

The differential equation of a swarm of smart particles (insects, birds, robots, etc.) can be obtained using two axioms:

1. there is a steady density μ for the fluid dynamics
2. the flow velocity has the density gradient as a module

the Euler's equation (see Fluid mechanics of Landau Lifshitz for some ideas used here):

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho \mathbf{v}) \\ \rho \frac{d\mathbf{v}}{dt} &= -\nabla p \implies \frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \cdot \nabla) \mathbf{v} - \frac{1}{\rho} \nabla p\end{aligned}$$

I simplify the equations using a force in each volumes that is proportional to the gradient of the density of smart particles:

$$\mathbf{F}_{dV} = -\alpha(\rho) \nabla \rho (\rho dV)$$

so that the intelligent particles move towards points with optimal density μ (this is the force that the fluid apply to the volume dV , but I consider this force like the force of the swarm of ρdV particles):

$$\alpha(\rho) = \frac{w^2}{\mu} \text{sgn}(\rho - \mu)$$

so that this is the force in the swarm (like a bird that change direction using a force to obtain the optimal density).

I write the Euler's equation for this force (I consider a constant pressure on an infinitesimal volume dV that it is equal to the density gradient):

$$\begin{aligned}\partial_t \rho &= -\nabla \cdot (\rho \mathbf{v}) \\ \rho d_t \mathbf{v} &= -\alpha(\rho) \rho \nabla \rho \implies \partial_t \mathbf{v} = -(\mathbf{v} \cdot \nabla) \mathbf{v} - \alpha(\rho) \nabla \rho\end{aligned}$$

I solve this equations for little perturbation of the density:

$$\rho = \mu + \epsilon$$

so that:

$$\begin{aligned}\partial_t \epsilon &= -\mu \nabla \cdot \mathbf{v} \\ \partial_t \mathbf{v} &= -\alpha(\rho) \nabla \epsilon\end{aligned}$$

to simplify the equations (like in the sound wave):

$$\mathbf{v} = \nabla \phi$$

so that:

$$\begin{aligned}\partial_t \epsilon &= -\mu \Delta \phi \\ \partial_t \nabla \phi &= -\alpha(\rho) \nabla \epsilon \\ \epsilon &= -\frac{1}{\alpha(\rho)} \partial_t \phi \\ \partial_{tt}^2 \phi - \alpha(\rho) \mu \Delta \phi &= 0 \\ \partial_{tt}^2 \phi - w^2 \operatorname{sgn}(\rho - \mu) \Delta \phi &= 0\end{aligned}$$

there are two regions; the internal region that has a sound wave solution with velocity w , and an external region that has exponential decay with mean lifetime of w .

The perturbative solution near the border of the swarm are sound waves of particles with constant velocity w , and out of the swarm the solution are an exponential space decay; a little perturbation because of draft, or predators, lead to an attraction at the border.