question 468: A Simple Identity for Pi

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abstract

This note presents a trivial formula for pi:

$$\pi = 3.14159265 \ldots$$

1. Introduction

The number π is defined by

$$\pi = \int_0^1 \frac{\sqrt{1-\sqrt{x}}}{\sqrt[4]{x^3}}\, dx = 3.14159265 \ldots \tag{1}$$

This note presents *a* simple formula for π .

2. Identity

Let $n \in \mathbb{N} = \{1, 2, 3, 4, \ldots\}$, $0 \le \theta \le \pi/2$, then

$$\pi\left(\binom{2n}{n} 2^{-2n-1} - \frac{(\cos\theta)^{2n}}{2}\right) + \theta(\cos\theta)^{2n} = \theta\binom{2n}{n}2^{-2n} + 2^{-2n}\sum_{k=0}^{n-1}\binom{2n}{k}\frac{\sin((2n-2k)\theta)}{n-k} - 2n\sum_{k=0}^{\infty}\binom{2k}{k}\frac{2^{-2k}(\cos\theta)^{2k+2n+1}}{(2k+1)(2k+2n+1)} \tag{2}$$

Examples : $\theta = \pi/4$, $\theta = \pi/3$, $\theta = \pi/6$, $n \in \mathbb{N}$

$$\pi\left(\binom{2n}{n} - 2^n\right) = 4\sum_{k=0}^{n-1}\binom{2n}{k}\frac{1}{n-k}\sin\left(\frac{(n-k)\pi}{2}\right) - 2^{n+2}\sqrt{2}\, n\sum_{k=0}^{\infty}\binom{2k}{k}\frac{2^{-3k}}{(2k+1)(2k+2n+1)} \tag{3}$$

$$\pi\left(\binom{2n}{n} - 1\right) = 6\sum_{k=0}^{n-1}\binom{2n}{k}\frac{1}{n-k}\sin\left(\frac{(2n-2k)\pi}{3}\right) - 6n\sum_{k=0}^{\infty}\binom{2k}{k}\frac{2^{-4k}}{(2k+1)(2k+2n+1)} \tag{4}$$

$$\pi\left(\binom{2n}{n} - 3^n\right) = 3\sum_{k=0}^{n-1}\binom{2n}{k}\frac{1}{n-k}\sin\left(\frac{(n-k)\pi}{3}\right) - 3^{n+1}\sqrt{3}\, n\sum_{k=0}^{\infty}\binom{2k}{k}\frac{(3/16)^k}{(2k+1)(2k+2n+1)} \tag{5}$$

References

A. Beckmann, P.: A History of Pi . 3rd ed. New York , Dorset Press , 1989.

B. Blatner, D.: The Joy of Pi . New York , Walker , 1997.