BOUNDARY MATRICES AND THE MARCUS-DE OLIVEIRA DETERMINANTAL CONJECTURE*

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Abstract. We present notes on the Marcus-de Oliveira conjecture. The conjecture concerns the region in the complex plane covered by the determinants of the sums of two normal matrices with prescribed eigenvalues. Call this region Δ. This paper focuses on boundary matrices of Δ. We prove 3 theorems regarding these boundary matrices. We propose 2 conjectures related to the Marcus-de Oliveira conjecture.

Key words. determinantal conjecture, Marcus-de Oliveira, determinants, normal matrices, convex-hull

AMS subject classifications. 15A15, 15A16

1. Introduction.

Marcus [3] and de Oliveira [2] made the following conjecture. Given two normal matrices A and B with prescribed eigenvalues $a_1, a_2 ... a_n$ and $b_1, b_2 ... b_n$ respectively, $det(A + B)$ lies within the region:

$$co\Big\{ \prod (a_i + b_{\sigma(i)}) \Big\}$$

where $\sigma \in S_n$. co denotes the convex hull of the $n!$ points in the complex plane. As described in [1], the problem can be restated as follows. Given two diagonal matrices, $A_0 = diag(a_1, a_2 ... a_n)$ and $B_0 = diag(b_1, b_2 ... b_n)$, let:

$$\Delta = \Big\{ det(A_0 + UB_0U^*) : U \in U(n) \Big\} \tag{1.1}$$

where $U(n)$ is the set of $n \times n$ unitary matrices. Then we can write the conjecture as:

CONJECTURE 1.1 (Marcus-de Oliveira Conjecture).

$$\Delta \subseteq co\Big\{ \prod (a_i + b_{\sigma(i)}) \Big\} \tag{1.2}$$

Let

$$R(U) = det(A_0 + UB_0U^*). \tag{1.3}$$

Then the points forming the convex hull are at $R(P_0), R(P_1) ... R(P_{n!-1})$, where the P's are the $n \times n$ permutation matrices. We will refer to these as **permutation points** from now on.

The paper is organized as follows. In section 2 we define terms that will be used in the rest of the paper. These terms are necessary to state our main results. In section 3, we state our 3 main theorems. section 4 provides a proof of the first theorem. section 5 provides a proof of the second, and section 6 provides a proof of the third. In section 7, we state 2 conjectures. In section 8, we conclude.

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2. Terms and definitions.

2.1. Boundary points and matrices.

- Given a point P on $\partial\Delta$ (the boundary of Δ) and given a unitary matrix U such that R(U)=P, we call U a **boundary matrix** of Δ. See (1.3)
- A **regular boundary point** is a point where the boundary is smooth.
- A non-permutation boundary matrix for a regular boundary point is called a **regular boundary matrix**.

2.2. Properties of unitary matrices given A_0 and B_0.

In this section, we define four properties of unitary matrices that will be very useful when examining boundary matrices of Δ. These properties will be referred to throughout the paper in relation to a given unitary matrix U.

The first three of these properties are matrices related to U. These matrices are defined in [1], p.27. They provide a language to talk about unitary matrices within the context of the determinantal conjecture.

B-matrix

$$B = UB_0U^* \tag{2.1}$$

C-matrix

$$C = A_0 + UB_0U^* \tag{2.2}$$

Using (1.3), $R(U) = det(C)$

F-matrix

$$F = BC^{-1} - C^{-1}B$$

We can change the F-matrix into a more useful form:

$$F = (C - A_0)C^{-1} - C^{-1}(C - A_0)$$

$$F = C^{-1}A_0 - A_0C^{-1} \tag{2.3}$$

The F-matrix is only defined when C is invertible or equivalently $R(U) \neq 0$.

Since A_0 is diagonal, we see that F is a zero-diagonal matrix.

As demonstrated in [1], p.27, the F-matrix is 0 if and only if U is a permutation matrix.

The fourth property is conditional. Given a unitary matrix U with $R(U) \neq 0$ and with F-matrix $F \neq 0$. let $T = tr(ZF)$, where Z is any skew-hermitian matrix. T is a complex number and can be seen as a vector in the complex plane. If for all possible skew-hermitian matrices Z, all values of T are either parallel or anti-parallel, then we say that U is **trace-argument constant**. We take the zero-vector as being parallel to any vector.

2.3. Additional matrix related definitions.

- An **essentially-hermitian** matrix is a matrix that can be written as $e^{i\theta}H + \lambda I$ where θ is real, H is hermitian, λ is complex and I is the identity matrix. Equivalently an essentially-hermitian matrix is a normal matrix with collinear eigenvalues. This definition comes from [4].

3. Main Results.

THEOREM 3.1. *Every regular boundary matrix U of Δ with $R(U) \neq 0$ is trace-argument constant.*

THEOREM 3.2. *$\partial\Delta$ is smooth at all non-zero, non-permutation points.*

THEOREM 3.3. *Given a unitary matrix that is trace-argument constant, its F-matrix is essentially-hermitian with $\lambda = 0$.*

4. Proof of Theorem 3.1. Our aim is to examine boundary matrices of Δ. Towards this aim, it is useful to consider smooth unitary matrix functions going through these boundary matrices and see how they behave under (1.3). For this reason, we introduce the functional form of (1.3).

$$R(t) = \det(A_0 + U(t)B_0U^*(t)) \tag{4.1}$$

where t is real and $U(t)$ is some smooth function of unitary matrices.

Suppose $U(t)$ goes through a boundary matrix of interest, U_0 at t = 0.

Every unitary matrix can be written as an exponential of a skew-hermitian matrix. So we can write:

$U(t) = e^{S(t)}U_0$, where $S(t)$ is a smooth function of skew hermitian matrices with $S(0) = 0$.

Every choice of $S(t)$ with $S(0) = 0$, gives us every possible $U(t)$ that passes through U_0 at $t = 0$.

We wish to examine $U(t)$ and $R(t)$ near $t = 0$.

For small Δt,

$U(\Delta t) = (e^{S(\Delta t)})U_0$

$U(\Delta t) = (e^{S(0)+(\Delta t)S'(0)})U_0$

$U(\Delta t) = (e^{(\Delta t)S'(0)})U_0$

If we take the above function and plug it into $R(t)$ we'll get $R(\Delta t)$, but it won't be in a form useful to us. We use a result from [1], p.27 for this purpose. In order to state this result within the context of this paper, we first need the functional forms of the B-matrix, C-matrix, F-matrix (these were defined in section 2):

$$B(t) = U(t)B_0U^*(t) \tag{4.2}$$

$$C(t) = A_0 + B(t) \tag{4.3}$$

$$F(t) = C^{-1}(t)A_0 - A_0C^{-1}(t) \tag{4.4}$$

Now we can state the result from [1]:

When $F(0) \neq 0$,

$$R(\Delta t) = R(0) + (\Delta t)\det(C(0))tr(S'(0)F(0)) + O((\Delta t)^2) \tag{4.5}$$

$$R'(0) = det(C(0))tr(S'(0)F(0)) \tag{4.6}$$

If $F(0) = 0$ then U_0 is a permutation matrix and hence not a regular boundary matrix (section 2). Our concern here is with regular boundary matrices so we will assume $F(0) \neq 0$.

Note that $C(0)$ is just the C-matrix of U_0 and $F(0)$ is just the F-matrix of U_0. Also, $F(0)$ is only defined as long as $R(0) \neq 0$.

Assume U_0 is a regular boundary matrix with $R(0) \neq 0$. Then the tangent line to the curve $R(t)$ at $t = 0$ must remain the same regardless of our choice of $S(t)$. This is illustrated in Figure 1 where the closed curve indicates $\partial\Delta$. $R'(0)$ can be seen as a vector in the complex plane. So all possible values of $R'(0)$ are either parallel or anti-parallel.

$S'(0)$ is a skew hermitian matrix since the difference of skew-hermitian matrices is also skew-hermitian. $S'(0)$ can turn out to be any skew-hermitian matrix.

Proof. Suppose we choose an arbitrary skew-hermitian matrix and multiply each element of the matrix by t. Then we get a smooth function of skew-hermitian matrices $S(t)$ with $S(0) = 0$ such that $S'(0)$ is the skew-hermitian matrix we initially chose. □

So we can rewrite $R'(0)$ without any reference to the $S(t)$ function:

$$R'(0) = det(C(0))tr(ZF(0)) \tag{4.7}$$

where Z is a skew-hermitian matrix. Since all values of $R'(0)$ are either parallel or anti-parallel, all values of $tr(ZF(0))$ are parallel or anti-parallel, regardless of the choice of Z. That gives us Theorem 3.1.

5. Proof of Theorem 3.2. In [1], p.26, Theorem 4, Bebiano and Queiró prove that if within the neighborhood of a non-zero point $z \in \partial\Delta$, Δ is contained within an angle less than π, then z must be a permutation point. We extend this result to show that at a non-zero, non-permutation point, the boundary is smooth.

Proof. Given we have a non-zero, non-permutation point $z \in \partial\Delta$. Proof by contradiction. Assume the boundary is not smooth at z. Since we know that Δ is not contained within an angle less than π by Bebiano and Queiró's result, we know that within the neighborhood of z, Δ is contained within some angle greater than π. So we

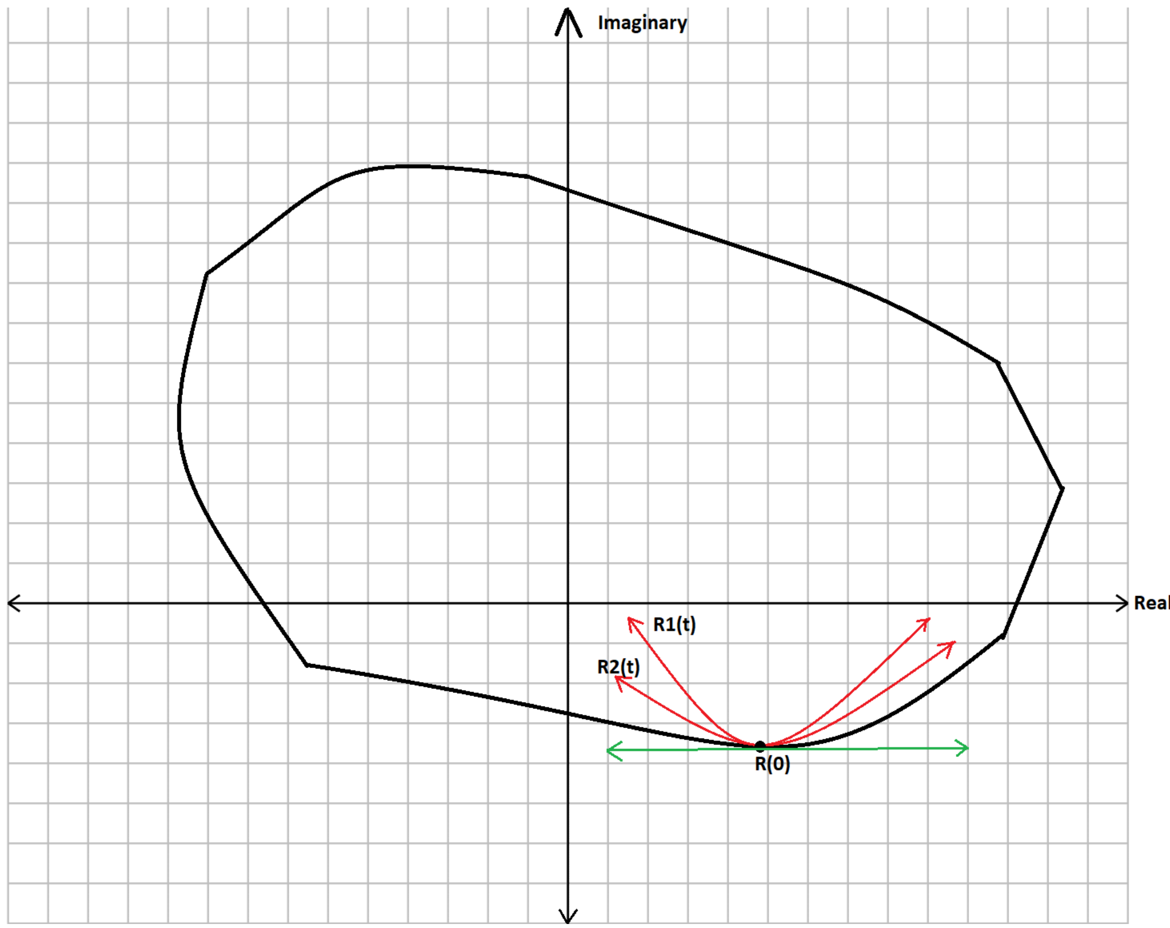

FIG. 1. *Region Δ with tangents at a boundary point*

can find two smooth functions $R_1(t) \subseteq \Delta$ and $R_2(t) \subseteq \Delta$ such that $R_1(0) = R_2(0) = z$ and $R_1'(0)$ is not parallel or anti-parallel to $R_2'(0)$.

Let U be a boundary matrix for z and let F be the F-matrix of U. Then using (4.6),

$R_1'(0) = det(C)tr(Z_1F)$

$R_2'(0) = det(C)tr(Z_2F)$

where Z_1 and Z_2 are two skew-hermitian matrices. But since $R_1'(0)$ and $R_2'(0)$ are not parallel or anti-parallel, they form a basis for all the complex numbers as a vector space over the real numbers.

So $V = a \times det(C)tr(Z_1F) + b \times det(C)tr(Z_2F)$ goes in any direction depending on the choice of real numbers a and b.

$V = det(C)(a \times tr(Z_1F) + b \times tr(Z_2F))$

$V = det(C)tr((a \times Z_1 + b \times Z_2)F)$

$Z_n = a \times Z_1 + b \times Z_2$ is also a skew-hermitian matrix.

So given any direction, there exists a skew-hermitian matrix Z_n such that $det(C)tr(Z_nF)$ goes in that direction. Hence there exists a smooth function $R_n(t) \subseteq \Delta$ such that $R_n(0) = z$, and $R_n'(0)$ is parallel or anti-parallel to that direction.

So there are functions going through z in all directions, contained within Δ. So z is not a boundary point. We arrive at a contradiction. □

This proves Theorem 3.2.

6. Proof of Theorem 3.3. For n = 3, we define the following 12 skew-hermitian matrices with zero diagonal:

$$Z_{12} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Z_{13} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad Z_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Z_{21} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Z_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad Z_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$Z_{12,i} = Z_{21,i} = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Z_{13,i} = Z_{31,i} = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \qquad Z_{23,i} = Z_{32,i} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$$

Note that the commas do not indicate tensors. They're just used here as a label to distinguish imaginary and real matrices.

We define Z_{ab} and $Z_{ab,i}$ similarly for all $n > 3$, where $a \neq b$. For a given n we have $n(n-1)$ real matrices and $n(n-1)$ imaginary matrices.

Given a trace-argument constant unitary matrix U with F-matrix F. Suppose $F_{ab} = F_{ab,r} + iF_{ab,i}$

where $F_{ab,r}$ and $F_{ab,i}$ are real numbers.

$tr(Z_{ab}F) = F_{ab} - F_{ba}$

$tr(Z_{ab,i}F) = (F_{ab} + F_{ba})i$

Substitute in for F_{ab} and F_{ba}

$tr(Z_{ab}F) = (F_{ab,r} - F_{ba,r}) + i(F_{ab,i} - F_{ba,i})$

$tr(Z_{ab,i}F) = (-F_{ab,i} - F_{ba,i}) + i(F_{ab,r} + F_{ba,r})$

Since U is trace-argument constant,

$(F_{ab,i} - F_{ba,i})(-F_{ab,i} - F_{ba,i}) = (F_{ab,r} + F_{ba,r})(F_{ab,r} - F_{ba,r})$

We can simplify this to get:

$F_{ab,r}^2 + F_{ab,i}^2 = F_{ba,r}^2 + F_{ba,i}^2$

$|F_{ab}| = |F_{ba}|$

We can write:

$F_{ab} = |F_{ab}|\angle\theta_{ab}$

$F_{ba} = |F_{ab}|\angle\theta_{ba}$

slope of $tr(Z_{ab}F)$:

$\frac{\sin(\theta_{ab})-\sin(\theta_{ba})}{\cos(\theta_{ab})-\cos(\theta_{ba})} = -\cot(\frac{\theta_{ab}+\theta_{ba}}{2})$

similarly,

slope of $tr(Z_{cd}F) = -\cot(\frac{\theta_{cd}+\theta_{dc}}{2})$, where $c \neq d$

$\cot(\frac{\theta_{cd}+\theta_{dc}}{2}) = \cot(\frac{\theta_{ab}+\theta_{ba}}{2})$

therefore either:

$\frac{\theta_{cd}+\theta_{dc}}{2} = \frac{\theta_{ab}+\theta_{ba}}{2}$

or,

$\frac{\theta_{cd}+\theta_{dc}}{2} = \frac{\theta_{ab}+\theta_{ba}}{2} + \pi$

For some specific x, y where $x \neq y$

let $\beta = \frac{\theta_{xy}+\theta_{yx}}{2}$

let $H = e^{-i\beta}F$

For any $a \neq b$,

$H_{ab} = |H_{ab}|\angle\alpha_{ab}$

$\frac{\alpha_{ab}+\alpha_{ba}}{2} = 0$ or π

Therefore H is zero-diagonal, with transpositional elements of equal magnitude and opposite arguments. Therefore H is hermitian.

We can write F as:

$F = e^{i\beta}H$

This completes our proof of Theorem 3.3.

7. Conjectures. Before we state our conjectures we define a region Δ_S which is a restriction of Δ. See (1.1).

$$\Delta_S = \left\{det(A_0 + OB_0O^*) : O \in O(n)\right\} \tag{7.1}$$

where $O(n)$ is the set of $n \times n$ real orthogonal matrices.

As proven in [5], p.207, theorem 4.4.7, a matrix is normal and symmetric if and only if it is diagonalizable by a real orthogonal matrix.

Therefore Δ_S is the set of determinants of sums of normal, symmetric matrices with prescribed eigenvalues. We know Δ_S contains all the permutation points.

CONJECTURE 7.1 (Restricted Marcus-de Oliveira Conjecture).

$$\Delta_S \subseteq co\Big\{ \prod (a_i + b_{\sigma(i)}) \Big\}$$

The above conjecture is supported by computational experiments.

CONJECTURE 7.2 (Boundary Conjecture).

$$\partial\Delta \subseteq \partial\Delta_S$$

THEOREM 7.3. *If the boundary conjecture is true, the restricted Marcus-de Oliveira conjecture imples the full Marcus-de Oliveira conjecture.*

Proof. The unitary group and the real orthogonal group are compact subsets of the $n \times n$ complex matrices. Since a continuous image of a compact set is compact, Δ and Δ_S are compact subsets of the complex plane. Hence they are both closed by the Heine-Borel theorem.

Suppose we know Conjecture 7.1 is true. Then Δ_S along with its boundary is within the convex-hull. Suppose we also know that Conjecture 7.2 is true. Then we know that $\partial\Delta$ is inside the convex-hull. Can we have a unitary matrix U such that $R(U)$ is outside the convex-hull? No, because that would mean we have points of Δ on both the inside and outside of $\partial\Delta$. This is impossible since Δ is a closed set. So Δ is within the convex hull proving Conjecture 1.1. ∎

8. Conclusion. We hope that further analysis on boundary matrices of Δ, either by expanding on the results in this paper, or novel research, leads to a proof of the Boundary Conjecture. Then proving the full Marcus-de Oliveira conjecture would amount to proving the restricted conjecture. Whether the restricted conjecture is any easier to prove is unknown, but it's an avenue worth exploring.

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