Identities for second order recurrence sequences*

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Abstract

We derive several identities for arbitrary homogeneous second order recurrence sequences with constant coefficients. The results are then applied to present a harmonized study of six well known integer sequences, namely the Fibonacci sequence, the sequence of Lucas numbers, the Jacobsthal sequence, the Jacobsthal-Lucas sequence, the Pell sequence and the Pell-Lucas sequence.

Contents

1 Introduction

Our aim in writing this paper is to derive several identities for arbitrary second order recurrence sequences with constant coefficients. As a concrete illustration of how our results may be put to use, we will derive identities for the integer sequences mentioned in the abstract and defined below.

The Fibonacci numbers, F_n, and the Lucas numbers, L_n, are defined, for $n \in \mathbb{Z}$, as usual, through the recurrence relations $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$, $F_0 = 0$, $F_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$

*AMS Classification: 11B37, 11B39, 65B10

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$(n \geq 2)$, $L_0 = 2$, $L_1 = 1$, with $F_{-n} = (-1)^{n-1}F_n$ and $L_{-n} = (-1)^n L_n$. Exhaustive discussion of the properties of Fibonacci and Lucas numbers can be found in Vajda [8] and in Koshy [5].

The Jacobsthal numbers, J_n, and the Jacobsthal-Lucas numbers, j_n, are defined, for $n \in \mathbb{Z}$, through the recurrence relations $J_n = J_{n-1} + 2J_{n-2}$ $(n \geq 2)$, $J_0 = 0$, $J_1 = 1$ and $j_n = j_{n-1} + 2j_{n-2}$ $(n \geq 2)$, $j_0 = 2$, $j_1 = 1$, with $J_{-n} = (-1)^{n-1}2^{-n}J_n$ and $j_{-n} = (-1)^n 2^{-n} j_n$. Horadam [4] and Aydin [2] are good reference materials on the Jacobsthal and associated sequences.

The Pell numbers, P_n, and Pell-Lucas numbers, Q_n, are defined, for $n \in \mathbb{Z}$, through the recurrence relations $P_n = 2P_{n-1} + P_{n-2}$ $(n \geq 2)$, $P_0 = 0$, $P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}$ $(n \geq 2)$, $Q_0 = 2$, $Q_1 = 1$, with $P_{-n} = (-1)^{n-1}P_n$ and $Q_{-n} = (-1)^n Q_n$. Koshy [6], Horadam [3] and Patel and Shrivastava [7] are useful source materials on Pell and Pell-Lucas numbers.

Note that, in this paper, apart from in the binomial summation identities where the upper limit must be non-negative, the upper limit in the summation identities is allowed to take on negative values once we adopt the summation convention that, if $k < 0$ then

$$\sum_{r=0}^{k} f_r \equiv -\sum_{r=k+1}^{-1} f_r\,,$$

as long as f_r is not singular in the summation interval.

Here is a couple of results to whet the reader's appetite for reading on:

From Corollary 2:

$$\begin{aligned} F_{n+h}L_{n+k} - F_n L_{n+h+k} &= (-1)^n F_h L_k\,,\\ J_{n+h}j_{n+k} - J_n j_{n+h+k} &= (-1)^n 2^n J_h j_k\,,\\ P_{n+h}Q_{n+k} - P_n Q_{n+h+k} &= (-1)^n P_h Q_k\,. \end{aligned}$$

From Theorem 5:

$$(-1)^u L_u^2 + (-1)^v L_v^2 + (-1)^w L_w^2 = (-1)^w L_u L_v L_w + 4\,,$$

$$(-1)^u 2^v j_u^2 + (-1)^v 2^u j_v^2 + (-1)^w j_w^2 = (-1)^w j_u j_v j_w + 2^{w+2}$$

and

$$(-1)^u Q_u^2 + (-1)^v Q_v^2 + (-1)^w Q_w^2 = (-1)^w Q_u Q_v Q_w + 4\,,$$

for integers u, v, w such that $u + v = w$.

From Theorem 8, for nonnegative integer k and arbitrary integers a, b, c, d, e, m for which the denominator does not vanish:

$$\begin{aligned} &\sum_{r=0}^{k} \binom{k}{r} \left(\frac{F_{d-c}F_{e-b} - F_{e-c}F_{d-b}}{F_{d-a}F_{e-c} - F_{e-a}F_{d-c}}\right)^r F_{m-(b-c)k+(b-a)r}\\ &\qquad = \left(\frac{F_{d-a}F_{e-b} - F_{e-a}F_{d-b}}{F_{d-a}F_{e-c} - F_{e-a}F_{d-c}}\right)^k F_m\,, \end{aligned}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{J_{d-c}J_{e-b}-J_{e-c}J_{d-b}}{J_{d-a}J_{e-c}-J_{e-a}J_{d-c}}\right)^{r}J_{m-(b-c)k+(b-a)r}$$
$$=\left(\frac{J_{d-a}J_{e-b}-J_{e-a}J_{d-b}}{J_{d-a}J_{e-c}-J_{e-a}J_{d-c}}\right)^{k}J_{m}\,,$$
$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{P_{d-c}P_{e-b}-P_{e-c}P_{d-b}}{P_{d-a}P_{e-c}-P_{e-a}P_{d-c}}\right)^{r}P_{m-(b-c)k+(b-a)r}$$
$$=\left(\frac{P_{d-a}P_{e-b}-P_{e-a}P_{d-b}}{P_{d-a}P_{e-c}-P_{e-a}P_{d-c}}\right)^{k}P_{m}\,.$$

2 Main results

2.1 Identities

Lemma 1. *Let $\{X_m\}$ and $\{Y_m\}$, $m\in\mathbb{Z}$, be homogeneous second order recurrence sequences with constant coefficients. Let $\{X_m\}$ and $\{Y_m\}$ possess the same recurrence relation. Let $\Delta_{xy}=X_{d-a}Y_{e-b}-X_{e-a}Y_{d-b}$. Then, the identity:*

$$\begin{aligned}&(X_{d-a}Y_{e-b}-X_{e-a}Y_{d-b})X_{m-c}\\&\quad=(X_{d-c}Y_{e-b}-X_{e-c}Y_{d-b})X_{m-a}\\&\qquad+(X_{d-a}X_{e-c}-X_{e-a}X_{d-c})Y_{m-b}\,,\end{aligned}$$

holds for arbitrary integers a, b, c, d, e and m for which $\Delta_{xy}\neq 0$.

Proof. By hypothesis, $\{X_m\}$ and $\{Y_m\}$ have the same recurrence relations, therefore we seek a relation of the following type:

$$X_{m-c}=\lambda_1X_{m-a}+\lambda_2Y_{m-b}\,, \tag{2.1}$$

between any three numbers X_{m-c}, X_{m-a} and Y_{m-b}, where a, b and c are fixed integers and λ_1 and λ_2 are suitable constants. Evaluating (2.1) at $m=d$ and at $m=e$ produces two equations:

$$X_{d-c}=\lambda_1X_{d-a}+\lambda_2Y_{d-b} \tag{2.2}$$

and

$$X_{e-c}=\lambda_1X_{e-a}+\lambda_2Y_{e-b}\,, \tag{2.3}$$

to be solved simultaneously for the constants λ_1 and λ_2. Solutions exist if

$$\Delta_{xy}=\begin{vmatrix}X_{d-a}&Y_{d-b}\\X_{e-a}&Y_{e-b}\end{vmatrix}=X_{d-a}Y_{e-b}-X_{e-a}Y_{d-b}\neq 0\,.$$

The result follows from substituting into (2.1) the λ_1 and λ_2 found from solving (2.2) and (2.3).

□

Lemma 2. *Let $\{X_m\}$, $m\in\mathbb{Z}$, be a homogeneous second order recurrence sequence with constant coefficients. Then, the following identity holds for arbitrary integers a, b, c, d, e and m:*

$$\begin{aligned}&(X_{d-a}X_{e-b}-X_{e-a}X_{d-b})X_{m-c}\\&\quad=(X_{d-c}X_{e-b}-X_{e-c}X_{d-b})X_{m-a}\\&\qquad+(X_{d-a}X_{e-c}-X_{e-a}X_{d-c})X_{m-b}\,.\end{aligned}$$

Proof. Let $\Delta_{xx} = X_{d-a}X_{e-b} - X_{e-a}X_{d-b} \neq 0$ and proceed as in the proof of Lemma 1. We have

$$\begin{aligned}(X_{d-a}X_{e-b} &- X_{e-a}X_{d-b})X_{m-c}\\ &= (X_{d-c}X_{e-b} - X_{e-c}X_{d-b})X_{m-a}\\ &\quad + (X_{d-a}X_{e-c} - X_{e-a}X_{d-c})X_{m-b}\,.\end{aligned} \tag{2.4}$$

But we will now prove that the identity (2.4) continues to hold even if $\Delta_{xx} = 0$. Let

$$\Delta_1 = X_{d-c}X_{e-b} - X_{e-c}X_{d-b}, \quad \Delta_2 = X_{d-a}X_{e-c} - X_{e-a}X_{d-c}\,.$$

There are six possible situations in which Δ_{xx} can vanish. We consider them in turn.

1. $X_{d-a} = 0 = X_{e-a}$, in which case $d = e \Rightarrow \Delta_1 = \Delta_2 = 0$ and hence identity (2.4) remains valid.

2. $X_{e-b} = 0 = X_{d-b}$, in which case, again, $d = e \Rightarrow \Delta_1 = \Delta_2 = 0$ and hence identity (2.4) remains valid

3. $X_{d-a} = 0 = X_{d-b}$, in which case $b = a$ and the right side of identity (2.4) reads

$$(X_{d-c}X_{e-a} - X_{e-c}X_{d-a})X_{m-a} + (X_{d-a}X_{e-c} - X_{e-a}X_{d-c})X_{m-a}\,,$$

   which evaluates to zero, so that identity (2.4) remains valid.

4. $X_{e-b} = 0 = X_{d-b}$, in which case $e = d$ and the right side of identity (2.4) reads

$$(X_{d-c}X_{d-b} - X_{d-c}X_{d-b})X_{m-a} + (X_{d-a}X_{d-c} - X_{d-a}X_{d-c})X_{m-b}\,,$$

   which evaluates to zero, so that identity (2.4) remains valid.

5. $X_{d-a} = X_{e-a}$ and $X_{e-b} = X_{d-b}$, in which case, again, $d = e \Rightarrow \Delta_1 = \Delta_2 = 0$ and hence identity (2.4) remains valid.

6. $X_{d-a} = X_{d-b}$ and $X_{e-b} = X_{e-a}$, in which case, as in case 3, $b = a$ and hence identity (2.4) remains valid.

Thus we see that identity (2.4) is valid regardless of the nature of Δ_{xx}, so that the identity holds for all integers.

□

Lemma 3. *Let $\{X_m\}$, $m \in \mathbb{Z}$, be a homogeneous second order recurrence sequence with constant coefficients. Then, the following identity holds for arbitrary integers a, b, c and m:*

$$\begin{aligned}({X_0}^2 &- X_{b-a}X_{a-b})X_{m-c}\\ &= (X_{a-c}X_0 - X_{b-c}X_{a-b})X_{m-a}\\ &\quad + (X_0X_{b-c} - X_{b-a}X_{a-c})X_{m-b}\,.\end{aligned}$$

2.2 Summation identities

The following identities are obtained by making appropriate substitutions from Lemmata 1 and 2 into Lemmata 1 and 2 of [1].

Lemma 4. *Let $\{X_m\}$ and $\{Y_m\}$, $m \in \mathbb{Z}$, be homogeneous second order recurrence sequences with constant coefficients. Let $\{X_m\}$ and $\{Y_m\}$ possess the same recurrence relation. Let $\Delta_{xy} = X_{d-a}Y_{e-b} - X_{e-a}Y_{d-b}$, $\Delta_1 = X_{d-c}Y_{e-b} - X_{e-c}Y_{d-b}$ and $\Delta_2 = X_{d-a}X_{e-c} - X_{e-a}X_{d-c}$. Then, the following identity holds for arbitrary integers a, b, c, d, e, m and k for which $\Delta_{xy} \neq 0$, $\Delta_1 \neq 0$, $\Delta_2 \neq 0$:*

$$\begin{aligned}
&\sum_{r=0}^{k} \left(\frac{X_{d-a}Y_{e-b} - X_{e-a}Y_{d-b}}{X_{d-c}Y_{e-b} - X_{e-c}Y_{d-b}} \right)^r Y_{m-k(a-c)-b+c+(a-c)r} \\
&\qquad = \left(\frac{X_{d-a}Y_{e-b} - X_{e-a}Y_{d-b}}{X_{d-a}X_{e-c} - X_{e-a}X_{d-c}} \right) \left(\frac{X_{d-a}Y_{e-b} - X_{e-a}Y_{d-b}}{X_{d-c}Y_{e-b} - X_{e-c}Y_{d-b}} \right)^k X_m \\
&\qquad\quad - \left(\frac{X_{d-c}Y_{e-b} - X_{e-c}Y_{d-b}}{X_{d-a}X_{e-c} - X_{e-a}X_{d-c}} \right) X_{m-(k+1)(a-c)} \,.
\end{aligned} \tag{2.5}$$

Lemma 5. *Let $\{X_m\}$, $m \in \mathbb{Z}$, be a homogeneous second order recurrence sequence with constant coefficients. Let $\Delta_1 = X_{d-c}X_{e-b} - X_{e-c}X_{d-b}$ and $\Delta_2 = X_{d-a}X_{e-c} - X_{e-a}X_{d-c}$. Then, the following identities hold for integer k and arbitrary integers a, b, c, d, e and m for which $\Delta_1 \neq 0$ and $\Delta_2 \neq 0$:*

$$\begin{aligned}
&\sum_{r=0}^{k} \left(\frac{X_{d-a}X_{e-b} - X_{e-a}X_{d-b}}{X_{d-c}X_{e-b} - X_{e-c}X_{d-b}} \right)^r X_{m-k(a-c)-b+c+(a-c)r} \\
&\qquad = \left(\frac{X_{d-a}X_{e-b} - X_{e-a}X_{d-b}}{X_{d-a}X_{e-c} - X_{e-a}X_{d-c}} \right) \left(\frac{X_{d-a}X_{e-b} - X_{e-a}X_{d-b}}{X_{d-c}X_{e-b} - X_{e-c}X_{d-b}} \right)^k X_m \\
&\qquad\quad - \left(\frac{X_{d-c}X_{e-b} - X_{e-c}X_{d-b}}{X_{d-a}X_{e-c} - X_{e-a}X_{d-c}} \right) X_{m-(k+1)(a-c)} \,,
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&\sum_{r=0}^{k} \left(\frac{X_{d-a}X_{e-b} - X_{e-a}X_{d-b}}{X_{d-a}X_{e-c} - X_{e-a}X_{d-c}} \right)^r X_{m-k(b-c)-a+c+(b-c)r} \\
&\qquad = \left(\frac{X_{d-a}X_{e-b} - X_{e-a}X_{d-b}}{X_{d-c}X_{e-b} - X_{e-c}X_{d-b}} \right) \left(\frac{X_{d-a}X_{e-b} - X_{e-a}X_{d-b}}{X_{d-a}X_{e-c} - X_{e-a}X_{d-c}} \right)^k X_m \\
&\qquad\quad - \left(\frac{X_{d-a}X_{e-c} - X_{e-a}X_{d-c}}{X_{d-c}X_{e-b} - X_{e-c}X_{d-b}} \right) X_{m-(k+1)(b-c)}
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&\sum_{r=0}^{k} \left(\frac{X_{e-a}X_{d-c} - X_{d-a}X_{e-c}}{X_{d-c}X_{e-b} - X_{e-c}X_{d-b}} \right)^r X_{m-k(a-b)+b-c+(a-b)r} \\
&\qquad = \left(\frac{X_{d-a}X_{e-c} - X_{e-a}X_{d-c}}{X_{d-a}X_{e-b} - X_{e-a}X_{d-b}} \right) \left(\frac{X_{e-a}X_{d-c} - X_{d-a}X_{e-c}}{X_{d-c}X_{e-b} - X_{e-c}X_{d-b}} \right)^k X_m \\
&\qquad\quad + \left(\frac{X_{d-c}X_{e-b} - X_{e-c}X_{d-b}}{X_{d-a}X_{e-b} - X_{e-a}X_{d-b}} \right) X_{m-(k+1)(a-b)} \,.
\end{aligned} \tag{2.8}$$

2.3 Binomial summation identities

The following identities are obtained by making appropriate substitutions from the identity of Lemma 2 into the identities of Lemma 3 of [1].

Lemma 6. *Let $\{X_m\}$, $m \in \mathbb{Z}$, be a homogeneous second order recurrence sequence with constant coefficients. Let $\Delta_1 = X_{d-c}X_{e-b} - X_{e-c}X_{d-b}$ and $\Delta_2 = X_{d-a}X_{e-c} - X_{e-a}X_{d-c}$. Then, the following identity holds for positive integer k and arbitrary integers a, b, c, d, e and m for which $\Delta_1 \neq 0$ and $\Delta_2 \neq 0$:*

$$\begin{aligned}\sum_{r=0}^{k} \binom{k}{r} \left(\frac{X_{d-c}X_{e-b} - X_{e-c}X_{d-b}}{X_{d-a}X_{e-c} - X_{e-a}X_{d-c}}\right)^r X_{m-(b-c)k+(b-a)r} \\ = \left(\frac{X_{d-a}X_{e-b} - X_{e-a}X_{d-b}}{X_{d-a}X_{e-c} - X_{e-a}X_{d-c}}\right)^k X_m\,,\end{aligned} \tag{2.9}$$

$$\begin{aligned}\sum_{r=0}^{k} \binom{k}{r} \left(\frac{X_{e-a}X_{d-b} - X_{d-a}X_{e-b}}{X_{d-a}X_{e-c} - X_{e-a}X_{d-c}}\right)^r X_{m+(a-b)k+(b-c)r} \\ = \left(\frac{X_{d-c}X_{e-b} - X_{e-c}X_{d-b}}{X_{e-a}X_{d-c} - X_{d-a}X_{e-c}}\right)^k X_m\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\sum_{r=0}^{k} \binom{k}{r} \left(\frac{X_{e-a}X_{d-b} - X_{d-a}X_{e-b}}{X_{d-c}X_{e-b} - X_{e-c}X_{d-b}}\right)^r X_{m+(b-a)k+(a-c)r} \\ = \left(\frac{X_{d-a}X_{e-c} - X_{e-a}X_{d-c}}{X_{e-c}X_{d-b} - X_{d-c}X_{e-b}}\right)^k X_m\,.\end{aligned} \tag{2.11}$$

3 Applications and examples

We now employ the results of the previous section to give a combined study of six well known integer sequences. First we give a modified version of Lemma 1 that allows the removal of the Δ_{xy} condition.

Lemma 7. *Let $\{X_m\}$ and $\{Y_m\}$, $m \in \mathbb{Z}$, be homogeneous second order recurrence sequences with constant coefficients. Let $\{X_m\}$ and $\{Y_m\}$ possess the same recurrence relation. Let $Y_m \neq 0$ for all integers m. Finally, let $\{X_m\}$ and $\{Y_m\}$ have at most three members in common. Then, the identity:*

$$\begin{aligned}&(X_{d-a}Y_{e-b} - X_{e-a}Y_{d-b})X_{m-c} \\ &\quad = (X_{d-c}Y_{e-b} - X_{e-c}Y_{d-b})X_{m-a} \\ &\qquad + (X_{d-a}X_{e-c} - X_{e-a}X_{d-c})Y_{m-b}\,,\end{aligned}$$

holds for arbitrary integers a, b, c, d, e and m.

Proof. Let $\Delta_{xy} = X_{d-a}Y_{e-b} - X_{e-a}Y_{d-b}$. According to Lemma 1 we have

$$\begin{aligned}&(X_{d-a}Y_{e-b} - X_{e-a}Y_{d-b})X_{m-c} \\ &\quad = (X_{d-c}Y_{e-b} - X_{e-c}Y_{d-b})X_{m-a} \\ &\qquad + (X_{d-a}X_{e-c} - X_{e-a}X_{d-c})Y_{m-b}\,,\end{aligned} \tag{3.1}$$

provided that $\Delta_{xy} \neq 0$. But we will now prove that the identity (3.1) continues to hold even if $\Delta_{xy} = 0$. Let

$$\Delta_1 = X_{d-c}Y_{e-b} - X_{e-c}Y_{d-b}, \quad \Delta_2 = X_{d-a}X_{e-c} - X_{e-a}X_{d-c}\,.$$

Δ_{xy} vanishes under the following conditions:

1. $X_{d-a} = 0 = X_{e-a}$, in which case $d = e \Rightarrow \Delta_1 = \Delta_2 = 0$ and hence identity (3.1) remains valid.

2. $X_{d-a} = X_{e-a}$ and $Y_{e-b} = Y_{d-b}$, in which case, again, $d = e \Rightarrow \Delta_1 = \Delta_2 = 0$ and hence identity (3.1) remains valid.

Thus we see that identity (3.1) is valid regardless of the nature of Δ_{xy}, so that the identity holds for all integers.

□

3.1 Identities

Our first set of results comes from choosing an appropriate (X, Y) pair, in each case, from the set $\{F, L, J, j, P, Q\}$ and using it in Lemma 7.

Theorem 1. *The following identities hold for arbitrary integers a, b, c, d, e and m:*

$$\begin{aligned}&(F_{d-a}L_{e-b} - F_{e-a}L_{d-b})F_{m-c}\\ &\quad= (F_{d-c}L_{e-b} - F_{e-c}L_{d-b})F_{m-a}\\ &\qquad+ (F_{d-a}F_{e-c} - F_{e-a}F_{d-c})L_{m-b}\,,\end{aligned} \tag{3.2}$$

$$\begin{aligned}&(L_{d-a}F_{e-b} - L_{e-a}F_{d-b})L_{m-c}\\ &\quad= (L_{d-c}F_{e-b} - L_{e-c}F_{d-b})L_{m-a}\\ &\qquad+ (L_{d-a}L_{e-c} - L_{e-a}L_{d-c})F_{m-b}\,,\end{aligned} \tag{3.3}$$

$$\begin{aligned}&(J_{d-a}j_{e-b} - J_{e-a}j_{d-b})J_{m-c}\\ &\quad= (J_{d-c}j_{e-b} - J_{e-c}j_{d-b})J_{m-a}\\ &\qquad+ (J_{d-a}J_{e-c} - J_{e-a}J_{d-c})j_{m-b}\,,\end{aligned} \tag{3.4}$$

$$\begin{aligned}&(j_{d-a}J_{e-b} - j_{e-a}J_{d-b})j_{m-c}\\ &\quad= (j_{d-c}J_{e-b} - j_{e-c}J_{d-b})j_{m-a}\\ &\qquad+ (j_{d-a}j_{e-c} - j_{e-a}j_{d-c})J_{m-b}\,,\end{aligned} \tag{3.5}$$

$$\begin{aligned}&(P_{d-a}Q_{e-b} - P_{e-a}Q_{d-b})P_{m-c}\\ &\quad= (P_{d-c}Q_{e-b} - P_{e-c}Q_{d-b})P_{m-a}\\ &\qquad+ (P_{d-a}P_{e-c} - P_{e-a}P_{d-c})Q_{m-b}\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&(Q_{d-a}P_{e-b} - Q_{e-a}P_{d-b})Q_{m-c}\\ &\quad= (Q_{d-c}P_{e-b} - Q_{e-c}P_{d-b})Q_{m-a}\\ &\qquad+ (Q_{d-a}Q_{e-c} - Q_{e-a}Q_{d-c})P_{m-b}\,.\end{aligned} \tag{3.7}$$

To demonstrate how known identities may be recovered (and further new ones discovered), set $m = c$ in identities (3.2), (3.4) and (3.6) of Theorem 1 to obtain the following result.

Corollary 2. *The following identities hold for integers a, b, c, d and e:*

$$(F_{d-c}L_{e-b} - F_{e-c}L_{d-b})F_{c-a} = (F_{e-a}F_{d-c} - F_{d-a}F_{e-c})L_{c-b}\,, \tag{3.8}$$

$$(J_{d-c}j_{e-b} - J_{e-c}j_{d-b})J_{c-a} = (J_{e-a}J_{d-c} - J_{d-a}J_{e-c})j_{c-b} \tag{3.9}$$

and

$$(P_{d-c}Q_{e-b} - P_{e-c}Q_{d-b})P_{c-a} = (P_{e-a}P_{d-c} - P_{d-a}P_{e-c})Q_{c-b}\,. \tag{3.10}$$

Upon setting $e = a$ in the identities of Corollary 2 and using $F_{a-c} = (-1)^{a-c-1}F_{c-a}$, $J_{a-c} = (-1)^{a-c-1}2^{a-c}J_{c-a}$ and $P_{a-c} = (-1)^{a-c-1}P_{c-a}$ we obtain

$$F_{d-c}L_{a-b} - F_{a-c}L_{d-b} = (-1)^{a-c}F_{d-a}L_{c-b}\,, \tag{3.11}$$

$$J_{d-c}j_{a-b} - J_{a-c}j_{d-b} = (-1)^{a-c}2^{a-c}J_{d-a}j_{c-b} \tag{3.12}$$

and

$$P_{d-c}Q_{a-b} - P_{a-c}Q_{d-b} = (-1)^{a-c}P_{d-a}Q_{c-b}\,. \tag{3.13}$$

In order to write the above identities using three parameters, we set $a = d - h$, $b = d - n - h - k$ and $c = d - n - h$, obtaining

$$F_{n+h}L_{n+k} - F_nL_{n+h+k} = (-1)^nF_hL_k\,, \tag{3.14}$$

$$J_{n+h}j_{n+k} - J_nj_{n+h+k} = (-1)^n2^nJ_hj_k \tag{3.15}$$

and

$$P_{n+h}Q_{n+k} - P_nQ_{n+h+k} = (-1)^nP_hQ_k\,. \tag{3.16}$$

Setting $c = b$ in the identities (3.11), (3.12) and (3.13), we have

$$F_{d-b}L_{a-b} - F_{a-b}L_{d-b} = (-1)^{a-b}2F_{d-a}\,, \tag{3.17}$$

$$J_{d-b}j_{a-b} - J_{a-b}j_{d-b} = (-1)^{a-b}2^{a-b+1}J_{d-a} \tag{3.18}$$

and

$$P_{d-b}Q_{a-b} - P_{a-b}Q_{d-b} = (-1)^{a-b}2P_{d-a}\,. \tag{3.19}$$

Two parameter forms are obtained by setting $d - b = u$ and $a - b = v$, giving

$$F_uL_v - F_vL_u = (-1)^v2F_{u-v}\,, \tag{3.20}$$

$$J_uj_v - J_vj_u = (-1)^v2^{v+1}J_{u-v} \tag{3.21}$$

and

$$P_uQ_v - P_vQ_u = (-1)^v2P_{u-v}\,. \tag{3.22}$$

Setting $b = 0$, $c = -a$ in identities (3.11), (3.12) and (3.13) gives

$$F_{d+a} - (-1)^aF_{d-a} = F_aL_d\,, \tag{3.23}$$

$$J_{d+a} - (-1)^a2^aJ_{d-a} = J_aj_d \tag{3.24}$$

and

$$P_{d+a} - (-1)^a P_{d-a} = P_a Q_d\,. \tag{3.25}$$

Choosing $b = c = 0$, $e = a + d$ in the identities in Corollary 2 and making use of the identities (3.17), (3.18) and (3.19), we obtain Catalan's identities:

$$F_d{}^2 - F_{d-a} F_{d+a} = (-1)^{d-a} F_a{}^2\,, \tag{3.26}$$

$$J_d{}^2 - J_{d-a} J_{d+a} = (-1)^{d-a} 2^{d-a} J_a{}^2 \tag{3.27}$$

and

$$P_d{}^2 - P_{d-a} P_{d+a} = (-1)^{d-a} P_a{}^2\,. \tag{3.28}$$

Note that identity (2.23) of Horadam [4] is a special case of identity (3.27) while identity (30) of [3] is a special case of (3.28).

Upon setting $d = 0$, $c = -a$ in Corollary 2 and making use of identities (3.53), (3.54) and (3.53), we obtain:

$$F_e L_{a+b} + (-1)^b F_a L_{e-b} = F_{e+a} L_b\,, \tag{3.29}$$

$$J_e j_{a+b} + (-1)^b 2^b J_a j_{e-b} = J_{e+a} j_b \tag{3.30}$$

and

$$P_e Q_{a+b} + (-1)^b P_a Q_{e-b} = P_{e+a} Q_b\,. \tag{3.31}$$

Puttig $e = a$ in (3.29) — (3.31) produces

$$L_{a+b} + (-1)^b L_{a-b} = L_a L_b \tag{3.32}$$

$$j_{a+b} + (-1)^b 2^b j_{a-b} = j_a j_b \tag{3.33}$$

$$Q_{a+b} + (-1)^b Q_{a-b} = Q_a L_b\,, \tag{3.34}$$

while using $b = 0$ in the identities gives

$$F_e L_a + F_a L_e = 2F_{e+a}\,, \tag{3.35}$$

$$J_e j_a + J_a j_e = 2J_{e+a} \tag{3.36}$$

and

$$P_e Q_a + P_a Q_e = 2P_{e+a}\,. \tag{3.37}$$

Finally, setting $e = b$ in the same identities (3.29) — (3.31) gives

$$F_{a+b} L_b - F_b L_{a+b} = (-1)^b 2F_a\,, \tag{3.38}$$

$$J_{a+b} j_b - J_b j_{a+b} = (-1)^b 2^{b+1} J_a \tag{3.39}$$

and

$$P_{a+b} Q_b - P_b Q_{a+b} = (-1)^b 2P_a\,. \tag{3.40}$$

The choice $e = 2u + b$, $a = b$, $d = b$ and $c = b + u$ in Corollary 2 yields the identities:

$$L_{2u} + (-1)^u 2 = L_u{}^2\,, \tag{3.41}$$

$$j_{2u} + (-1)^u 2^{u+1} = j_u{}^2 \tag{3.42}$$

and

$$Q_{2u} + (-1)^u 2 = Q_u{}^2\,. \tag{3.43}$$

Note that identities (3.41), (3.42) and (3.43) can also be obtained directly from identities (3.11), (3.12) and (3.13) by setting $b = a$, $c = a + u$ and $d = a + 2u$.

Lemma 2 invites the following results.

Theorem 3. *The following identities hold for integers a, b, c, d, e and m:*

$$\begin{aligned}&(F_{d-a}F_{e-b}-F_{e-a}F_{d-b})F_{m-c}\\&\quad=(F_{d-c}F_{e-b}-F_{e-c}F_{d-b})F_{m-a}\\&\qquad+(F_{d-a}F_{e-c}-F_{e-a}F_{d-c})F_{m-b}\,,\end{aligned}\tag{3.44}$$

$$\begin{aligned}&(L_{d-a}L_{e-b}-L_{e-a}L_{d-b})L_{m-c}\\&\quad=(L_{d-c}L_{e-b}-L_{e-c}L_{d-b})L_{m-a}\\&\qquad+(L_{d-a}L_{e-c}-L_{e-a}L_{d-c})L_{m-b}\,,\end{aligned}\tag{3.45}$$

$$\begin{aligned}&(J_{d-a}J_{e-b}-J_{e-a}J_{d-b})J_{m-c}\\&\quad=(J_{d-c}J_{e-b}-J_{e-c}J_{d-b})J_{m-a}\\&\qquad+(J_{d-a}J_{e-c}-J_{e-a}J_{d-c})J_{m-b}\,,\end{aligned}\tag{3.46}$$

$$\begin{aligned}&(j_{d-a}j_{e-b}-j_{e-a}j_{d-b})j_{m-c}\\&\quad=(j_{d-c}j_{e-b}-j_{e-c}j_{d-b})j_{m-a}\\&\qquad+(j_{d-a}j_{e-c}-j_{e-a}j_{d-c})j_{m-b}\,,\end{aligned}\tag{3.47}$$

$$\begin{aligned}&(P_{d-a}P_{e-b}-P_{e-a}P_{d-b})P_{m-c}\\&\quad=(P_{d-c}P_{e-b}-P_{e-c}P_{d-b})P_{m-a}\\&\qquad+(P_{d-a}P_{e-c}-P_{e-a}P_{d-c})P_{m-b}\end{aligned}\tag{3.48}$$

and

$$\begin{aligned}&(Q_{d-a}Q_{e-b}-Q_{e-a}Q_{d-b})Q_{m-c}\\&\quad=(Q_{d-c}Q_{e-b}-Q_{e-c}Q_{d-b})Q_{m-a}\\&\qquad+(Q_{d-a}Q_{e-c}-Q_{e-a}Q_{d-c})Q_{m-b}\,.\end{aligned}\tag{3.49}$$

Setting $m=b$ in identities (3.44), (3.46) and (3.48) gives the next set of results.

Corollary 4. *The following identities hold for integers a, b, c, d and e:*

$$(F_{d-a}F_{e-b}-F_{e-a}F_{d-b})F_{b-c}=(F_{d-c}F_{e-b}-F_{e-c}F_{d-b})F_{b-a}\,,\tag{3.50}$$

$$(J_{d-a}J_{e-b}-J_{e-a}J_{d-b})J_{b-c}=(J_{d-c}J_{e-b}-J_{e-c}J_{d-b})J_{b-a}\tag{3.51}$$

and

$$(P_{d-a}P_{e-b}-P_{e-a}P_{d-b})P_{b-c}=(P_{d-c}P_{e-b}-P_{e-c}P_{d-b})P_{b-a}\,.\tag{3.52}$$

Using $b=0$, $c=-a$ and $e=a$ in the identities in Corollary 4, we obtain:

$$F_{d+a}+(-1)^aF_{d-a}=F_dL_a\,,\tag{3.53}$$

$$J_{d+a}+(-1)^a2^aJ_{d-a}=J_dj_a\tag{3.54}$$

and

$$P_{d+a}+(-1)^aP_{d-a}=P_dQ_a\,.\tag{3.55}$$

Putting $d=a$ in each case, the identities in Corollary 4 reduce to

$$F_{a-c}F_{e-b}-F_{e-c}F_{a-b}=(-1)^{a-b}F_{e-a}F_{b-c}\,,\tag{3.56}$$

$$J_{a-c}J_{e-b} - J_{e-c}J_{a-b} = (-1)^{a-b}2^{a-b}J_{e-a}J_{b-c} \tag{3.57}$$

and

$$P_{a-c}P_{e-b} - P_{e-c}P_{a-b} = (-1)^{a-b}P_{e-a}P_{b-c}\,. \tag{3.58}$$

Using $a = e + h$, $b = e - n - k$, $c = e - n$, identities (3.56) — (3.58) can also be written

$$F_{n+h}F_{n+k} - F_nF_{n+h+k} = (-1)^nF_hF_k\,, \tag{3.59}$$

$$J_{n+h}J_{n+k} - J_nJ_{n+h+k} = (-1)^n2^nJ_hJ_k \tag{3.60}$$

and

$$P_{n+h}P_{n+k} - P_nP_{n+h+k} = (-1)^nP_hP_k\,. \tag{3.61}$$

Theorem 5. *The following identities hold for all integers a, b and c:*

$$\begin{aligned}&(-1)^{a-b}L_{a-b}^2 + (-1)^{b-c}L_{b-c}^2 + (-1)^{a-c}L_{a-c}^2\\&\quad= (-1)^{a-c}L_{a-b}L_{b-c}L_{a-c} + 4\,,\end{aligned} \tag{3.62}$$

$$\begin{aligned}&(-1)^{a-b}2^{b-a}j_{a-b}^2 + (-1)^{b-c}2^{c-b}j_{b-c}^2 + (-1)^{a-c}2^{c-a}j_{a-c}^2\\&\quad= (-1)^{a-c}2^{c-a}j_{a-b}j_{b-c}j_{a-c} + 4\end{aligned} \tag{3.63}$$

and

$$\begin{aligned}&(-1)^{a-b}Q_{a-b}^2 + (-1)^{b-c}Q_{b-c}^2 + (-1)^{a-c}Q_{a-c}^2\\&\quad= (-1)^{a-c}Q_{a-b}Q_{b-c}Q_{a-c} + 4\,.\end{aligned} \tag{3.64}$$

Proof. Set $m = c$ in Lemma 3 and use $X = L$, $X = j$ and $X = Q$, in turn. □

Note that, for integers u, v, w such that $u + v = w$, the identities in Theorem 5 can also be written

$$(-1)^uL_u^2 + (-1)^vL_v^2 + (-1)^wL_w^2 = (-1)^wL_uL_vL_w + 4\,, \tag{3.65}$$

$$(-1)^u2^vj_u^2 + (-1)^v2^uj_v^2 + (-1)^wj_w^2 = (-1)^wj_uj_vj_w + 2^{w+2} \tag{3.66}$$

and

$$(-1)^uQ_u^2 + (-1)^vQ_v^2 + (-1)^wQ_w^2 = (-1)^wQ_uQ_vQ_w + 4\,. \tag{3.67}$$

3.2 Weighted sums

Choosing an appropriate (X, Y) pair, in each case, from the set $\{F, L, J, j, P, Q\}$ and using it in Lemma 4 we have the next set of results.

Theorem 6. *The following identities hold for any integer k and arbitrary integers a, b, c, d, e, m for which the denominator does not vanish:*

$$\begin{aligned}&\sum_{r=0}^k\left(\frac{F_{d-a}L_{e-b} - F_{e-a}L_{d-b}}{F_{d-c}L_{e-b} - F_{e-c}L_{d-b}}\right)^rL_{m-k(a-c)-b+c+(a-c)r}\\&\quad= \left(\frac{F_{d-a}L_{e-b} - F_{e-a}L_{d-b}}{F_{d-a}F_{e-c} - F_{e-a}F_{d-c}}\right)\left(\frac{F_{d-a}L_{e-b} - F_{e-a}L_{d-b}}{F_{d-c}L_{e-b} - F_{e-c}L_{d-b}}\right)^kF_m\\&\qquad - \left(\frac{F_{d-c}L_{e-b} - F_{e-c}L_{d-b}}{F_{d-a}F_{e-c} - F_{e-a}F_{d-c}}\right)F_{m-(k+1)(a-c)}\,,\end{aligned} \tag{3.68}$$

$$\sum_{r=0}^{k}\left(\frac{L_{d-a}F_{e-b}-L_{e-a}F_{d-b}}{L_{d-c}F_{e-b}-L_{e-c}F_{d-b}}\right)^{r}F_{m-k(a-c)-b+c+(a-c)r}$$
$$=\left(\frac{L_{d-a}F_{e-b}-L_{e-a}F_{d-b}}{L_{d-a}L_{e-c}-L_{e-a}L_{d-c}}\right)\left(\frac{L_{d-a}F_{e-b}-L_{e-a}F_{d-b}}{L_{d-c}F_{e-b}-L_{e-c}F_{d-b}}\right)^{k}L_{m} \tag{3.69}$$
$$-\left(\frac{L_{d-c}F_{e-b}-L_{e-c}F_{d-b}}{L_{d-a}L_{e-c}-L_{e-a}L_{d-c}}\right)L_{m-(k+1)(a-c)}\,,$$

$$\sum_{r=0}^{k}\left(\frac{J_{d-a}j_{e-b}-J_{e-a}j_{d-b}}{J_{d-c}j_{e-b}-J_{e-c}j_{d-b}}\right)^{r}j_{m-k(a-c)-b+c+(a-c)r}$$
$$=\left(\frac{J_{d-a}j_{e-b}-J_{e-a}j_{d-b}}{J_{d-a}J_{e-c}-J_{e-a}J_{d-c}}\right)\left(\frac{J_{d-a}j_{e-b}-J_{e-a}j_{d-b}}{J_{d-c}j_{e-b}-J_{e-c}j_{d-b}}\right)^{k}J_{m} \tag{3.70}$$
$$-\left(\frac{J_{d-c}j_{e-b}-J_{e-c}j_{d-b}}{J_{d-a}J_{e-c}-J_{e-a}J_{d-c}}\right)J_{m-(k+1)(a-c)}\,,$$

$$\sum_{r=0}^{k}\left(\frac{j_{d-a}J_{e-b}-j_{e-a}J_{d-b}}{j_{d-c}J_{e-b}-j_{e-c}J_{d-b}}\right)^{r}J_{m-k(a-c)-b+c+(a-c)r}$$
$$=\left(\frac{j_{d-a}J_{e-b}-j_{e-a}J_{d-b}}{j_{d-a}j_{e-c}-j_{e-a}j_{d-c}}\right)\left(\frac{j_{d-a}J_{e-b}-j_{e-a}J_{d-b}}{j_{d-c}J_{e-b}-j_{e-c}J_{d-b}}\right)^{k}j_{m} \tag{3.71}$$
$$-\left(\frac{j_{d-c}J_{e-b}-j_{e-c}J_{d-b}}{j_{d-a}j_{e-c}-j_{e-a}j_{d-c}}\right)j_{m-(k+1)(a-c)}\,,$$

$$\sum_{r=0}^{k}\left(\frac{P_{d-a}Q_{e-b}-P_{e-a}Q_{d-b}}{P_{d-c}Q_{e-b}-P_{e-c}Q_{d-b}}\right)^{r}Q_{m-k(a-c)-b+c+(a-c)r}$$
$$=\left(\frac{P_{d-a}Q_{e-b}-P_{e-a}Q_{d-b}}{P_{d-a}P_{e-c}-P_{e-a}P_{d-c}}\right)\left(\frac{P_{d-a}Q_{e-b}-P_{e-a}Q_{d-b}}{P_{d-c}Q_{e-b}-P_{e-c}Q_{d-b}}\right)^{k}P_{m} \tag{3.72}$$
$$-\left(\frac{P_{d-c}Q_{e-b}-P_{e-c}Q_{d-b}}{P_{d-a}P_{e-c}-P_{e-a}P_{d-c}}\right)P_{m-(k+1)(a-c)}$$

and

$$\sum_{r=0}^{k}\left(\frac{Q_{d-a}P_{e-b}-Q_{e-a}P_{d-b}}{Q_{d-c}P_{e-b}-Q_{e-c}P_{d-b}}\right)^{r}P_{m-k(a-c)-b+c+(a-c)r}$$
$$=\left(\frac{Q_{d-a}P_{e-b}-Q_{e-a}P_{d-b}}{Q_{d-a}Q_{e-c}-Q_{e-a}Q_{d-c}}\right)\left(\frac{Q_{d-a}P_{e-b}-Q_{e-a}P_{d-b}}{Q_{d-c}P_{e-b}-Q_{e-c}P_{d-b}}\right)^{k}Q_{m} \tag{3.73}$$
$$-\left(\frac{Q_{d-c}P_{e-b}-Q_{e-c}P_{d-b}}{Q_{d-a}Q_{e-c}-Q_{e-a}Q_{d-c}}\right)Q_{m-(k+1)(a-c)}\,.$$

Using $X=F$, $X=L$, $X=J$, $X=j$, $X=P$, $X=Q$, in turn, in Lemma 5 gives the next results.

Theorem 7. *The following identities hold for any integer k and arbitrary integers a, b, c,* *d, e, m for which the denominator does not vanish:*

$$\sum_{r=0}^{k}\left(\frac{F_{d-a}F_{e-b}-F_{e-a}F_{d-b}}{F_{d-c}F_{e-b}-F_{e-c}F_{d-b}}\right)^{r}F_{m-k(a-c)-b+c+(a-c)r}$$
$$=\left(\frac{F_{d-a}F_{e-b}-F_{e-a}F_{d-b}}{F_{d-a}F_{e-c}-F_{e-a}F_{d-c}}\right)\left(\frac{F_{d-a}F_{e-b}-F_{e-a}F_{d-b}}{F_{d-c}F_{e-b}-F_{e-c}F_{d-b}}\right)^{k}F_{m} \tag{3.74}$$
$$-\left(\frac{F_{d-c}F_{e-b}-F_{e-c}F_{d-b}}{F_{d-a}F_{e-c}-F_{e-a}F_{d-c}}\right)F_{m-(k+1)(a-c)}\,,$$

$$\sum_{r=0}^{k}\left(\frac{F_{d-a}F_{e-b}-F_{e-a}F_{d-b}}{F_{d-a}F_{e-c}-F_{e-a}F_{d-c}}\right)^{r}F_{m-k(b-c)-a+c+(b-c)r}$$
$$=\left(\frac{F_{d-a}F_{e-b}-F_{e-a}F_{d-b}}{F_{d-c}F_{e-b}-F_{e-c}F_{d-b}}\right)\left(\frac{F_{d-a}F_{e-b}-F_{e-a}F_{d-b}}{F_{d-a}F_{e-c}-F_{e-a}F_{d-c}}\right)^{k}F_{m} \tag{3.75}$$
$$-\left(\frac{F_{d-a}F_{e-c}-F_{e-a}F_{d-c}}{F_{d-c}F_{e-b}-F_{e-c}F_{d-b}}\right)F_{m-(k+1)(b-c)}\,,$$

$$\sum_{r=0}^{k}\left(\frac{F_{e-a}F_{d-c}-F_{d-a}F_{e-c}}{F_{d-c}F_{e-b}-F_{e-c}F_{d-b}}\right)^{r}F_{m-k(a-b)+b-c+(a-b)r}$$
$$=\left(\frac{F_{d-a}F_{e-c}-F_{e-a}F_{d-c}}{F_{d-a}F_{e-b}-F_{e-a}F_{d-b}}\right)\left(\frac{F_{e-a}F_{d-c}-F_{d-a}F_{e-c}}{F_{d-c}F_{e-b}-F_{e-c}F_{d-b}}\right)^{k}F_{m} \tag{3.76}$$
$$+\left(\frac{F_{d-c}F_{e-b}-F_{e-c}F_{d-b}}{F_{d-a}F_{e-b}-F_{e-a}F_{d-b}}\right)F_{m-(k+1)(a-b)}\,,$$

$$\sum_{r=0}^{k}\left(\frac{L_{d-a}L_{e-b}-L_{e-a}L_{d-b}}{L_{d-c}L_{e-b}-L_{e-c}L_{d-b}}\right)^{r}L_{m-k(a-c)-b+c+(a-c)r}$$
$$=\left(\frac{L_{d-a}L_{e-b}-L_{e-a}L_{d-b}}{L_{d-a}L_{e-c}-L_{e-a}L_{d-c}}\right)\left(\frac{L_{d-a}L_{e-b}-L_{e-a}L_{d-b}}{L_{d-c}L_{e-b}-L_{e-c}L_{d-b}}\right)^{k}L_{m} \tag{3.77}$$
$$-\left(\frac{L_{d-c}L_{e-b}-L_{e-c}L_{d-b}}{L_{d-a}L_{e-c}-L_{e-a}L_{d-c}}\right)L_{m-(k+1)(a-c)}\,,$$

$$\sum_{r=0}^{k}\left(\frac{L_{d-a}L_{e-b}-L_{e-a}L_{d-b}}{L_{d-a}L_{e-c}-L_{e-a}L_{d-c}}\right)^{r}L_{m-k(b-c)-a+c+(b-c)r}$$
$$=\left(\frac{L_{d-a}L_{e-b}-L_{e-a}L_{d-b}}{L_{d-c}L_{e-b}-L_{e-c}L_{d-b}}\right)\left(\frac{L_{d-a}L_{e-b}-L_{e-a}L_{d-b}}{L_{d-a}L_{e-c}-L_{e-a}L_{d-c}}\right)^{k}L_{m} \tag{3.78}$$
$$-\left(\frac{L_{d-a}L_{e-c}-L_{e-a}L_{d-c}}{L_{d-c}L_{e-b}-L_{e-c}L_{d-b}}\right)L_{m-(k+1)(b-c)}\,,$$

$$\sum_{r=0}^{k}\left(\frac{L_{e-a}L_{d-c}-L_{d-a}L_{e-c}}{L_{d-c}L_{e-b}-L_{e-c}L_{d-b}}\right)^{r}L_{m-k(a-b)+b-c+(a-b)r}$$
$$=\left(\frac{L_{d-a}L_{e-c}-L_{e-a}L_{d-c}}{L_{d-a}L_{e-b}-L_{e-a}L_{d-b}}\right)\left(\frac{L_{e-a}L_{d-c}-L_{d-a}L_{e-c}}{L_{d-c}L_{e-b}-L_{e-c}L_{d-b}}\right)^{k}L_{m} \tag{3.79}$$
$$+\left(\frac{L_{d-c}L_{e-b}-L_{e-c}L_{d-b}}{L_{d-a}L_{e-b}-L_{e-a}L_{d-b}}\right)L_{m-(k+1)(a-b)}\,,$$

$$\sum_{r=0}^{k}\left(\frac{J_{d-a}J_{e-b}-J_{e-a}J_{d-b}}{J_{d-c}J_{e-b}-J_{e-c}J_{d-b}}\right)^{r}J_{m-k(a-c)-b+c+(a-c)r}$$
$$=\left(\frac{J_{d-a}J_{e-b}-J_{e-a}J_{d-b}}{J_{d-a}J_{e-c}-J_{e-a}J_{d-c}}\right)\left(\frac{J_{d-a}J_{e-b}-J_{e-a}J_{d-b}}{J_{d-c}J_{e-b}-J_{e-c}J_{d-b}}\right)^{k}J_{m}$$
$$-\left(\frac{J_{d-c}J_{e-b}-J_{e-c}J_{d-b}}{J_{d-a}J_{e-c}-J_{e-a}J_{d-c}}\right)J_{m-(k+1)(a-c)}\,, \tag{3.80}$$

$$\sum_{r=0}^{k}\left(\frac{J_{d-a}J_{e-b}-J_{e-a}J_{d-b}}{J_{d-a}J_{e-c}-J_{e-a}J_{d-c}}\right)^{r}J_{m-k(b-c)-a+c+(b-c)r}$$
$$=\left(\frac{J_{d-a}J_{e-b}-J_{e-a}J_{d-b}}{J_{d-c}J_{e-b}-J_{e-c}J_{d-b}}\right)\left(\frac{J_{d-a}J_{e-b}-J_{e-a}J_{d-b}}{J_{d-a}J_{e-c}-J_{e-a}J_{d-c}}\right)^{k}J_{m}$$
$$-\left(\frac{J_{d-a}J_{e-c}-J_{e-a}J_{d-c}}{J_{d-c}J_{e-b}-J_{e-c}J_{d-b}}\right)J_{m-(k+1)(b-c)}\,, \tag{3.81}$$

$$\sum_{r=0}^{k}\left(\frac{J_{e-a}J_{d-c}-J_{d-a}J_{e-c}}{J_{d-c}J_{e-b}-J_{e-c}J_{d-b}}\right)^{r}J_{m-k(a-b)+b-c+(a-b)r}$$
$$=\left(\frac{J_{d-a}J_{e-c}-J_{e-a}J_{d-c}}{J_{d-a}J_{e-b}-J_{e-a}J_{d-b}}\right)\left(\frac{J_{e-a}J_{d-c}-J_{d-a}J_{e-c}}{J_{d-c}J_{e-b}-J_{e-c}J_{d-b}}\right)^{k}J_{m}$$
$$+\left(\frac{J_{d-c}J_{e-b}-J_{e-c}J_{d-b}}{J_{d-a}J_{e-b}-J_{e-a}J_{d-b}}\right)J_{m-(k+1)(a-b)}\,, \tag{3.82}$$

$$\sum_{r=0}^{k}\left(\frac{j_{d-a}j_{e-b}-j_{e-a}j_{d-b}}{j_{d-c}j_{e-b}-j_{e-c}j_{d-b}}\right)^{r}j_{m-k(a-c)-b+c+(a-c)r}$$
$$=\left(\frac{j_{d-a}j_{e-b}-j_{e-a}j_{d-b}}{j_{d-a}j_{e-c}-j_{e-a}j_{d-c}}\right)\left(\frac{j_{d-a}j_{e-b}-j_{e-a}j_{d-b}}{j_{d-c}j_{e-b}-j_{e-c}j_{d-b}}\right)^{k}j_{m}$$
$$-\left(\frac{j_{d-c}j_{e-b}-j_{e-c}j_{d-b}}{j_{d-a}j_{e-c}-j_{e-a}j_{d-c}}\right)j_{m-(k+1)(a-c)}\,, \tag{3.83}$$

$$\sum_{r=0}^{k}\left(\frac{j_{d-a}j_{e-b}-j_{e-a}j_{d-b}}{j_{d-a}j_{e-c}-j_{e-a}j_{d-c}}\right)^{r}j_{m-k(b-c)-a+c+(b-c)r}$$
$$=\left(\frac{j_{d-a}j_{e-b}-j_{e-a}j_{d-b}}{j_{d-c}j_{e-b}-j_{e-c}j_{d-b}}\right)\left(\frac{j_{d-a}j_{e-b}-j_{e-a}j_{d-b}}{j_{d-a}j_{e-c}-j_{e-a}j_{d-c}}\right)^{k}j_{m}$$
$$-\left(\frac{j_{d-a}j_{e-c}-j_{e-a}j_{d-c}}{j_{d-c}j_{e-b}-j_{e-c}j_{d-b}}\right)j_{m-(k+1)(b-c)}\,, \tag{3.84}$$

$$\sum_{r=0}^{k}\left(\frac{j_{e-a}j_{d-c}-j_{d-a}j_{e-c}}{j_{d-c}j_{e-b}-j_{e-c}j_{d-b}}\right)^{r}j_{m-k(a-b)+b-c+(a-b)r}$$
$$=\left(\frac{j_{d-a}j_{e-c}-j_{e-a}j_{d-c}}{j_{d-a}j_{e-b}-j_{e-a}j_{d-b}}\right)\left(\frac{j_{e-a}j_{d-c}-j_{d-a}j_{e-c}}{j_{d-c}j_{e-b}-j_{e-c}j_{d-b}}\right)^{k}j_{m}$$
$$+\left(\frac{j_{d-c}j_{e-b}-j_{e-c}j_{d-b}}{j_{d-a}j_{e-b}-j_{e-a}j_{d-b}}\right)j_{m-(k+1)(a-b)}\,, \tag{3.85}$$

$$\sum_{r=0}^{k}\left(\frac{P_{d-a}P_{e-b}-P_{e-a}P_{d-b}}{P_{d-c}P_{e-b}-P_{e-c}P_{d-b}}\right)^{r}P_{m-k(a-c)-b+c+(a-c)r}$$
$$=\left(\frac{P_{d-a}P_{e-b}-P_{e-a}P_{d-b}}{P_{d-a}P_{e-c}-P_{e-a}P_{d-c}}\right)\left(\frac{P_{d-a}P_{e-b}-P_{e-a}P_{d-b}}{P_{d-c}P_{e-b}-P_{e-c}P_{d-b}}\right)^{k}P_{m} \tag{3.86}$$
$$-\left(\frac{P_{d-c}P_{e-b}-P_{e-c}P_{d-b}}{P_{d-a}P_{e-c}-P_{e-a}P_{d-c}}\right)P_{m-(k+1)(a-c)}\,,$$

$$\sum_{r=0}^{k}\left(\frac{P_{d-a}P_{e-b}-P_{e-a}P_{d-b}}{P_{d-a}P_{e-c}-P_{e-a}P_{d-c}}\right)^{r}P_{m-k(b-c)-a+c+(b-c)r}$$
$$=\left(\frac{P_{d-a}P_{e-b}-P_{e-a}P_{d-b}}{P_{d-c}P_{e-b}-P_{e-c}P_{d-b}}\right)\left(\frac{P_{d-a}P_{e-b}-P_{e-a}P_{d-b}}{P_{d-a}P_{e-c}-P_{e-a}P_{d-c}}\right)^{k}P_{m} \tag{3.87}$$
$$-\left(\frac{P_{d-a}P_{e-c}-P_{e-a}P_{d-c}}{P_{d-c}P_{e-b}-P_{e-c}P_{d-b}}\right)P_{m-(k+1)(b-c)}\,,$$

$$\sum_{r=0}^{k}\left(\frac{P_{e-a}P_{d-c}-P_{d-a}P_{e-c}}{P_{d-c}P_{e-b}-P_{e-c}P_{d-b}}\right)^{r}P_{m-k(a-b)+b-c+(a-b)r}$$
$$=\left(\frac{P_{d-a}P_{e-c}-P_{e-a}P_{d-c}}{P_{d-a}P_{e-b}-P_{e-a}P_{d-b}}\right)\left(\frac{P_{e-a}P_{d-c}-P_{d-a}P_{e-c}}{P_{d-c}P_{e-b}-P_{e-c}P_{d-b}}\right)^{k}P_{m} \tag{3.88}$$
$$+\left(\frac{P_{d-c}P_{e-b}-P_{e-c}P_{d-b}}{P_{d-a}P_{e-b}-P_{e-a}P_{d-b}}\right)P_{m-(k+1)(a-b)}\,,$$

$$\sum_{r=0}^{k}\left(\frac{Q_{d-a}Q_{e-b}-Q_{e-a}Q_{d-b}}{Q_{d-c}Q_{e-b}-Q_{e-c}Q_{d-b}}\right)^{r}Q_{m-k(a-c)-b+c+(a-c)r}$$
$$=\left(\frac{Q_{d-a}Q_{e-b}-Q_{e-a}Q_{d-b}}{Q_{d-a}Q_{e-c}-Q_{e-a}Q_{d-c}}\right)\left(\frac{Q_{d-a}Q_{e-b}-Q_{e-a}Q_{d-b}}{Q_{d-c}Q_{e-b}-Q_{e-c}Q_{d-b}}\right)^{k}Q_{m} \tag{3.89}$$
$$-\left(\frac{Q_{d-c}Q_{e-b}-Q_{e-c}Q_{d-b}}{Q_{d-a}Q_{e-c}-Q_{e-a}Q_{d-c}}\right)Q_{m-(k+1)(a-c)}\,,$$

$$\sum_{r=0}^{k}\left(\frac{Q_{d-a}Q_{e-b}-Q_{e-a}Q_{d-b}}{Q_{d-a}Q_{e-c}-Q_{e-a}Q_{d-c}}\right)^{r}Q_{m-k(b-c)-a+c+(b-c)r}$$
$$=\left(\frac{Q_{d-a}Q_{e-b}-Q_{e-a}Q_{d-b}}{Q_{d-c}Q_{e-b}-Q_{e-c}Q_{d-b}}\right)\left(\frac{Q_{d-a}Q_{e-b}-Q_{e-a}Q_{d-b}}{Q_{d-a}Q_{e-c}-Q_{e-a}Q_{d-c}}\right)^{k}Q_{m} \tag{3.90}$$
$$-\left(\frac{Q_{d-a}Q_{e-c}-Q_{e-a}Q_{d-c}}{Q_{d-c}Q_{e-b}-Q_{e-c}Q_{d-b}}\right)Q_{m-(k+1)(b-c)}$$

and

$$\sum_{r=0}^{k}\left(\frac{Q_{e-a}Q_{d-c}-Q_{d-a}Q_{e-c}}{Q_{d-c}Q_{e-b}-Q_{e-c}Q_{d-b}}\right)^{r}Q_{m-k(a-b)+b-c+(a-b)r}$$
$$=\left(\frac{Q_{d-a}Q_{e-c}-Q_{e-a}Q_{d-c}}{Q_{d-a}Q_{e-b}-Q_{e-a}Q_{d-b}}\right)\left(\frac{Q_{e-a}Q_{d-c}-Q_{d-a}Q_{e-c}}{Q_{d-c}Q_{e-b}-Q_{e-c}Q_{d-b}}\right)^{k}Q_{m} \tag{3.91}$$
$$+\left(\frac{Q_{d-c}Q_{e-b}-Q_{e-c}Q_{d-b}}{Q_{d-a}Q_{e-b}-Q_{e-a}Q_{d-b}}\right)Q_{m-(k+1)(a-b)}\,.$$

3.3 Weighted binomial sums

Using $X=F$, $X=L$, $X=J$, $X=j$, $X=P$, $X=Q$, in turn, in Lemma 6 gives the next results.

Theorem 8. *The following identities hold for nonnegative integer k and arbitrary integers a, b, c, d, e, m for which the denominator does not vanish:*

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{F_{d-c}F_{e-b}-F_{e-c}F_{d-b}}{F_{d-a}F_{e-c}-F_{e-a}F_{d-c}}\right)^{r}F_{m-(b-c)k+(b-a)r}=\left(\frac{F_{d-a}F_{e-b}-F_{e-a}F_{d-b}}{F_{d-a}F_{e-c}-F_{e-a}F_{d-c}}\right)^{k}F_{m}\,,\tag{3.92}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{F_{e-a}F_{d-b}-F_{d-a}F_{e-b}}{F_{d-a}F_{e-c}-F_{e-a}F_{d-c}}\right)^{r}F_{m+(a-b)k+(b-c)r}=\left(\frac{F_{d-c}F_{e-b}-F_{e-c}F_{d-b}}{F_{e-a}F_{d-c}-F_{d-a}F_{e-c}}\right)^{k}F_{m}\,,\tag{3.93}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{F_{e-a}F_{d-b}-F_{d-a}F_{e-b}}{F_{d-c}F_{e-b}-F_{e-c}F_{d-b}}\right)^{r}F_{m+(b-a)k+(a-c)r}=\left(\frac{F_{d-a}F_{e-c}-F_{e-a}F_{d-c}}{F_{e-c}F_{d-b}-F_{d-c}F_{e-b}}\right)^{k}F_{m}\,,\tag{3.94}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{L_{d-c}L_{e-b}-L_{e-c}L_{d-b}}{L_{d-a}L_{e-c}-L_{e-a}L_{d-c}}\right)^{r}L_{m-(b-c)k+(b-a)r}=\left(\frac{L_{d-a}L_{e-b}-L_{e-a}L_{d-b}}{L_{d-a}L_{e-c}-L_{e-a}L_{d-c}}\right)^{k}L_{m}\,,\tag{3.95}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{L_{e-a}L_{d-b}-L_{d-a}L_{e-b}}{L_{d-a}L_{e-c}-L_{e-a}L_{d-c}}\right)^{r}L_{m+(a-b)k+(b-c)r}=\left(\frac{L_{d-c}L_{e-b}-L_{e-c}L_{d-b}}{L_{e-a}L_{d-c}-L_{d-a}L_{e-c}}\right)^{k}L_{m}\,,\tag{3.96}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{L_{e-a}L_{d-b}-L_{d-a}L_{e-b}}{L_{d-c}L_{e-b}-L_{e-c}L_{d-b}}\right)^{r}L_{m+(b-a)k+(a-c)r}=\left(\frac{L_{d-a}L_{e-c}-L_{e-a}L_{d-c}}{L_{e-c}L_{d-b}-L_{d-c}L_{e-b}}\right)^{k}L_{m}\,,\tag{3.97}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{J_{d-c}J_{e-b}-J_{e-c}J_{d-b}}{J_{d-a}J_{e-c}-J_{e-a}J_{d-c}}\right)^{r}J_{m-(b-c)k+(b-a)r}=\left(\frac{J_{d-a}J_{e-b}-J_{e-a}J_{d-b}}{J_{d-a}J_{e-c}-J_{e-a}J_{d-c}}\right)^{k}J_{m}\,,\tag{3.98}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{J_{e-a}J_{d-b}-J_{d-a}J_{e-b}}{J_{d-a}J_{e-c}-J_{e-a}J_{d-c}}\right)^{r}J_{m+(a-b)k+(b-c)r}=\left(\frac{J_{d-c}J_{e-b}-J_{e-c}J_{d-b}}{J_{e-a}J_{d-c}-J_{d-a}J_{e-c}}\right)^{k}J_{m}\,,\tag{3.99}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{J_{e-a}J_{d-b}-J_{d-a}J_{e-b}}{J_{d-c}J_{e-b}-J_{e-c}J_{d-b}}\right)^{r}J_{m+(b-a)k+(a-c)r} = \left(\frac{J_{d-a}J_{e-c}-J_{e-a}J_{d-c}}{J_{e-c}J_{d-b}-J_{d-c}J_{e-b}}\right)^{k}J_{m}\,, \tag{3.100}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{j_{d-c}j_{e-b}-j_{e-c}j_{d-b}}{j_{d-a}j_{e-c}-j_{e-a}j_{d-c}}\right)^{r}j_{m-(b-c)k+(b-a)r} = \left(\frac{j_{d-a}j_{e-b}-j_{e-a}j_{d-b}}{j_{d-a}j_{e-c}-j_{e-a}j_{d-c}}\right)^{k}j_{m}\,, \tag{3.101}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{j_{e-a}j_{d-b}-j_{d-a}j_{e-b}}{j_{d-a}j_{e-c}-j_{e-a}j_{d-c}}\right)^{r}j_{m+(a-b)k+(b-c)r} = \left(\frac{j_{d-c}j_{e-b}-j_{e-c}j_{d-b}}{j_{e-a}j_{d-c}-j_{d-a}j_{e-c}}\right)^{k}j_{m}\,, \tag{3.102}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{j_{e-a}j_{d-b}-j_{d-a}j_{e-b}}{j_{d-c}j_{e-b}-j_{e-c}j_{d-b}}\right)^{r}j_{m+(b-a)k+(a-c)r} = \left(\frac{j_{d-a}j_{e-c}-j_{e-a}j_{d-c}}{j_{e-c}j_{d-b}-j_{d-c}j_{e-b}}\right)^{k}j_{m}\,, \tag{3.103}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{P_{d-c}P_{e-b}-P_{e-c}P_{d-b}}{P_{d-a}P_{e-c}-P_{e-a}P_{d-c}}\right)^{r}P_{m-(b-c)k+(b-a)r} = \left(\frac{P_{d-a}P_{e-b}-P_{e-a}P_{d-b}}{P_{d-a}P_{e-c}-P_{e-a}P_{d-c}}\right)^{k}P_{m}\,, \tag{3.104}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{P_{e-a}P_{d-b}-P_{d-a}P_{e-b}}{P_{d-a}P_{e-c}-P_{e-a}P_{d-c}}\right)^{r}P_{m+(a-b)k+(b-c)r} = \left(\frac{P_{d-c}P_{e-b}-P_{e-c}P_{d-b}}{P_{e-a}P_{d-c}-P_{d-a}P_{e-c}}\right)^{k}P_{m}\,, \tag{3.105}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{P_{e-a}P_{d-b}-P_{d-a}P_{e-b}}{P_{d-c}P_{e-b}-P_{e-c}P_{d-b}}\right)^{r}P_{m+(b-a)k+(a-c)r} = \left(\frac{P_{d-a}P_{e-c}-P_{e-a}P_{d-c}}{P_{e-c}P_{d-b}-P_{d-c}P_{e-b}}\right)^{k}P_{m}\,, \tag{3.106}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{Q_{d-c}Q_{e-b}-Q_{e-c}Q_{d-b}}{Q_{d-a}Q_{e-c}-Q_{e-a}Q_{d-c}}\right)^{r}Q_{m-(b-c)k+(b-a)r} = \left(\frac{Q_{d-a}Q_{e-b}-Q_{e-a}Q_{d-b}}{Q_{d-a}Q_{e-c}-Q_{e-a}Q_{d-c}}\right)^{k}Q_{m}\,, \tag{3.107}$$

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{Q_{e-a}Q_{d-b}-Q_{d-a}Q_{e-b}}{Q_{d-a}Q_{e-c}-Q_{e-a}Q_{d-c}}\right)^{r}Q_{m+(a-b)k+(b-c)r} = \left(\frac{Q_{d-c}Q_{e-b}-Q_{e-c}Q_{d-b}}{Q_{e-a}Q_{d-c}-Q_{d-a}Q_{e-c}}\right)^{k}Q_{m} \tag{3.108}$$

and

$$\sum_{r=0}^{k}\binom{k}{r}\left(\frac{Q_{e-a}Q_{d-b}-Q_{d-a}Q_{e-b}}{Q_{d-c}Q_{e-b}-Q_{e-c}Q_{d-b}}\right)^{r}Q_{m+(b-a)k+(a-c)r} = \left(\frac{Q_{d-a}Q_{e-c}-Q_{e-a}Q_{d-c}}{Q_{e-c}Q_{d-b}-Q_{d-c}Q_{e-b}}\right)^{k}Q_{m}\,. \tag{3.109}$$

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