 8\ g^{\mu_2\nu_2}\ X^{\mu_1\nu_1}. \tag{8.8}$$

$$[\ X^{\mu_1\mu_2\mu_3},\ X^{\nu_1\nu_2}\ ]_* \ = \ 12\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\nu_2}\ \pm\ .... \tag{8.9}$$

$$[\ X^{\mu_1\mu_2\mu_3},\ X^{\nu_1\nu_2\nu_3}\ ]_* \ = \ -\,36\ G^{\mu_1\mu_2\ \nu_1\nu_2}\ X^{\mu_3\nu_3}\ \pm\ ...... \tag{8.10}$$

$$[\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2}\ ]_* \ = \ -\,16\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\mu_4\nu_2}\ \pm ...... \tag{8.11}$$

$$[\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2}\ ]_* \ = \ -\,16\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\mu_4\nu_2}\ +\ 16\ g^{\mu_1\nu_2}\ X^{\mu_2\mu_3\mu_4\nu_1}\ -\ .... \tag{8.12}$$

$$[\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2\nu_3}\ ]_* \ = \ 48\ G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4}\ -\ 48\ G^{\mu_1\mu_2\mu_4\ \nu_1\nu_2\nu_3}\ X^{\mu_3}\ +\ ..... \tag{8.13}$$

$$[\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2\nu_3\nu_4}\ ]_* \ = \ 192\ G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4\nu_4}\ -\ .... \tag{8.14}$$

etc...... where

$$G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n} \ = \ g^{\mu_1\nu_1}\ g^{\mu_2\nu_2}\ ....\ g^{\mu_n\nu_n}\ +\ \textit{signed permutations} \tag{8.15}$$

The metric components $G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n}$ in C-space can also be written as a determinant of the $n \times n$ matrix $\mathbf{G}$ whose entries are $g^{\mu_I \nu_J}$

$$det\ \mathbf{G}_{n\times n} \ = \ \frac{1}{n!}\ \epsilon_{i_1 i_2 .....i_n}\ \epsilon_{j_1 j_2 ....j_n}\ g^{\mu_{i_1}\nu_{j_1}}\ g^{\mu_{i_2}\nu_{j_2}}\ ....\ g^{\mu_{i_n}\nu_{j_n}}. \qquad (8.16)$$

$i_1, i_2, ..., i_n \subset I = 1, 2, ..., D$ and $j_1, j_2, ..., j_n \subset J = 1, 2, ..., D$. One must also include in the C-space metric G^{MN} the (Clifford) scalar-scalar component G^{00} (that could be related to the dilaton field) and the pseudo-scalar/pseudo-scalar component $G^{\mu_1\mu_2.....\mu_D\ \nu_1\nu_2......\nu_D}$ (that could be related to the axion field).

One must emphasize that when the spacetime metric components $g_{\mu\nu}$ are *no* longer *constant*, the noncommutative algebra among the poly-vector-valued coordinates in $D = 4$, does *not* longer *obey* the Jacobi identities. For this reason we restrict our construction to a flat spacetime background $g_{\mu\nu} = \eta_{\mu\nu}$.

The noncommutative conditions on the polyvector coordinates in condensed notation can be written as

$$[\ X^M,\ X^N\ ]_* \ = \ X^M * X^N \ - \ X^N * X^M \ = \ \Omega^{MN}(X) \ = \ f^{MN}{}_L\ X^L \ = \ f^{MNL}\ X_L \quad (8.17)$$

the structure constants f^{MNL} are antisymmetric under the exchange of polyvector valued indices. An immediate consequence of the noncommutativity of coordinates is

$$[\ \hat{X}^{\mu_1},\ \hat{X}^{\mu_2}\ ] \ = \ 2\ \hat{X}^{\mu_1\mu_2} \ \Rightarrow\ \Delta X^\mu\ \Delta X^\nu \ \geq \ \frac{1}{2}\ |\ <\ \hat{X}^{\mu\nu}\ >\ | = \ X^{\mu\nu} \qquad (8.18)$$

Hence, the bivector area coordinates $X^{\mu\nu}$ in C-space can be seen as a measure of the noncommutative nature of the "quantized" spacetime coordinates $\hat{X}^\mu$.

After using the relations, from eqs-(8.5-8.15),

$$[X^2, X^3] \ = \ 2\ X^{23}, \quad \{\ X^1,\ X^{23}\ \} \ = \ 2\ X^{123}. \qquad (8.19)$$

one gets finally

$$[X^1, X^2, X^3] \ = \ 2\ X^{123} \ + \ cyclic\ permutations \ = \ 6\ X^{123}. \qquad (8.20)$$

since $X^{123} = X^{231} = X^{312} = -X^{132} = ...$

The 4-ary commutator is defined as

$$[X^1, X^2, X^3, X^4] \ = \ X^1\ [X^2, X^3, X^4] \ - \ X^2\ [X^3, X^4, X^1] \ +$$
$$X^3\ [X^4, X^1, X^2] \ - \ X^4\ [X^1, X^2, X^3] \ =$$
$$\frac{1}{2}\ \{\ X^1,\ [X^2, X^3, X^4]\ \} \ + \ \frac{1}{2}\ [\ X^1,\ [X^2, X^3, X^4]\ ] \ - \ .... \ =$$
$$3\ \{\ X^1,\ X^{234}\ \} \ + \ 3\ [\ X^1,\ X^{234}\ ] \ - \ .... \ =$$
$$6\ X^{1234} \ + \ 18\ (\ g^{12}\ X^{34} \ + \ g^{13}\ X^{42} \ + \ g^{14}\ X^{23}\ ) \ - \ .... \ = \ 24\ X^{1234} \qquad (8.21)$$

due to the cancellations

$$(g^{12} X^{34} + g^{13} X^{42} + g^{14} X^{23}) - (g^{23} X^{41} + g^{24} X^{13} + g^{21} X^{34}) +$$

$$(g^{34} X^{12} + g^{31} X^{24} + g^{32} X^{41}) - (g^{41} X^{23} + g^{42} X^{31} + g^{43} X^{12}) = 0. \quad (8.22)$$

resulting from the conditions $X^{\mu\nu} = -X^{\nu\mu}$, $g^{\mu\nu} = g^{\nu\mu}$ after recurring to the (anti) commutators

$$[X^1, X^{234}] = 2 X^{1234}, \quad \{X^1, X^{234}\} = 6 (g^{12} X^{34} + g^{13} X^{42} + g^{14} X^{23}). \quad (8.23)$$

and the conditions $X^{1234} = -X^{2341} = X^{3412} = -X^{4123}$. For example, given a Noncommutative Clifford space in $D = 4$, one arrives at

$$[X^1, X^2] = 2 X^{12}, [X^1, X^2, X^3] = 6 X^{123}, [X^1, X^2, X^3, X^4] = 24 X^{1234}. \quad (8.24)$$

where X^1, X^2, X^3, X^4 is a shorthand notation for $X^{\mu_1}, X^{\mu_2}, X^{\mu_3}, X^{\mu_4}$. Therefore, one finds that the poly-vector coordinates $X^{\mu_1\mu_2}, X^{\mu_1\mu_2\mu_3}, X^{\mu_1\mu_2\mu_3\mu_4}$ can be seen, respectively, as the binary, ternary and 4-ary commutators of the non-commuting vector coordinates X^μ. In the general case, using the noncommutative algebra in Clifford spaces one arrives by recursion at

$$[X^1, X^2, ..., X^n] = n! X^{123.....n}. \quad (8.25)$$

This n-ary commutator interpretation of the poly-vector valued coordinates of a noncommutative Clifford space warrants further investigation.

At this stage it is important to emphasize that the Noncommutative Clifford-valued poly-vector coordinates algebra does *not* satisfy the Nambu-Filipov conditions which can be written as

$$\mathcal{D}_{[X^1,X^2]} [Y^1, Y^2, Y^3] = [X^1, X^2, [Y^1, Y^2, Y^3]] =$$

$$[[X^1, X^2, Y^1], Y^2, Y^3] + [Y^1, [X^1, X^2, Y^2], Y^3] + [Y^1, Y^2, [X^1, X^2, Y^3]]. \quad (8.25a)$$

$$[X^1, X^2, ..., X^{n-1}, [Y^1, Y^2, ..., Y^n]] =$$

$$[[X^1, X^2, ..., X^{n-1}, Y^1], Y^2, ..., Y^n] +$$

$$[Y^1, [X^1, X^2, ..., X^{n-1}, Y^2], Y^3, ..., Y^n] + ... +$$

$$[Y^1, Y^2, ..., Y^{n-1}, [X^1, X^2, ..., X^{n-1}, Y^n]]. \quad (8.25b)$$

For n-ary brackets, Nambu showed that the Jacobian, the classical Nambu-Poison bracket (NPB)

$$\{X^1, X^2, ..., X^n\}_{NPB} = \epsilon^{i_1 i_2 .....i_n} \partial_{i_1} X^1 \partial_{i_2} X^2 .... \partial_{i_n} X^n. \quad (8.26)$$

satisfies the Nambu-Filippov special conditions, [73], [75]. The NPB is antisymmetric under the exchange of any pair of entries and satisfies the analog of the Liebnitz rule. It is not difficult to see that

$$[\ X^1,\ X^2,\ [X^3,\ X^4,\ X^5]\ ]\ \neq$$
$$[\ [X^1, X^2, X^3],\ X^4,\ X^5\ ] + [\ X^3,\ [X^1, X^2, X^4],\ X^5\ ] + [\ X^3,\ X^4,\ [X^1, X^2, X^5]\ ]. \quad (8.27)$$

The main reason being that the ternary commutator

$$[X^1, X^2, X^3]\ =\ 6\ X^{123}\ \neq\ \sum_i\ f^{123}{}_i\ X^i. \quad (8.28)$$

Naturally, the Jacobi identity is satisfied

$$[\ X^1,\ [X^2,\ X^3]\ ]\ =\ [\ [X^1,\ X^2],\ X^3\ ]\ +\ [\ X^2,\ [X^1,\ X^3]\ ]. \quad (8.29)$$

n-ary algebras are relevant to the large N limit of covariant Matrix Models based on generalized n-th power matrices (hyper-matrices) [78] $\mathbf{X}_{i_1 i_2 ......i_n}$, that are extensions of square, cubic, quartic, ... matrices (hyper-matrices). These Matrix models bear a relationship to Eguchi-Schild p-brane actions for $p+1 = n$. The range of indices is $i_1, i_2, ..., i_n \subset I = 1, 2, ...N$. The n-ary commutator of n generalized n-th power matrices (hyper matrices) in the large $N \to \infty$ has a correspondence with the Nambu-brackets (NB) as follows

$$[\ \mathbf{X}^1,\ \mathbf{X}^2,\ ....,\ \mathbf{X}^n\ ]_{i_1 i_2 ......i_n}\ \to\ \{\ X^1,\ X^2,\ ....,\ X^n\ \}_{NB}. \quad (8.30)$$

by replacing the hyper matrix $\mathbf{X}_{i_1 i_2 ......i_n}$ in the large $N \to \infty$ limit for the c-function of n-variables $X(\sigma^1, \sigma^2, ..., \sigma^n)$. The trace operation in the large N limit has a correspondence with the integral $\int d^n\sigma$ so that

$$Trace\ \Big(\ [\ \mathbf{X}^1,\ \mathbf{X}^2,\ ....,\ \mathbf{X}^n\ ]^2\ \Big)\ \to\ \int\ d^n\sigma\ \{\ X^1,\ X^2,\ ....,\ X^n\ \}^2_{NB}. \quad (8.31)$$

recovering in this fashion the Eguchi-Schild p-brane actions for $p+1 = n$. The fermionic version of (8.31) is

$$\int\ d^n\sigma\ \bar\Psi\ \Gamma_{12....n-1}\ \{\ X^1,\ X^2,\ ....,\ X^{n-1}\ , \Psi\}. \quad (8.32)$$

Covariant (super) brane actions based on n-ary structures and generalized matrix models have been recently constructed by [72]. The authors [70] have shown that the light-cone gauge-fixed action of a super p-brane belongs to a *new* kind of supersymmetric gauge theory of p-volume preserving diffeomorphisms (diffs) associated with the p-spatial dimensions of the extended object. These authors conjectured that this new kind of supersymmetric gauge theory must be related to an infinite-dim *nonabelian* antisymmetric gauge theory. It was recently shown in [71] how this new theory should be part of an underlying antisymmetric nonabelian tensorial gauge field theory of $p+1$-dimensional diffs (upon supersymmetrization) associated with the world volume evolution of the p-brane.

Ternary algebraic structures appearing in various domains of theoretical and mathematical physics were reviewed by [69], like the notion of quark algebraic confinement based on a Z_3 -graded matrix algebra over the complex field $\mathbf{C}$. A generalization of noncommutative geometry and gauge theories based on *ternary* Z_3-graded structures was constructed by [69]. The usual Z_2-graded structures such as Grassmann, Lie and Clifford algebras are generalized to the Z_3-graded case leading to *hypersymmetry* which is a Z_3 graded generalization of supersymmetry. The de Rham complex with the differential operator d satisfies the condition $d^3 = 0$ instead of $d^2 = 0$. Ternary generalizations of Clifford algebras were defined by the relations [69]

$$Q^a \; Q^b \; Q^c \;=\; \omega \; Q^b \; Q^c \; Q^a \;+\; \omega^2 \; Q^c \; Q^a \; Q^b \;+\; 3 \; \rho^{abc} \; \mathbf{1} \tag{8.33}$$

where ω is the cubic root of unity $e^{i2\pi/3}$ and ρ^{abc} is the analog of a cubic metric (a cubic matrix) obeying the conditions

$$\rho^{abc} \;+\; \omega \; \rho^{bca} \;+\; \omega^2 \; \rho^{cab} \;=\; 0. \tag{8.34}$$

Our whole construction of C-spaces [1] based on ordinary Clifford algebras can be extended to ternary Clifford algebras. By replacing the cubic roots of unity for the N-th roots of unity and the cubic metric for $\rho^{a_1 a_2 \ldots\ldots a_n}$ one can define the N-ary generalizations of Clifford algebras. In [74] and references therein one can find a generalization of n-ary Nambu algebras and beyond.

The canonical Moyal noncommutative (but associative) star product is defined as

$$(f \;*\; g)(x,p) \;=\; \left(e^{\frac{i\hbar\omega_{ij}}{2!} \; \partial_{Z'_i} \;\wedge\; \partial_{Z''_j}} \right) \; f(Z') \; g(Z'')|_{Z=Z'=Z''}. \tag{8.35}$$

where the derivatives are evaluated at $Z = Z' = Z''$ and the phase coordinates are defined by $Z = (x, p)$; $Z' = (x', p'')$; $Z'' = (x'', p'')$. By analogy one can define the ternary $\bullet$ product of three functions of x, y, z in terms of a deformation parameter κ as

$$(f \;\bullet\; g \;\bullet\; h)(x,y,z) \;=\; \left(e^{\frac{i\kappa\epsilon_{ijk}}{3!} \; \partial_{X'_i} \;\wedge\; \partial_{X''_j} \;\wedge\; \partial_{X'''_k}} \right) \; f(X') \; g(X'') \; h(X'''). \tag{8.36}$$

where the derivatives are evaluated at $X_i = X'_i = X''_i = X'''_i$; the range of indices is $i = 1, 2, 3$. The coordinates are defined by

$$X_i = x, y, z; \; X'_i = x', y', z'; \; X''_i = x'', y'', z''; \; X'''_i = x''', y''', z'''. \tag{8.37}$$

The author [83] has also proposed such ternary product. The n-ary extension of (8.36) is straightforward. It remains to be seen whether or not the ternary $\bullet$ product obeys the ternary associativity condition

$$A \bullet B \bullet (C \bullet D \bullet E) \;=\; A \bullet (B \bullet C \bullet D) \bullet E \;=\; (A \bullet B \bullet C) \bullet D \bullet E. \tag{8.38}$$

The Moyal canonical star product can also be recast in integral form as [75]

$$ (f \ * \ g)(x,p) \ = \ (\frac{1}{\pi\hbar})^2 \int \ du_1 \ du_2 \ dv_1 \ dv_2 \ e^{\frac{2i}{\hbar} \ \Delta(u_i,v_i)} \ \times $$

$$ f(x+u_1, p+v_1) \ g(x+u_2, p+v_2). \tag{8.39} $$

where the integral limits are $-\infty, +\infty$ and the kernel of the exponential is given by the determinant

$$ \Delta(u_i, v_i) = det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}. \tag{8.40} $$

The analog of the integral (8.39) for the ternary case is

$$ (f \ \bullet \ g \ \bullet \ h)(x,y,z) \ = \ (\frac{1}{\kappa})^3 \int \ du_1 du_2 du_3 dv_1 dv_2 dv_3 dw_1 dw_2 dw_3 \ e^{\frac{2\pi i}{\kappa} \ \Delta(u_i,v_i,w_i)} \ \times $$

$$ f(x+u_1, y+v_1, z+w_1) \ g(x+u_2, y+v_2, z+w_2) \ h(x+u_3, y+v_3, z+w_3). \tag{8.41} $$

where the kernel of the exponential is given by the determinant

$$ \Delta(u_i, v_i, w_i) = det \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}. \tag{8.42} $$

However, the latter integral expression (8.41) for the putative ternary $\bullet$ product does not appear to yield the same expression as the ternary $\bullet$ product provided by eq-(8.36). In the Weyl-Wigner-Groenewold-Moyal (WWGM) deformation quantization procedure, the operator/function in classical phase space correspondence $\hat A(\hat q, \hat p) \leftrightarrow A(q,p;\hbar)$ is given by [75]

$$ A(q,p) \ = \ \mathbf{W}[\ \hat A(\hat q, \hat p) \ ] \ = \ \int \ dy \ e^{\frac{-2i\pi py}{\hbar}} \ < q+y| \ \hat A(\hat q, \hat p) \ |q-y> . \tag{8.43} $$

such that the WWGM map of the product of two Weyl-ordered operators $\hat A(\hat q, \hat p) \ \hat B(\hat q, \hat p)$ into the star product of their *symbols* $A(q,p;\hbar) \ * B(q,p;\hbar)$ obeys the relations

$$ \mathbf{W}(\ \hat A(\hat q, \hat p) \ \hat B(\hat q, \hat p) \ ) \ = \ A(q,p,\hbar) \ * \ B(q,p,\hbar) \ \Rightarrow $$

$$ \mathbf{W}(\ [\ \hat A(\hat q, \hat p), \ \hat B(\hat q, \hat p \ ] \ ) \ = \ \{ \ A(q,p,\hbar), \ B(q,p,\hbar) \ \}_* \ = \ A \ * \ B \ - \ B \ * \ A. \tag{8.44} $$

Given the noncommutative ternary $\bullet$ product of three functions of x, y, z as shown in eq-(8.36), and which is associated with the deformation quantization of ternary structures [83], the immediate question is how to generalize the WWGM map (8.44) in the binary star product case to the ternary $\bullet$ product case. In particular, how to map the Nambu-Heisenberg n-ary commutation relations of linear operators into the $deformed$ Nambu-Poisson brackets of their corresponding symbols. For instance, to find the correspondence

$$ \{ \ A, \ B, \ C \ \}_\bullet \ = \ A \bullet B \bullet C \ \pm \ permutations \ \leftrightarrow \ [\ \hat A, \ \hat B, \ \hat C \ ] \tag{8.45} $$

such that the Nambu-Weyl-Heisenberg *ternary* commutation relations among a *triad* of canonical "conjugate" operators has a one-to-one correspondence to the *deformed* Nambu-Poisson brackets of their symbols as follows

$$[\ \hat{A},\ \hat{B},\ \hat{C}\ ]\ =\ i\ \kappa\ \mathbf{I}\ \leftrightarrow\ \{\ A,\ B,\ C\ \}_{\bullet}\ =\ i\ \kappa. \tag{8.46}$$

The deformation parameter κ appearing in (8.36) plays now the role of Planck's constant h in (8.46). To find the linear operator $\hat{A}(\hat{x},\hat{y},\hat{z}) \leftrightarrow A(x,y,z)$ correspondence such that the relations (8.45,8.46) are obeyed in conjunction with the Nambu-Filippov fundamental identity [73], etc .... is a very *challenging* problem; i.e. to construct a Hypermatrix formulation of QM based on a deformation quantization of Nambu-Poisson classical mechanics. For example, the ternary product of three Hypermatrices which preserves the rank is

$$\delta_{i_3 j_1}\ \delta_{j_3 k_1}\ \delta_{k_3 i_1}\ A_{i_1 i_2 i_3}\ B_{j_1 j_2 j_3}\ C_{k_1 k_2 k_3}\ =\ (ABC)_{i_2 j_2 k_2}. \tag{8.47}$$

Following Heisenberg's formulation of ordinary QM, the large $N = \infty$ limit of a Hypermatrix should correspond to an operator in a Hilbert space. It is warranted to pursue these ideas further to see whether or not one can construct a Hypermatrix formulation/extension of QM.

To conclude this section we must emphasize that the quantization of Nambu mechanics is notoriously difficult. The geometric interpretation of quantized Nambu-Poisson structures in terms of noncommutative geometries has been recently studied by [77] where an extension of the usual axioms of quantization, in which classical Nambu-Poisson structures are translated to n-Lie algebras at the quantum level, were described. It was demonstrated that this generalized procedure matches an extension of the Berezin-Toeplitz quantization that is a mixture of geometric quantization and deformation quantization. It was not the aim of [77] to solve the problem of quantizing Nambu mechanics but merely to find geometric interpretations of operator algebras in terms of quantized algebras of functions which are endowed with an n-Lie bracket. That is, the authors [77] solved the *kinematical* problem of quantizing Nambu mechanics, which consists of providing a quantization prescription mapping classical observables to quantum operators, but the dynamical problem of deriving quantum dynamics from the classical Nambu mechanics was *not* solved. Other approaches to the quantization of Nambu mechanics is the Zariski quantization [76]

9 Concluding Remarks, Beyond Clifford Algebras, Generalized Geometries

This tour through the developments of the Extended Relativity in Clifford spaces was based entirely on orthogonal Clifford algebras. Symplectic Clifford algebras involving commutators instead of anti-commutators are as important [85]. An extended orthogonal-symplectic Clifford Algebraic formalism was developed in [84] which allowed the novel construction of a graded Clifford gauge field theory of gravity. It has a direct relationship to higher spin gauge fields, bimetric gravity, antisymmetric metrics and biconnections. In one particular case it allows a plausible mechanism to cancel the cosmological constant contribution to the action.

The possibility of embedding these orthogonal-symplectic Clifford algebras into an $infinite$ dimensional algebra, coined the super-Clifford algebra was also described in [84]. Some physical applications of the geometry of super-Clifford spaces to generalized super-geometries, double field theories, U-duality, $11D$ supergravity, M-theory, and E_7, E_8, E_{11} algebras were briefly outlined.

A concise overview of the physical and mathematical structures underpinning the appearence of non-associative deformations of geometry (gravity) in non-geometric string theory can be found in [86]. In particular the role played by L_∞ algebras in these developments. Extended geometry (generalized diffeomorphisms) is the framework unifying double geometry (double field theory), exceptional geometry (exceptional field theory), non-geometric string theory, $\cdots$ [87]. The L_∞ algebras for the extended geometry has been recently examined by [88] in terms of Borcherds superalgebras. For this reason it is imperative to search for non-associative generalizations of orthogonal and symplectic Clifford algebras.

Another very important topic that we did not explore is the quantum group deformations of Clifford algebras which are relevant to the symmetries of noncommutative spacetime [92], [93]. κ-deformations of the Poincare algebra were introduced by [89] . The Clifford-Hopf κ-deformed quantum Poincare algebra was constructed in [91]. An impending project is to furnish quantum group deformations of orthogonal-symplectic and super-Clifford algebras. A lot of work remains ahead, mainly in incorporating the extended relativity theory in Clifford spaces within the framework of generalized geometries and tensor hierarchy algebras.

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APPENDIX A

In this Appendix we shall write the (anti) commutator relations [30] for the Clifford algebra generators.

$$\frac{1}{2}\,\{\,\gamma_a,\ \gamma_b\,\}\ =\ g_{ab}\,\mathbf{1};\ \ \frac{1}{2}\,[\,\gamma_a,\ \gamma_b\,]\ =\ \gamma_{ab}\ =\ -\ \gamma_{ba},\quad a,b=1,2,3,\cdots,m \qquad (A.1)$$

$$[\,\gamma_a,\ \gamma_{bc}\,]\ =\ 2\ g_{ab}\,\gamma_c\ -2\ g_{ac}\,\gamma_b,\ \ \{\,\gamma_a,\ \gamma_{bc}\,\}\ =\ 2\ \gamma_{abc} \qquad (A.2)$$

$$[\,\gamma_{ab},\ \gamma_{cd}\,]\ =\ -2\ g_{ac}\,\gamma_{bd}\ +2\ g_{ad}\,\gamma_{bc}\ -2\ g_{bd}\,\gamma_{ac}\ +2\ g_{bc}\,\gamma_{ad} \qquad (A.3)$$

In general one has [30]

$$pq=\mathbf{odd},\ \ [\gamma_{m_1m_2....m_p},\ \gamma^{n_1n_2....n_q}]\ =\ 2\,\gamma^{n_1n_2....n_q}_{m_1m_2....m_p}-\frac{2p!q!}{2!(p-2)!(q-2)!}\,\delta^{[n_1n_2}_{[m_1m_2}\,\gamma^{n_3....n_q]}_{m_3......m_p]}+$$

$$\frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[n_1....n_4}_{[m_1....m_4}\ \gamma^{n_5....n_q]}_{m_5......m_p]}\ -\ .... \qquad (A.4)$$

$$pq = \mathbf{even},\ \{\ \gamma_{m_1m_2....m_p},\ \gamma^{n_1n_2....n_q}\ \}\ =\ 2\ \gamma^{n_1n_2....n_q}_{m_1m_2....m_p} - \frac{2p!q!}{2!(p-2)!(q-2)!}\ \delta^{[n_1n_2}_{[m_1m_2}\ \gamma^{n_3....n_q]}_{m_3......m_p]} +$$

$$\frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[n_1....n_4}_{[m_1....m_4}\ \gamma^{n_5....n_q]}_{m_5......m_p]}\ -\ .... \qquad (A.5)$$

$$pq = \mathbf{even},\ \ [\gamma_{m_1m_2....m_p},\ \gamma^{n_1n_2....n_q}]\ =\ \frac{(-1)^{p-1}2p!q!}{1!(p-1)!(q-1)!}\delta^{[n_1}_{[m_1}\ \gamma^{n_2....n_q]}_{m_2....m_p]}\ -$$

$$\frac{(-1)^{p-1}2p!q!}{3!(p-3)!(q-3)!}\ \delta^{[n_1n_2n_3}_{[m_1m_2m_3}\ \gamma^{n_4....n_q]}_{m_4......m_p]}\ +\ .... \qquad (A.6)$$

$$pq = \mathbf{odd},\ \ \{\ \gamma_{m_1m_2....m_p},\ \gamma^{n_1n_2....n_q}\ \}\ =\ \frac{(-1)^{p-1}2p!q!}{1!(p-1)!(q-1)!}\delta^{[n_1}_{[m_1}\ \gamma^{n_2....n_q]}_{m_2....m_p]}\ -$$

$$\frac{(-1)^{p-1}2p!q!}{3!(p-3)!(q-3)!}\ \delta^{[n_1n_2n_3}_{[m_1m_2m_3}\ \gamma^{n_4....n_q]}_{m_4......m_p]}\ +\ .... \qquad (A.7)$$

The generalized Kronecker delta is defined as the determinant

$$\delta^{a_1a_2.....a_k}_{b_1b_2.....b_k}\ \equiv\ det\begin{pmatrix} \delta^{a_1}_{b_1} & \ldots & \ldots & \delta^{a_1}_{b_k} \\ \delta^{a_2}_{b_1} & \ldots & \ldots & \delta^{a_2}_{b_k} \\ ------------ & & -------------- & \\ \delta^{a_k}_{b_1} & \ldots & \ldots & \delta^{a_k}_{b_k} \end{pmatrix} \qquad (A.8)$$

APPENDIX B

In this appendix we shall derive the expression for the analog of the torsionless Levi-Civita connection in C-space. Given a symmetric metric $g_{MN} = g_{NM}$ and setting the nonmetricity Q_{KMN} to zero gives

$$\nabla_K g_{MN}\ =\ \partial_K g_{MN}\ -\ \Gamma^L_{KM}\ g_{LN}\ -\ \Gamma^L_{KN}\ g_{ML}\ =\ 0 \qquad (B.1)$$

Performing a cyclic index permutation yields

$$\nabla_M g_{NK}\ =\ \partial_M g_{NK}\ -\ \Gamma^L_{MN}\ g_{LK}\ -\ \Gamma^L_{MK}\ g_{NL}\ =\ 0 \qquad (B.2)$$

$$\nabla_N g_{KM}\ =\ \partial_N g_{KM}\ -\ \Gamma^L_{NK}\ g_{LM}\ -\ \Gamma^L_{NM}\ g_{KL}\ =\ 0 \qquad (B.3)$$

adding eqs-(B.2, B.3) and subtracting eq-(B1) leads to

$$\partial_M g_{NK} + \partial_N g_{KM} - \partial_K g_{MN} = 2\, \Gamma^L_{(MN)}\, g_{LK} + 2\, \Gamma^L_{[MK]}\, g_{LK} + 2\, \Gamma^L_{[NK]}\, g_{LK} \quad (B.4)$$

where

$$\Gamma^L_{(MN)} \equiv \frac{1}{2} (\Gamma^L_{MN} + \Gamma^L_{NM}) \quad (B5)$$

$$\Gamma^L_{[MK]} \equiv \frac{1}{2} (\Gamma^L_{MK} - \Gamma^L_{KM}), \quad \Gamma^L_{[NK]} \equiv \frac{1}{2} (\Gamma^L_{NK} - \Gamma^L_{KN}) \quad (B.6)$$

when the Torsion is zero one has

$$T^L_{MK} = \Gamma^L_{MK} - \Gamma^L_{KM} - f^L_{MK} = 0 \Rightarrow 2\, \Gamma^L_{[MK]} = f^L_{MK} \quad (B.7)$$

such that eq-(B.4) becomes

$$\partial_M g_{NK} + \partial_N g_{KM} - \partial_K g_{MN} = 2\, \Gamma^L_{(MN)}\, g_{LK} + f^L_{MK}\, g_{LN} + f^L_{NK}\, g_{LM} \quad (B.8)$$

and from eq-(B.8) one can then deduce that the *symmetric* part of the connection is given by

$$\Gamma^L_{(MN)} = \frac{1}{2} g^{LK} [(\partial_M g_{NK} + \partial_N g_{MK} - \partial_K g_{MN}) + (f_{MKN} + f_{NKM})] \quad (B.9)$$

therefore, by adding the antisymmetric part of the connection $\Gamma^L_{[MN]} = \frac{1}{2} f^L_{MN}$ to the symmetric part $\Gamma^L_{(MN)}$ one obtains finally the full expression for the analog of the torsionless Levi-Civita connection in C-space

$$^{(lc)}\Gamma^L_{MN} = \Gamma^L_{(MN)} + \Gamma^L_{[MN]} = \{ {}^{L}_{MN} \} + \frac{1}{2} g^{LK} (f_{MKN} + f_{NKM} + f_{MNK}) \quad (B.10)$$

where

$$\{ {}^{L}_{MN} \} \equiv \frac{1}{2} g^{LK} (\partial_M g_{NK} + \partial_N g_{MK} - \partial_K g_{MN}) \quad (B.11)$$

APPENDIX C

In this Appendix we will perform the variation of $\mathbf{R}_{MJ}$. The Ricci tensor is given by

$$\mathbf{R}_{MJ} = \partial_M\, \Gamma_{NJ}{}^N - \partial_N\, \Gamma_{MJ}{}^N - \Gamma_{MJ}{}^L\, \Gamma_{NL}{}^N + \Gamma_{NJ}{}^L\, \Gamma_{ML}{}^N - f_{MN}{}^L\, \Gamma_{LJ}{}^N \quad (C.1)$$

the variation yields

$$\delta\mathbf{R}_{MJ} = \partial_M\, \delta\Gamma_{NJ}{}^N - \partial_N\, \delta\Gamma_{MJ}{}^N - \delta(\Gamma_{MJ}{}^L\, \Gamma_{NL}{}^N) + \delta(\Gamma_{NJ}{}^L\, \Gamma_{ML}{}^N) - \delta(f_{MN}{}^L\, \Gamma_{LJ}{}^N) \quad (C.2)$$

Given

$$\nabla_M(\delta\Gamma^N_{NJ}) \ = \ \partial_M(\delta\Gamma^N_{NJ}) \ - \ \Gamma^L_{MJ}\delta\Gamma^N_{NL} \qquad (C.3)$$

$$\nabla_N(\delta\Gamma^N_{MJ}) \ = \ \partial_N(\delta\Gamma^N_{MJ}) \ - \ \Gamma^L_{NM}\delta\Gamma^N_{LJ} \ - \ \Gamma^L_{NJ}\delta\Gamma^N_{ML} \ + \ \Gamma^N_{NL}\delta\Gamma^L_{MJ} \qquad (C.4)$$

it allows to express the partial derivatives in terms of the covariant derivatives plus connection terms such that

$$\partial_M \ \delta{\Gamma_{NJ}}^N \ - \ \partial_N \ \delta{\Gamma_{MJ}}^N \ = \ \nabla_M(\delta\Gamma^N_{NJ}) \ - \ \nabla_N(\delta\Gamma^N_{MJ}) \ + \\ \Gamma^L_{MJ}\delta\Gamma^N_{NL} \ - \ \Gamma^L_{NM}\delta\Gamma^N_{LJ} \ - \ \Gamma^L_{NJ}\delta\Gamma^N_{ML} \ + \ \Gamma^N_{NL}\delta\Gamma^L_{MJ} \qquad (C.5)$$

Inserting (C.5) into (C.2) and after collecting terms one arrives at

$$\delta\mathbf{R}_{MJ} \ = \ \nabla_M(\delta\Gamma^N_{NJ}) \ - \ \nabla_N(\delta\Gamma^N_{MJ}) \ + \ \Gamma^N_{ML} \ \delta(\ \Gamma^L_{NJ} \ ) \ - \ \Gamma^L_{NM} \ \delta(\Gamma^N_{LJ}) \ - \ \delta(\ {f_{ML}}^N \ \Gamma^L_{NJ} \ ) \qquad (C.6)$$

By relabeling the dummy indices $L \leftrightarrow N$ in the term $\Gamma^L_{NM} \ \delta(\Gamma^N_{LJ})$ of eq-(C.6) it becomes $\Gamma^N_{LM} \ \delta(\Gamma^L_{NJ})$ yielding finally

$$\delta\mathbf{R}_{MJ} \ = \ \nabla_M(\delta\Gamma^N_{NJ}) \ - \ \nabla_N(\delta\Gamma^N_{MJ}) \ + 2 \ \Gamma^N_{[ML]} \ \delta(\ \Gamma^L_{NJ} \ ) \ - \ \delta(\ {f_{ML}}^N \ \Gamma^L_{NJ} \ ) \qquad (C.7)$$

as promised.

APPENDIX D

We will show that the curvature expression (2.23) transforms homogeneously under coordinate transformations when the connection transforms as

$$\tilde{\Gamma}^K_{MN} \ = \ \Gamma^{K'}_{M'N'} \ \frac{\partial X^{M'}}{\partial \tilde{X}^M} \ \frac{\partial X^{N'}}{\partial \tilde{X}^N} \ \frac{\partial \tilde{X}^K}{\partial X^{K'}} \ + \ \frac{\partial^2 X^P}{\partial \tilde{X}^M \partial \tilde{X}^N} \ \frac{\partial \tilde{X}^K}{\partial X^P} \qquad (D.1)$$

Writing eq-(D.1) as $\tilde{\Gamma}^K_{MN} = \hat{\Gamma}^K_{MN} + I^K_{MN}$, in terms of the homogeneous $\hat{\Gamma}^K_{MN}$ and inhomogeneous I^K_{MN} parts, respectively, leads us to show that the inhomogenous terms appearing in $\tilde{R}^K_{MNJ}$ must vanish. These are given by

$$\tilde{\partial}_M \ I^K_{NJ} \ - \ \tilde{\partial}_N \ I^K_{MJ} \ - \ \hat{\Gamma}^L_{MJ} \ I^K_{NL} \ + \ \hat{\Gamma}^L_{NJ} \ I^K_{ML} \ - \ I^L_{MJ} \ \hat{\Gamma}^K_{NL} \\ + \ I^L_{NJ} \ \hat{\Gamma}^K_{ML} \ - \ I^L_{MJ} \ I^K_{NL} \ + \ I^L_{NJ} \ I^K_{ML} \ - \ \tilde{f}^L_{MN} \ I^K_{LJ} \ - \\ \Gamma^{K'}_{M'J'} \ \frac{\partial}{\partial \tilde{X}^N} \ \left(\frac{\partial X^{M'}}{\partial \tilde{X}^M} \ \frac{\partial X^{J'}}{\partial \tilde{X}^J} \ \frac{\partial \tilde{X}^K}{\partial X^{K'}} \right) \ + \ \Gamma^{K'}_{N'J'} \ \frac{\partial}{\partial \tilde{X}^M} \ \left(\frac{\partial X^{N'}}{\partial \tilde{X}^N} \ \frac{\partial X^{J'}}{\partial \tilde{X}^J} \ \frac{\partial \tilde{X}^K}{\partial X^{K'}} \right) \qquad (D.2)$$

where the structure functions transform homogeneously as

$$\tilde{f}^L_{MN} \ = \ f^{L'}_{M'N'} \ \frac{\partial X^{M'}}{\partial \tilde{X}^M} \ \frac{\partial X^{N'}}{\partial \tilde{X}^N} \ \frac{\partial \tilde{X}^L}{\partial X^{L'}} \qquad (D.3)$$

Some of the inhomogenous terms vanish due to the contraction of indices. Let us take the term

$$\tilde{f}^L_{MN}\ I^K_{LJ}\ =\ f^{L'}_{M'N'}\ \frac{\partial X^{M'}}{\partial \tilde{X}^M}\ \frac{\partial X^{N'}}{\partial \tilde{X}^N}\ \frac{\partial \tilde{X}^L}{\partial X^{L'}}\ \frac{\partial^2 X^P}{\partial \tilde{X}^L \partial \tilde{X}^J}\ \frac{\partial \tilde{X}^K}{\partial X^P} \qquad (D.4)$$

In particular let us focus in the following term in (D.4) resulting from the contraction of the L indices and after using the chain rule of differentiation

$$\frac{\partial \tilde{X}^L}{\partial X^{L'}}\ \frac{\partial^2 X^P}{\partial \tilde{X}^L \partial \tilde{X}^J}\ =\ \frac{\partial^2 X^P}{\partial X^{L'} \partial \tilde{X}^J}\ =\ \frac{\partial^2 X^P}{\partial \tilde{X}^J \partial X^{L'}}\ =$$

$$\frac{\partial}{\partial \tilde{X}^J}\ \frac{\partial X^P}{\partial X^{L'}}\ =\ \frac{\partial}{\partial \tilde{X}^J}\ \delta^P_{L'}\ =\ 0 \qquad (D.5)$$

Hence, the terms $\tilde{f}^L_{MN}\ I^K_{LJ}$ in (D.4) vanish as a result of (D.5). Proceeding in a similar fashion on can show that terms in (D.2) like $\hat{\Gamma}^L_{NJ} I^K_{ML},\ -\hat{\Gamma}^L_{MJ} I^K_{NL},\ I^L_{NJ} I^K_{ML},\ -I^L_{MJ}\ I^K_{NL}$ vanish via similar contraction procedure as that obtained in eqs-(D.4, D.5).

The other terms (D.2) that do not vanish as a result of the contraction of indices will cancel out among each other. The following derivative terms cancel out

$$\tilde{\partial}_M\ I^K_{NJ}\ -\ \tilde{\partial}_N\ I^K_{MJ}\ =\ \frac{\partial^3 X^P}{\partial \tilde{X}^M \partial \tilde{X}^N \partial \tilde{X}^J}\ \frac{\partial \tilde{X}^K}{\partial X^P}\ -\ \frac{\partial^3 X^P}{\partial \tilde{X}^N \partial \tilde{X}^M \partial \tilde{X}^J}\ \frac{\partial \tilde{X}^K}{\partial X^P}\ =\ 0 \qquad (D.6)$$

after having used

$$\frac{\partial^2 \tilde{X}^K}{\partial \tilde{X}^M \partial X^P}\ =\ 0,\quad \frac{\partial^2 \tilde{X}^K}{\partial \tilde{X}^N \partial X^P}\ =\ 0 \qquad (D.7)$$

based on the results obtained in (D.5). Finally one can verify after a relabeling of dummy indices, using the results given by eqs-(D.5, D.7), and by having $(\partial^2 X^{M'}/\partial \tilde{X}^M \partial \tilde{X}^N) = (\partial^2 X^{M'}/\partial \tilde{X}^N \partial \tilde{X}^M)$ that the remaining terms of (D.2) cancel out exactly

$$-\ \Gamma^{K'}_{M'J'}\ \frac{\partial}{\partial \tilde{X}^N}\ \left(\frac{\partial X^{M'}}{\partial \tilde{X}^M}\ \frac{\partial X^{J'}}{\partial \tilde{X}^J}\ \frac{\partial \tilde{X}^K}{\partial X^{K'}} \right)\ +\ \Gamma^{K'}_{N'J'}\ \frac{\partial}{\partial \tilde{X}^M}\ \left(\frac{\partial X^{N'}}{\partial \tilde{X}^N}\ \frac{\partial X^{J'}}{\partial \tilde{X}^J}\ \frac{\partial \tilde{X}^K}{\partial X^{K'}} \right)\ +$$

$$I^L_{NJ}\ \hat{\Gamma}^K_{ML}\ -\ I^L_{MJ}\ \hat{\Gamma}^K_{NL}\ =\ 0 \qquad (D.8)$$

To sum up, the inhomogeneous terms (D.2) either vanish as a result of the contraction of indices or cancel out among each other as shown above. Finally one is left with the homogeneous terms. For example, like

$$\frac{\partial \Gamma^{K'}_{N'J'}}{\partial \tilde{X}^M}\ \frac{\partial X^{N'}}{\partial \tilde{X}^N}\ \frac{\partial X^{J'}}{\partial \tilde{X}^J}\ \frac{\partial \tilde{X}^K}{\partial X^{K'}}\ =$$

$$\frac{\partial \Gamma^{K'}_{N'J'}}{\partial X^{M'}}\ \frac{\partial X^{M'}}{\partial \tilde{X}^M}\ \frac{\partial X^{N'}}{\partial \tilde{X}^N}\ \frac{\partial X^{J'}}{\partial \tilde{X}^J}\ \frac{\partial \tilde{X}^K}{\partial X^{K'}} \qquad (D.9)$$

after using the chain rule of differentiation. Similar results follow for the other components of the curvature. Concluding, one has that the curvature transforms covariantly (homogeneously)

$$\tilde{R}^{K}_{MNJ} = R^{K'}_{M'N'J'} \frac{\partial X^{M'}}{\partial \tilde{X}^{M}} \frac{\partial X^{N'}}{\partial \tilde{X}^{N}} \frac{\partial X^{J'}}{\partial \tilde{X}^{J}} \frac{\partial \tilde{X}^{K}}{\partial X^{K'}} \qquad (D.10)$$

as expected where the expression for $\tilde{R}^{K}_{MNJ}$ is the same as eq-(2.23) by replacing all quantities for their tilde counterparts.

Appendix E : Measure in C-spaces

To finalize we shall discuss the measure issue in curved C-spaces. In a given coordinate system (a generalized Lorentz frame) the mixed-grade components of the metric g_{MN}, g^{MN}, beins E^{A}_{M}, inverse beins E^{M}_{A}, can be set to zero in order to considerably simplify the calculations; i.e. namely due to the very large diffeomorphism symmetry in C-space, one may choose a frame ("diagonal gauge") such that the *mixed* grade components of the metric g_{MN}, beins E^{A}_{M}, inverse beins E^{M}_{A} are zero. In this case the C-space metric components can be chosen to be given by the determinant expressions

$$g_{\mu_1\mu_2.....\mu_k \ \nu_1\nu_2.......\nu_k} (\mathbf{X}) =$$

$$det \begin{pmatrix} g_{\mu_1\nu_1}(\mathbf{X})) & \ldots & \ldots & g_{\mu_1\nu_k}(\mathbf{X}) \\ g_{\mu_2\nu_1}(\mathbf{X}) & \ldots & \ldots & g_{\mu_2\nu_k}(\mathbf{X}) \\ ------------ & & & -------------- \\ g_{\mu_k\nu_1}(\mathbf{X}) & \ldots & \ldots & g_{\mu_k\nu_k}(\mathbf{X}) \end{pmatrix} \qquad (E.1)$$

where $g_{\mu\nu}(\mathbf{X})$ is now a function of the Clifford polyvector-valued coordinates $\mathbf{X}$ which includes the ordinary vectorial coordinates x^{ρ}.

The metric component g_{ss} involving the scalar "directions" in C-space of the Clifford poly-vectors must also be included. It behaves like a Clifford scalar. The highest grade component $g_{[\mu_1\mu_2...\mu_D] \ [\nu_1\nu_2...\nu_D]}$ involves the pseudo-scalar "directions" in C-space. The latter scalar and pseudo-scalars might bear some connection to the dilaton and axion fields in Cosmology and particle physics.

The advantage of having $g_{MN} = 0$ if the grade of M is not the same as the grade of N is that the determinant of the C-space metric can be factorized as the product of determinants of matrices which are comprised of entries (blocks) given themselves by determinants like in eq-(E.1) . If an ordering prescription of indices is introduced, $\mu_1 < \mu_2 < ....\mu_n$ and $\nu_1 < \nu_2 < ....\nu_n$, the bivector-bivector components of the C-space metric in $D = 4$ dimensions $g_{\mu_1\mu_2 \ \nu_1\nu_2}$ can be arranged into an *ordered* square array of entries (blocks) given by a 6×6 matrix, since the number of independent bivector components in $D = 4$ is $4 \times 3/2 = 6$. For instance, the entries of the square 6×6 matrix $g_{\mu_1\mu_2 \ \nu_1\nu_2}$ are given themselves by determinants : $g_{12 \ 12} = g_{11}g_{22} - g_{12}g_{21}$; $g_{13 \ 13} = g_{11}g_{33} - g_{13}g_{31}$, ... etc, and such that its determinant is given by the ordinary determinant of an square 6×6 matrix.

The trivector-trivector components of the C-space metric in $D = 4$ dimensions $g_{\mu_1\mu_2\mu_3 \ \nu_1\nu_2\nu_3}$ can be arranged into an *ordered* square array of entries given by a 4×4 matrix, since the number of independent trivector components in $D = 4$ is $4 \times 3 \times 2/2 \times 3 = 4$. The entries $g_{\mu_1\mu_2\mu_3 \ \nu_1\nu_2\nu_3}$ of this square 4×4 matrix are given themselves by the determinants as shown in eq-(E.1). Following a similar procedure with the other C-space metric components, in this way one can write the measure of integration in $D = 4$ as the square root of the product of determinants

$$\mu_m(g_{MJ}) = \sqrt{|g|\ |det\ (g_{\mu\nu})|\ |det\ (g_{\mu_1\mu_2\ \nu_1\nu_2})|\ |det\ (g_{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3})|\ |det\ (g_{\mu_1\mu_2\mu_3\mu_4\ \nu_1\nu_2\nu_3\nu_4})|} \tag{E.2}$$

where g is the scalar-scalar part g_{ss} of the C-space metric and which must *not* be confused with $|det\ g_{\mu\nu}|$. The generalization to other dimensions is straightforward. Therefore, the integration measure in C-space would be

$$\int ds\ \prod dx^\mu\ \prod dx^{\mu_1\mu_2}\ \ldots\ dx^{\mu_1\mu_2\ldots\mu_D}\ \mu_m(g_{MJ}) \tag{E.3}$$

In the most general case one can have a C-space metric with non-vanishing mixed grade components such that the metric g_{MJ} components can be assembled into arrays of ordered *rectangular* matrices. The problem becomes that one cannot longer define a determinant of a rectangular matrix. One can also view the g_{MJ} as a hyper-matrix but the construction (if possible) of the hyper-determinant of the C-space metric (a hyper matrix) is a more difficult problem [13], [14]. To conclude we should mention the work involving two measures of integration and which possesses a number of attractive features [18] (and references therein). In addition to the standard measure $\sqrt{|det\ g_{\mu\nu}|}\ d^4x$ in $D = 4$, another measure of integration $\mathbf{\Phi} = d\phi_1 \wedge d\phi_2 \wedge d\phi_3 \wedge d\phi_4$ involving the four scalar fields $\phi_1, \phi_2, \phi_3, \phi_4$ as new dynamical variables was introduced. For details we refer to [18].

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