Question 460: A Formula for Pi

Edgar Valdebenito

abstract

This note presents a simple formula for pi.

1. INTRODUCTION. The number pi is defined by:

$$\pi = \int_0^1 \frac{1}{x^{3/4} + x^{5/4}} dx = 3.14159265... \quad (1)$$

This note presents a simple formula for pi.

2. FORMULA.

Let $n \in \mathbb{N} = \{1, 2, 3, 4, ...\}$, then

$$\pi = 2^{n+1} \left(\tan^{-1} \left(\frac{s_n}{2^{1-2^{-n}} + c_n} \right) + \tan^{-1} \left(\frac{s_n}{2^{1+2^{-n}} + c_n} \right) \right) \quad (2)$$

where

$$s_n = \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}}}_{n-radicals} \quad , n \in \mathbb{N} \quad (3)$$

$$c_n = \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}}}_{n-radicals} \quad , n \in \mathbb{N} \quad (4)$$

$$s_1 = \sqrt{2} \quad , \quad s_2 = \sqrt{2 - \sqrt{2}} \quad , \quad s_3 = \sqrt{2 - \sqrt{2 + \sqrt{2}}} \,, ... \quad (5)$$

$$c_1 = \sqrt{2} \quad , \quad c_2 = \sqrt{2 + \sqrt{2}} \quad , \quad c_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \,, ... \quad (6)$$

Examples:

$$m=1 \Rightarrow \quad \pi = 4\left(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right) \tag{7}$$

$$m=2 \Rightarrow \quad \pi = 8\left(\tan^{-1}\left(\frac{\sqrt{2-\sqrt{2}}}{\sqrt[4]{8}+\sqrt{2+\sqrt{2}}}\right) + \tan^{-1}\left(\frac{\sqrt{2-\sqrt{2}}}{2\sqrt[4]{2}+\sqrt{2+\sqrt{2}}}\right)\right) \tag{8}$$

$$m=3 \Rightarrow \quad \pi = 16\left(\tan^{-1}\left(\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt[4]{128}+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right) + \tan^{-1}\left(\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2\sqrt[8]{2}+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)\right) \tag{9}$$

Remark: $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, $-1 \le x \le 1$, is the arctangent function.

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