Refutation of the Solovay theorem

Abstract: Solovay's arithmetical completeness theorem for provability logic is refuted by showing the following are not tautologous: Löb's rule as an inference; Gödel's logic system (GL); Gödel's second incompleteness theorem; inconsistency claims of Peano arithmetic (PA); and inability to apply semantical completeness to results which are not contradictory and which are not tautologous.

From: Shah, A. (2013). Solovay's arithmetical completeness theorem for provability logic. University of Warwick. [Note: this paper is attributed to a student *number* with no email address.]

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated *proof* value, `F` as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). The 16-valued truth table is row-major and horizontal.

LET: p, q, r, s: placeholder; country; refugee; visa;
~ Not; & And; + Or; - Not Or; > Imply, greater than; < Not Imply, less than;
= Equivalent; % possibility, for one or some; # necessity, for all or every;
(p=p) tautology; (p@p) contradiction.

We evaluate Solovay's arithmetical completeness theorem from the narrative source, as edited:

> "Consider one who can only acquire a visa for a country only if proving that one will not remain in that country. Further, if one is not allowed back into a country after leaving it, then one will eventually reside somewhere else. Thus, the truthful one will remain in one's native country." (1.1)

(((~(r<q)>(p=p))>(r>~(s<q)))&((((r<q)>(r>q))>~(r<q))<(r>q)))>((r=(p=p))>(r<q)) ;
`TTTT TTTT TTTT TTTT` (1.2)

Remark: The location of the visa to be obtained while residing within a country or outside a country is irrelevant as to the same table result.

Eq. 1.2 as rendered is tautologous. However, its application to Gödel's logic system (GL), as *not* tautologous, is defective as shown below.

We evaluate the claims in the captioned as keyed to the text.

The assertion *it is necessary that (it is raining or it is not raining)* ([](*A* ∨ ¬*A*)) is true because it is either raining or it is not, and this is always true. (2.1.1)

#(A + ~A) = (A=A) ; `NNNN NNNN NNNN NNNN` (2.1.2)

However, the statement *it is necessarily raining or it is necessarily not raining* ([]*A* ∨ []¬*A*) is false. (2.2.1)

#A + #~A ; `NNNN NNNN NNNN NNNN` (2.2.2)

The intention of Eqs. 2.1.1 and 2.2.1 was to show the dual as #~A as not the negation as ~#A. However that point was lost because as rendered Eq. 2.1.2 is *not* tautologous but a truthity, and Eq. 2.2.2 is *not* contradictory but also a truthity. In other words, both equations are equivalent.

$([](A \lor \neg A)) = ([]A \lor []\neg A)$; (2.3.1)

#(A + ~A)= (#A + #~A); TTTT TTTT TTTT TTTT (2.3.2)

Eq. 2.3.2 is tautologous.

$[]([]A \rightarrow A) \leftrightarrow []A \leftrightarrow []([]A \land A)$ (Prp.3.23.1.1)

#(#A>A)=(#A=#(#A&A)) ; NNNN NNNN NNNN NNNN (Prp.3.23.1.2)

Eq. Prp.3.23.1.2 is *not* tautologous

In addition, the axiom form of the G in GL is defined as $[]([]A \rightarrow A) \rightarrow []A$. (Prp.3.23.2.1)

#(#A>A)>#A; CCTT CCTT CCTT CCTT (Prp.3.23.2.2)

Eq. Prp.3.23.2.2 is *not* tautologous.

For an axiomatic proof system, the rule of regularity is defined and derived: (Lem.3.5.1)

(A>B)=(#A>#B) ; TNTN TNTN TNTN TNTN (Lem.3.5.2)

Eq. Lem.3.5.2 is *not* tautologous.

$[]\bot \leftrightarrow []<>p$ (Lem.3.24.1)

#(p@p)=#%p ; NFNF NFNF NFNF NFNF (Lem.3.24.2)

Eq. Lem.3.24.2 is *not* tautologous.

The definition of GL is given as $[](p \leftrightarrow \neg[]p) \leftrightarrow [](p \leftrightarrow \neg[]\bot)$. (Thm.3.25.1)

#(p=~#p)=#(p=~(#(p@p))) ; TCTC TCTC TCTC TCTC (Thm.3.25.2)

Eq. Thm.3.25.2 is *not* tautologous.

For the arithmetical soundness of GL,"we define the *Löb Rule* to be the rule of inference in a modal logic axiomatic system which allows one to deduce A from $[]A \rightarrow A$" (Def.4.1.1.1)

(#A>A)>A ; FCNT FCNT FCNT FCNT (Def.4.1.1.2)

Eq. Def.4.1.1.2 is *not* tautologous.

"Peano arithmetic (PA) can prove that if arithmetic is consistent, then Peano arithmetic (PA) cannot prove its own consistency; this is Gödel's Second Incompleteness Theorem for PA, defined as ¬[]⊥→ ¬[]([]⊥ → ⊥)" (Cor.4.5.1)

~(#(A@A))>~(#(#(A@A)>(A@A))=(A=A)) ; `CCCC CCCC CCCC CCCC` (Cor.4.5.2)

Eq. Cor.4.5.2 is *not* tautologous. Therefore Gödel's Second Incompleteness Theorem is refuted.

PA can prove that if the inconsistency of arithmetic is not formally provable (in PA), then the consistency of arithmetic is undecidable. That is, not being able to formally prove the inconsistency of arithmetic implies that, firstly, it is not formally provable that arithmetic is consistent and, secondly, it is not formally provable that arithmetic is inconsistent. Hence, the formal unprovability of the inconsistency of arithmetic implies that the consistency of arithmetic is undecidable. ¬[][]⊥ → (¬[]¬[]⊥ ∧¬[][]⊥)" (Cor.4.6.1)

This translated to ~##(A@A)>(~#~#(A@A)&~##(A@A), and is rewritten as (A=(A&A)) > (~##A>(~#~#A&~##A)) ; `FFNN FFNN FFNN FFNN` (Cor.4.6.2)

Eq. Cor.4.6.2 *not* tautologous. The fore to further formalize and strengthen the Löb Rule is in vain.

The Eqs. above do not support *proof* in GL.

Should the Solovay arithmetical completeness theorem be invoked to show semantical completeness for the above, it similarly will not result in tautology. That means the Solovay theorem is refuted by extension.