Set Theory

A few philosophical concerns are in order before we discuss the topic. As humans, we have notions of right and wrong, truth and falsity, and meaning. Whether or not we admit, we *assign* meaning to things. We *decide* truth or falsity; we *choose* our logical system; we *judge* right or wrong. Yes, our complex world is *rarely* 'black and white', but the *precision* requirements of math require *us* to be precise and explicit in our: logical system, symbol system, and axiom systems. In this way, we develop a *precision system* with specific purposes and intent. In set theory, our domain is sets of 'things'. Initially, these will be general/non-unique as a required starting point. Then we will move toward progressively more complex forms of uniqueness. The purpose? To befuddle? No.. Again, we're attempting to arrive at some 'economical' set of statements that can function as a 'basis' for *all* set theory and math. Why economical (concise/efficient/succinct)? Why not? Why double or triple the 'work load' on set theorists and students? No, we need an *efficient* axiom system which: economizes statements, symbols, and complexity, maximizes student understanding, relevant sophistication, and maturity, and preferably this system will *jive* with our intuitions – making them precise.

..I'm presently taking a course in set theory with textbook by Hinman. I assume (trust) the text is strategically sound (most/all statements are logically valid and their connections are meaningful). But if the purpose of the (section of the) text is to impart students with an understanding of set theory and allow us to become independent thinkers in set theory, it does not fulfill it. In this respect, it appears counterproductive. Set theory is regarded as the core of mathematics and as such, is the most fundamental discipline. The axioms of set theory are the core of that. *How* we develop the axioms is *critical* to set theory and math.

Why axiomatize? It's a fair question.. I do not wish to delve into the historical reasons for the axiomatization of set theory. That will cause confusion. Suffice it to say, we can axiomatize something or not. There are certain benefits for doing this which you, as an independent reader, can investigate. If we don't, we must still *justify explicitly* why we believe what we believe. So a system that is not axiomatized must be able to 'produce' *explicitly and without contradiction* – the *same results* we'd expect from an axiomatized one.

Now, once we decide to axiomatize, we must follow a certain 'logical order' of dependency: *nothing* can be introduced such that it has *not been defined* – and – nothing can be *used* unless it is defined *first*. For instance, I cannot *express* equality unless I define what equality means *first*. This is *absolutely essential* in the development of set theory axioms.

Let us first 'lay out' our allowed logical symbols (and one more) of set theory:
→ ↔ ¬ ∧ ∨ fa te ( ) variables = ≡ ! e
with meaning:
if-then, if-and-only-if/iff, not, and, or, for-any, there-exists, such-that/applying-on, any symbol indicating something, equality, defined-equality, unique, element-of

We cannot use any other symbol, other than above, to create our axiom system. We will make English statements then attempt to 'rephrase' them into the logical+ symbols listed above. The point? To create a *precise* axiomatic system of *exact connected meaningful* statements which we can use to build set theory from .. The following axiom was questioned by my professor (the necessity). The purpose is to *make explicit* our notion of elements *before* that notion is used in set equivalence. It is *required* in a dependency framework.. We will find that this system is *most economical* in terms of repetition and symbol usage. We will state and prove a theorem regarding this.

Axiom 0: Existence: Sets exist; teX(yeX)
(reread in English: there-exists a set such-that it has an element.)

Axiom 1: Extensionality: Two sets are equal if they have the same elements;
faXfaY(faz(zeX↔zeY)→X=Y)
The reason we need Existence *before* Extensionality is because this Axiom requires the notion of 'same elements'. This *assumes* sets have elements. We *cannot use* this notion without defining it *first*. At this point, we can say the most primitive contradiction is ¬(x=x) but we will find something slightly less cumbersome very soon.

Axiom 2: Null-set/Empty-set: There is a *unique* set that is not a member of itself and we call that the 'null set' or 'empty set'; faXte!ϕ(¬(ϕeϕ)∧(ϕeX∨X=ϕ))
So the most primitive F (false) statement or contradiction is ϕeϕ.
Notice I cannot *state* Axiom 2 *before* Axiom 1 because I use the notion of *set equality* in Axiom 2. In other words, unless I define what 'set equality' is, I cannot use = meaningfully. Some might 'point out' that I use = in Axiom 1. But there is a hand-written convention I use which is difficult to put into typed text: def meaning 'by definition' where I write that over the arrow in Axiom 1. Only in this context, it is allowed. ..A helpful way to look at ϕ in set theory is analogous to constants relative to variables. It's *the* fundamental constant in set theory .. The following axiom is also new (in addition to Axiom 0). It's *required* because we need explicit precision in the *notion* before we use it later on. We also need some precision *notation* regarding sets.

Axiom 3: Containment: The containment of a set is the set of that set; faXte!Y(XeY)→Y≡{X}
Here, Y is not meant to be unique (in notation). What is defined here is the meaning of the *curly-braces*. Again, we assume a def above the arrow. This convention will prove extremely useful below. Now, we can actually 'do some work': We can show explicitly why ϕ={ϕ} is F. It may seem counter-intuitive, but if you write out the logical expression equivalent to ϕ, you see immediately that faXte!ϕ(¬(ϕeϕ)∧(ϕeX∨X=ϕ)) = {faXte!ϕ(¬(ϕeϕ)∧(ϕeX∨X=ϕ))} is equivalent to ϕeϕ, the statement that is *always* F .. An aside is relevant. Suppose we wanted to dispense with ϕ altogether. We could use {} to mean ϕ, but this may lead to confusion. It's not immediately obvious {}={{}} is F. Whereas ϕ={ϕ} is clearly F. Our minds *need symbols* to associate with *common notions*; the most common set in set theory is ϕ .. Why is ϕ={ϕ} clearly F? ϕ is the *only* set that does not contain itself. Why is ϕ the most common set? Because *every* non-empty set *contains* ϕ. For instance, the set {1,2}={ϕ,1,2}. Finally (although we have not introduced intersection), the intersection of *all* sets is ϕ .. Some readers might point out if {1,2}={ϕ,1,2}, this implies {}={ϕ} which is a contradiction. *This is precisely why we* ***cannot*** *use {} to mean ϕ.*

Axiom 4: Union: The union of two sets is formed by combining elements;
faXfaYte!Z(weZ↔(weX∨weY))→Z≡XUY
As with =, we're *defining* what U means (not the symbol Z). So we need to imagine a def above the arrow just above. This is not arbitrary. At the same time we develop our critical axioms, we introduce *convenient notation* to express notions associated with them. When I write ϕ, it is *short-hand* for the logical statement in Axiom 2. Notice that how we union two sets is by or-ing their elements. A fair question at this point is: what is ϕUϕ? If you asked this question yourself, you're paying attention. Since the null-set has no elements, combining 'no elements' with 'no elements' is still 'no elements' or in other words, ϕUϕ=ϕ. I leave this to the reader to prove in logical+ symbols. Hint: replace X and Y above by ϕ and realize the implications on Z.

The next axiom requires the notion of successor but do not restrict yourself to the integers because at this point, we cannot even talk about numbers. You might ask: why not? Look at the *axioms* above. Do we say *anything* about numbers? No. We talk about sets, elements, set equality, null-set, and union. There's nothing about numbers above anywhere.. Successor depends on union and so we must create an axiom for it .. The following axiom is 'new' in the sense other systems don't make it *explicit*.

Axiom 5: Successor: The successor of a set is the set union-ed with its containment;
faXte!S(X)(S(X)≡XU{X}≡{X,{X}}≡{{X},X})
Here, we introduce some especially convenient notation and associated notions. Notice we can start using 'standard' set notation: curly-braces and comma. We realize the order of set elements does not matter. The last part of the expression is strictly not required because of Extensionality, but it is stated for clarity. The comma is there to help separate elements; they are *not* part of the set. We're getting very close to something deeply profound.

Axiom 6: Infinity: Not all sets are finite; one in particular we call the natural numbers; it can be created from ϕ and S; there is a one-to-one correspondence between the naturals and the set created by repeated application of S on ϕ and its result;
te!X(ϕeX^fay(yeX→S(y)eX))→X≡**N** where the number of elements in **N** we label ∞
Again, there is a def above the right arrow. Again, X is not special; it's **N** and ∞ that are. The unique set mentioned above that is equivalent to the naturals is this:
S(S(S...(ϕ)))={ϕ,{ϕ},{ϕ,{ϕ}},{ϕ,{ϕ},{ϕ,{ϕ}}},{ϕ,{ϕ},{ϕ,{ϕ}},{ϕ,{ϕ},{ϕ,{ϕ}}}},...
0 1 2 3 4
Addendum: my professor objected to the use of ellipses because it's not well defined as other things are. I also could have injected another axiom about set *size* before this. That's 'overkill'. The fact we can generate a set with 1-to-1 correspondence with the naturals from S and ϕ indicates to me we're 'on the right track' about choice of 'essential' axioms. The logical statement just above *implies* the natural numbers because of its *existence*. There's *nothing* that says we *cannot* perform an operation *forever* such as successor in any allowed manner. In fact, this is an argument in my favor. Suppose there was something stopping us from performing S on S forever. That would imply some sort of allowable finiteness on the operation itself. But that's a contradiction *in itself*. Successor has *no* implied restrictions on *how many times* you use it. Therefore, the *nature* of Successor *implies* Infinity. So in reality, Axiom 6 is merely a 'corollary' to Axiom 5. We make it explicit and distinct because of the *centrality of the naturals* with respect to math .. There's a simple algorithm to create the set above. We start with the empty-set. We contain it. We add that to your list of sets, we contain that, we add that to your list of sets, we contain that,... Notice where I placed the integers and what they represent: they represent the number of elements in each subset: there are zero elements in the first, one element in the second, two elements in the third and so on. Now we have a simple way to ***count*** and this is *profound*. We also have a notion of ***infinity*** and this is also *profound*. What is even more-so is the fact we derived a direct analog of the naturals simply from the null-set and successor.. *Now* we can revisit your intuition above: S on **N**. What is S(0)? 1. S(1)? 2.. We can check this via containment: {0}=1, {1}=2, {2}=3.. The notation might be a bit confusing but if we 'read it out', not so bad.. Of course, that argument is somewhat imprecise because 3 also contains 1 not just 2, but serves the purpose of guiding our intuitions .. Note that we use the concept of *uniqueness* in Axioms 2 through 6; although this is not always the case, the concept visibly permeates set theory and therefore math. Finally, please note when I say 'X is not special', this simply means: we have *arbitrarily* chosen some symbol (variable) to *mean* a set with *certain* listed properties; those *properties* determine the *nature* of the uniqueness. So, if you were to write out your *own* axioms for set theory 'from scratch', you'd need to follow this pattern: *choose* a symbol set and *what they mean*, *decide* on which subset of symbols you need to express a *meaningful qualification of uniqueness*, and *make it explicit*. Of course, you must have a starting point

and so I was forced to make use of Existence and Extensionality to 'set the stage' for meaningful uniqueness. Essentially, this is the process of axiomatization. It's instructive to try it yourself. This makes you *acutely* aware of our decisions regarding: logical structure, meaning, and economy. I contest the system above is the *most economical* one in the sense – *any* other system devised with *equal requirements* would be a variation in symbols *only*..* The rest of the axioms will be stated in English only. It will be left to the reader to show how they 'pan out' in terms of logical+ notation.. Don't try to write all statements in primitive form: use the convenient notation we've developed above. Remember, *this* set of axioms *build* on each other!
*Of course, there is a more modern notion of fuzzy logic systems which are interesting and have some interesting applications in 'real life', but we assume a need here for 'black and white' / unequivocal truth/falsity.

Axiom 7: Pair: The set of two sets is a set.

Axiom 8: Powerset: The set of all subsets of a set is a set.

Axiom 9: Foundation: Every set has at least one epsilon-minimal element which has no sub-elements.

Axiom 10: Replacement: If φ is a formula which defines a function with domain A, the range of φ on A is a set.

These 11 axioms are sufficient to build set theory from the 'ground up'. Using logic+ and these 11 axioms, we can create a 'basis' for the theory of: functions, numbers, analysis, probability, algebra, and any other 'higher level' area. Based on logic+, set theory is indeed the 'Rosetta Stone' of mathematics and science. In spite of Gödel, Hilbert's Program *is* feasible: from *nothing*, we can create *everything*. We find, in the process of axiomatization, we *decide*: logic (and therefore *decidability*), the structure and *richness* of our system, and ultimate *applicability*. We decide the *meaning* of our system by our *choices*.

Theorem 1: The system above is *most economical* in terms of repetition and symbol usage.

Proof: First, we must define what we mean by economical. We define repetition to be: rephrasing of unique meaning. What we *mean* by symbol usage should be clear: what you mean by a particular symbol. So any other system you devise to mean the same things as above will *differ in symbols only*. Further, because the *structure* above is a *strictly dependent* structure, where one statement builds on previous, your system will either be dependent XOR not. If it is *not*, you will have to define *everything independently*. This is an *enormous and unnecessary* overhead (as inefficient as having to reprove *everything every time* you prove a theorem). So, as long as we choose an *appropriate* set of axioms, it becomes a matter of *ranking* them in an appropriate *dependent order* with *least* dependent starting first. Any less-interdependent list of axioms will have more overhead than above. QED