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A new special function and it's application in probability

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In this note we present a new special function that behaves like the error function and we provide an aproximated accurate closed form for it' s CDF in terms of both Chebyshev polynomials of the first kind and the error function.Also we provide it's series representation using pade approximant , We show a convincing numerical evidence about an accuracy of 10^{-6} for the approximants in the sense of the quadratic mean norm. A similar approach may be applied to other probability distributions.

KEYWORDS

New special function, *probability*, CDF, distribution, Chebychev polynomials, Series representation

1 | INTRODUCTION

Integrals of the error function see ([1]) occur in a great variety of applications usually in problems involving multiple integration where the integrand contains exponentials of the squares of the argument, example of applications can be cited from atomic physics astrophysics and statistical analysis ,It comes in my mind to seek for the integration of such functions $f(x)$ power it's antiderivative $g(x)$ We have got the below example (1.1) where it is the power of two distribution related to Normal distribution [01] as shown below such that $f(x)=e^{-x^2}$ and $g(x)=\mathrm{erf}(x)$

$$I(a)=\int_0^a \left(e^{-x^2}\right)^{\mathrm{erf}(x)} dx \tag{1.1}$$

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[1] In mathematics, the error function (also called the Gauss error function) is a special function (non-elementary) of sigmoid shape that occurs in probability, statistics, and partial differential equations describing diffusion. It is defined as:$\mathrm{Erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$. Of course, it is closely related to the normal cdf $\Phi(x)=P(N<x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}dt$ (where $N\sim N(0,1)$ is a standard normal) by the expression $\mathrm{Erf}=2\Phi(x\sqrt{2})-1$.

With $\mathrm{erf}(x)$ is called error function and it is defined below in (2.1)

$$\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt = \mathrm{erf}(x) \tag{1.2}$$

1.1 | Numerical approximation of $\int_0^a \left(e^{-x^2}\right)^{\mathrm{erf}(x)} dx$ in some ranges values

Now, if we really need a simple expression for $I(a)$ in some range of values, there are ways to get various approximations.

The function is very nice. It goes to its limit at ∞ very very fast. Here's below in Figure 1. the plot of $I(a)$ for $a \in [0, 10]$:

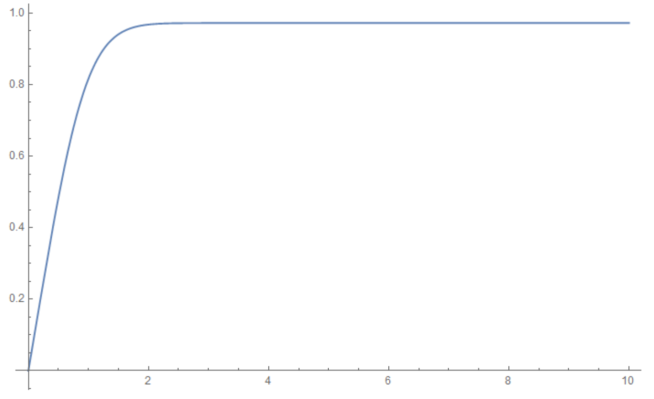

FIGURE 1 The plot of $I(a)$ for $a \in [0, 10]$

So (depending on the accuracy we need) we can easily take $I(a) = I(\infty)$ for $a > a_0$ with a_0 around 3 or 4.

Mathematica gives for the first 100 digits:

$I(\infty) = 0.972106992769178593151077875442391175554272$
1833855699009722910408441888759958220033410678218401258734

Now, what we can do for small a?

The function is so nice, we can just use the Taylor expansion around $a = 0$. The first term is:

$$I(a) \approx a$$

Here's the plot for $a \in [0, 1]$ The proof is simple. The Taylor series look like this:

$$I(a) = I(0) + I'(0)a + \frac{I''(0)}{2!}a^2 + \frac{I'''(0)}{3!}a^3 + \ldots$$

We can see that:

$$I(0) = 0$$

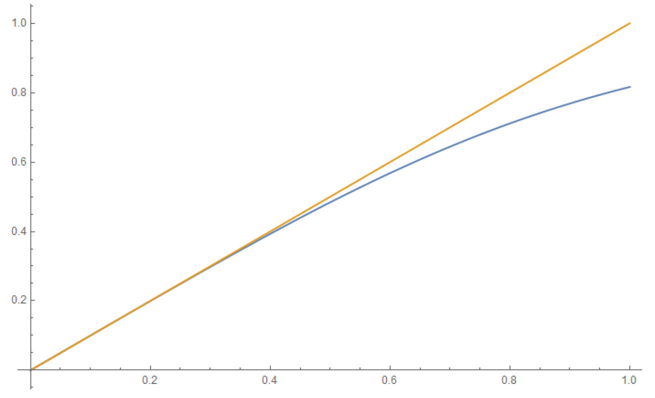

FIGURE 2 the plot of $I(a)$ for $a \in [0, 1]$

$$I'(0) = e^{-a^2 \operatorname{erf}(a)}\Big|_{a=0} = 1$$

Now let's find a better approximation by computing the higher derivatives:

$$I''(a) = \left(e^{-a^2 \operatorname{erf}(a)}\right)' = -\frac{2}{\sqrt{\pi}} a e^{-a^2(\operatorname{erf}(a)+1)} \left(\sqrt{\pi} e^{a^2} \operatorname{erf}(a) + a\right)$$

$$I''(0) = 0$$

—

we use Mathematica as a shortcut, but it's easy to do it by hand, if we remember that :

$$\operatorname{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

$$I'''(0) = 0$$

$$I^{IV}(0) = -\frac{12}{\sqrt{\pi}}$$

So our next approximation is:

$$I(a) \approx a - \frac{1}{2\sqrt{\pi}} a^4$$

The plot with both approximations (orange, green) and the function itself (blue) is given below: we can continue in the

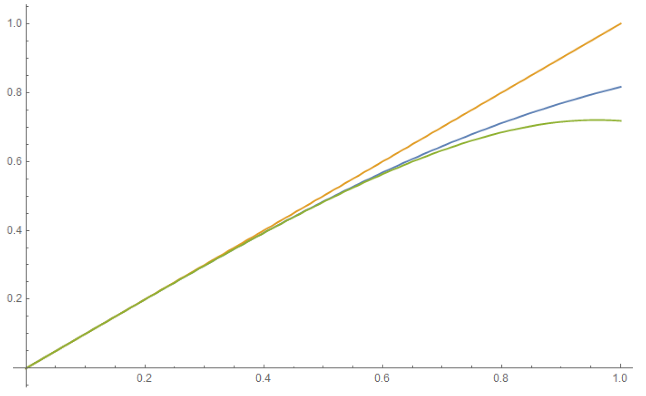

FIGURE 3

same way for higher derivatives. Now I admit that it's possible you need the values of $I(a)$ for all the possible a and with high precision, so the approximations won't do. Then wee need to turn to numerical integration (as Mathematica did for me to plot the function). Another way to approximate the function [02]is using its derivative:

$$\frac{dI}{da} = e^{-a^2 \operatorname{erf}(a)}$$

But this is an ordinary differential equation, which can be solved numerically.

As an illustration, here's a simple explicit Euler scheme for the step size h:

$$\frac{I(a+h) - I(a)}{h} = e^{-a^2 \operatorname{erf}(a)}$$

$$I(a+h) = I(a) + he^{-a^2 \operatorname{erf}(a)}$$

We can use an initial value $I(0) = 0$.

For $h = \frac{1}{10}$ we have the following result (red dots) compared to the exact function (blue line):

For $h = \frac{1}{50}$ see figure 5 :

This way can serve as a good alternative to numerical integration [03](depending on the context and the application of course). let us now to show the relationship between this function and other standard special functions (integral of error function) [04] as Error function and cumulative distribution function for normal distribution in the context of it uses , The function (1.1)

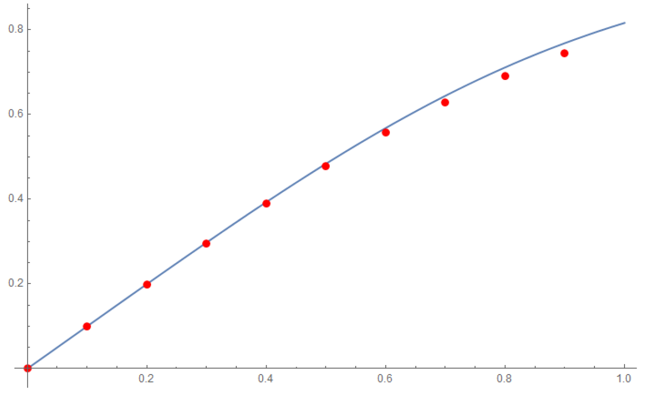

FIGURE 4 fig:plote

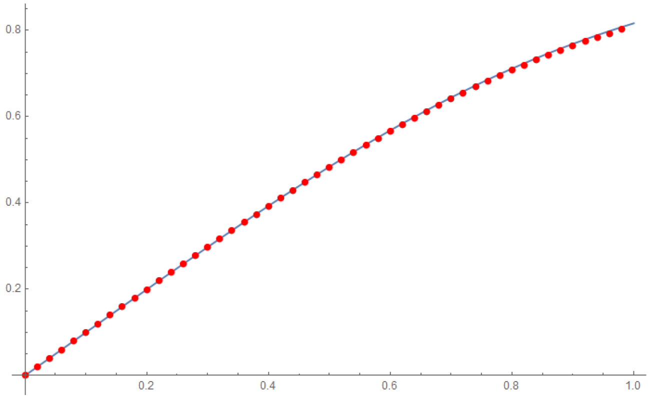

FIGURE 5 fig:plote

could be used to find values of complicated integral which are not available in any references of standard special functions and also not available to get them values in wolfram alpha for example :

$$\int_0^{+\infty} e^{x^2(1-2\Phi(x\sqrt{2}))} dx, \tag{1.3}$$

with $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} \mathrm{d}z.$ is CDF (cumulative distribution function for normal distribution) if someone was asked to find the value of this integral he w'd be confused how he can do it's evaluation because it is very complicated and probably he can't show that is convergent or not and for wolfram alpha as a best means of computation can't recognize at a least that ϕ is a cumulative normal distribution then no result w'd be obtained about the value of this integral. let us compute (2.2) using (1.1) and we will conclude that they have the same value and both are identical function and identical integral.

The well known formula which express the relationship between Error function and Cumulative density function see ([2]) is defined as:

$$\mathrm{Erf}(x) = 2(\Phi(x\sqrt{2}) - \Phi(0)) = 2\left(\Phi(x\sqrt{2}) - \frac{1}{2}\right) = 2\Phi(x\sqrt{2}) - 1. \tag{1.4}$$

and it is easy to check that is always hold for every real numbers by the following short proof

Proof By definition, the Error Function :

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t \tag{1.5}$$

Writing $t^2 = z^2/2$ implies $t = z/\sqrt{2}$ (because t is not negative), whence $\mathrm{d}t = \mathrm{d}z/\sqrt{2}$. The endpoints $t = 0$ and $t = x$ become $z = 0$ and $z = x\sqrt{2}$. To convert the resulting integral into something that looks like a cumulative distribution function (CDF), it must be expressed in terms of integrals that have lower limits of $-\infty$, thus:

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{2\pi}} \int_0^{x\sqrt{2}} e^{-z^2/2} \mathrm{d}z = 2\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x\sqrt{2}} e^{-z^2/2} \mathrm{d}z - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-z^2/2} \mathrm{d}z\right).$$

Those integrals on the right hand size are both values of the CDF of the standard Normal distribution,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} \mathrm{d}z.$$

Specifically,

$$\mathrm{Erf}(x) = 2(\Phi(x\sqrt{2}) - \Phi(0)) = 2\left(\Phi(x\sqrt{2}) - \frac{1}{2}\right) = 2\Phi(x\sqrt{2}) - 1. \tag{1.6}$$

[2]Cumulative distribution function for the normal distribution In probability theory and statistics, the cumulative distribution function (CDF, also cumulative density function) of a real-valued random variable X, or just distribution function of X, evaluated at x, is the probability that X will take a value less than or equal to x.If you have a quantity A that takes some value at random, the cumulative density function $F(x)$ gives the probability that X is less than or equal to x, that is:

$F(x) = P(A \leq x)$

In the case of a continuous distribution, it gives the area under the probability density function from minus infinity to x. Cumulative distribution functions are also used to specify the distribution of multivariate random variables.

Now since the LHS of (1.7) has a known value which it is $0.97210699\cdots$, then the right hand side also equal's :$0.97210699\cdots$ hence we came up with the following identity :

$$\int_0^a (e^{-x^2})^{\mathrm{Erf}(x)} dx = \int_0^a e^{x^2(1-2\Phi(x\sqrt{2}} dx) \tag{1.7}$$

Now we shall call the function defined in (1.1) zeraoulia$(x) = \int_0^X (e^{-t^2})^{\mathrm{erf}(t)} dt$ Since it's not refer to anyone and it has uknown analytic representation as elementary function using standard special functions and the R.H.S of (1.7) present another representation of zeraoulia function using CDF of the normal distribution.

Lemma 1 *zeraoulia*$(x) = \int_0^X (e^{-t^2})^{\mathrm{erf}(t)} dt$ *can't be expressed in terms of elementary function*

Proof It is a theorem of Liouville [06], reproven later with purely algebraic methods, that for rational functions f and g, g nonconstant, the antiderivative

$$\int [f(x)\exp(g(x))]\,\mathrm{d}x$$

can be expressed in terms of elementary functions if and only if there exists some rational function h such that it is a solution to the differential equation:

$$f = h' + hg$$

, now if we apply Liouville theorem we can come up with the following ODE :$1 = h'(x) + h(x)(-x^2\,\mathrm{erf}(x))$ with $g(x) = -x^2\,\mathrm{erf}(x))$ and $f(x) = 1$, It is first ordinary differential equation , The computation we made with wolfram alpha gives the following solution :

$$h(x) = c_1 \exp\left(\frac{e^{-x^2}\left(\sqrt{\pi}\, e^{x^2}\, x^3\, \mathrm{erf}(x) + x^2 + 1\right)}{3\sqrt{\pi}}\right) + \exp\left(\frac{e^{-x^2}\left(\sqrt{\pi}\, e^{x^2}\, x^3\, \mathrm{erf}(x) + x^2 + 1\right)}{3\sqrt{\pi}}\right) \int_1^x \exp\left(-\frac{e^{-\xi^2}\left(1 + \xi^2 + e^{\xi^2}\sqrt{\pi}\,\mathrm{erf}(\xi)\,\xi^3\right)}{3\sqrt{\pi}}\right) d\xi$$

FIGURE 6 Solution of the first order ordinary differentatial equation

with $h(0) = 1$, Really the function h can be written as :

$$h(x) = l(x)[c_1 + \int_1^x l(-\xi)d\xi] \tag{1.8}$$

Now it is clear at all that $l(x)$ is a transcendental function and the defined integral in the RHS of the $h(x)$ expression is also transcendental function because we have **Derivatives of rational functions are rational functions. Therefore, if the**

antiderivative is rational, then the original function was rational ,The function h is rational only at $x = 0$ and since $h(x) \neq 0$ as it is defined above in **FIGURE 6** then we have sum of two transcedental functions is always transcendental function , then $h(x)$ is not a rational function then we are done .

2 | A POSSIBLE APPROACH FORMULA FOR ZERAOULIA($+\infty$):

We may give here a possible approach formula for zeraoulia($+\infty$) which it is defined as :

$$zeraoulia(+\infty) = \int_0^{+\infty} \exp(-x^2 \operatorname{erf}(x))dx = 0.97210699\cdots$$

, The inverse symbolic calculator is unable to give us the representation of $0.97210699\cdots$ using standard special functions , But we have tried to give it's representation using error function representation as hypergeometric function [07] , We have :

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} x 1F1(\frac{1}{2}; \frac{3}{2}; -x^2) \tag{2.1}$$

with $1F1$ is the Kummer confluent hypergeometric function [07] , Now we have from (2.1) :

$$\int_0^{+\infty} \exp(-x^2 \operatorname{erf}(x))dx = \int_0^{+\infty} \exp(-\frac{2}{\sqrt{\pi}} x^3)^{1F1(\frac{1}{2};\frac{3}{2};-x^2)} dx \tag{2.2}$$

The RHS of (2.2) using (2.1) gives: $\frac{(\pi)^{\frac{1}{6}}\Gamma(\frac{4}{3})}{2^{\frac{1}{3}}}^{1F1(0.5;1.5;-x^2)}$, Hence we may choose $x = e\sqrt{\pi}$ and we can get finally :

$$zeraoulia(+\infty) \sim \frac{(\pi)^{\frac{1}{6}}\Gamma(\frac{4}{3})}{2^{\frac{1}{3}}}^{1F1(0.5;1.5;-\pi e^2))} = 0.97216864\cdots \tag{2.3}$$

Mathematica gives the nice approximation of (2.2) as shown below :

```
In[338]:=
α =
  (π^(1/6) Gamma[4 / 3] / 2^(1/3))^Hypergeometric1F1[0.5,1.5,-E^2 π];
β = NIntegrate[Exp[-x^2 Erf[x]], {x, 0, ∞}];
γ = α;
error = β - γ
Out[340]=
-0.0000616497
```

FIGURE 7 Error approximation for zeraoulia($+\infty$)

3 | SERIES REPRESENTATION OF ZERAOULIA FUNCTION :

We may try to find a series expansion in powers of t of

$$I(t)=\int_0^t \exp(-x^2\operatorname{erf}(x))dx=\sum_{p=1}^{\infty} c_p t^p.$$

The coefficients $c_p = p^{-1}d_{p-1}$ follow from the series expansion $e^{-x^2 \operatorname{erf} x} = \sum_{p=0}^{\infty} d_p x^p$, resulting in

$$I(t)=\sum_{p=1}^{\infty} c_p t^p = t-\frac{t^4}{2\sqrt{\pi}}+\frac{t^6}{9\sqrt{\pi}}+\frac{2t^7}{7\pi}-\frac{t^8}{40\sqrt{\pi}}-\frac{4t^9}{27\pi}+\frac{(\pi-28)t^{10}}{210\pi^{3/2}}+O\left(t^{11}\right).$$

The series $I(1)=\sum_{p=1}^{\infty} c_p$ seems to converge:

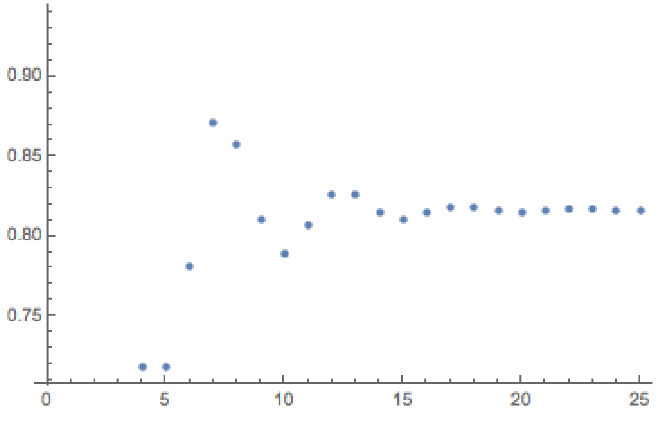

FIGURE 8 Plot of $I_N = \sum_{p=0}^{N} c_p$ as a function of N up to $N = 25$

The value of $I_{25} = 0.8162$ agrees with $I(1) = 0.816377$ to three decimal places. For $N = 50$ the agreement is up to six decimal places.But this didn't give us the power series closed form for n th term , we should use some approximations using approximation of error function and Pade approximant as shown in the following sections .

4 | SERIES REPRESENTATION OF : $\int_{-1}^{1} \operatorname{erf}(x)^n \, dx$ USING ERROR FUNCTION APPROXIMATION

We have the power series of :

$$e^{-x^2\operatorname{erf}(x)}=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}\operatorname{erf}^k(x)}{k!} \tag{4.1}$$

then from (4.1) we have :

$$\int_{-1}^{1} e^{-x^2 \operatorname{erf}(x)} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{-1}^{1} x^{2k} \operatorname{erf}^k(x) dx \tag{4.2}$$

Now the integral in R.H.S of (4.2) it's hard so much to evaluate using error function expression then we should use the following nice approximation :

$$(\operatorname{erf}(x))^2 \approx 1 - e^{-ax^2} \qquad \text{with} \qquad a = (1+\pi)^{2/3} \log^2(2)$$

, Here we can give a short proof to show the error function squared was approximated as well with the value of $a = (1+\pi)^{2/3} \log^2(2)$.

Proof We fully agree that

$$F(a) = \int_0^{\infty} \left(\operatorname{erf}(x)^2 - (1 - e^{-ax^2}) dx \right)^2$$

is minimum for $a \approx 1.23907$,According to $RIES$, this number seems to be much closer to

$$a = (1+\pi)^{2/3} \log^2(2) \approx 1.23907$$

than to $\frac{\pi^2}{8} \approx 1.23370$ even if this does make very large difference (the maximum error is reduced from 0.006 to 0.004 and the value of the integral $F(a)$ changes from 0.00002769 to 0.00002572). If we look for a still better approximation, we could consider $\log\left(1 - \operatorname{erf}(x)^2\right)$ (which, for sure, introduces a bias in the problem) and establish a Pade approximant and finally arrive to

$$\operatorname{erf}(x)^2 \approx 1 - \exp\left(-\frac{4}{\pi} \frac{1+\alpha x^2}{1+\beta x^2} x^2 \right)$$

where

$$\alpha = \frac{10 - \pi^2}{5(\pi - 3)\pi}$$

$$\beta = \frac{120 - 60\pi + 7\pi^2}{15(\pi - 3)\pi}$$

The value of the corresponding error function is 1.1568×10^{-7} that is to say almost 250 times smaller than with the initial formulation; the maximum error is 0.00035.

Now we are ready to approximate

$$I_n = \int_{-1}^{1} (\operatorname{erf}(x))^{2n} \, dx \tag{4.3}$$

$$J_n = \int_{-1}^{1} \left(1 - e^{-ax^2}\right)^n \, dx \tag{4.4}$$

for which the binomial expansion would be required (easy). This would give you things like:

$$J_1 = 2 - \frac{\sqrt{\pi}\operatorname{erf}\left(\sqrt{a}\right)}{\sqrt{a}} \tag{4.5}$$

$$J_2 = 2 - \frac{2\sqrt{\pi}\operatorname{erf}\left(\sqrt{a}\right)}{\sqrt{a}} + \frac{\sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\sqrt{2a}\right)}{\sqrt{a}} \tag{4.6}$$

$$J_3 = 2 - \frac{3\sqrt{\pi}\operatorname{erf}\left(\sqrt{a}\right)}{\sqrt{a}} + \frac{3\sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\sqrt{2a}\right)}{\sqrt{a}} - \frac{\sqrt{\frac{\pi}{3}}\operatorname{erf}\left(\sqrt{3a}\right)}{\sqrt{a}} \tag{4.7}$$

Now it is easy to get reccurence relation for J_n in (4.4), we take $t = \sqrt{akx} \implies dx = \frac{dt}{(\sqrt{ak})}$ we come up to : $erf(\sqrt{ak})$ which gives this general formula:

$$J_n = 2 + \sqrt{\frac{\pi}{a}} \sum_{k=1}^{n} (-1)^k \frac{\binom{n}{k}}{\sqrt{k}} \operatorname{erf}\left(\sqrt{ak}\right) \tag{4.8}$$

we produce below a short table for comparison

n	approximation	exact
1	0.591506	0.596751
2	0.279674	0.283168
3	0.151067	0.153256
4	0.0870954	0.0884650
5	0.0522216	0.0530855
6	0.0321485	0.0326982
7	0.0201718	0.0205243
8	0.0128409	0.0130686
9	0.00826756	0.00841548
10	0.00537202	0.00546863

we reused for this problem our approach with the same Padé approximants and obtained as approximations:

$$I_n = \frac{2}{2n+1}\left(\frac{4}{\pi}\right)^n {}_2F_1\left(2n, \frac{2n+1}{2}; \frac{2n+3}{2}; -\frac{1}{3}\right) \tag{4.9}$$

$$I_n = \frac{2}{2n+1}\left(\frac{4}{\pi}\right)^n F_1\left(\frac{2n+1}{2}; -2n, 2n; \frac{2n+3}{2}; \frac{1}{30}, -\frac{3}{10}\right) \tag{4.10}$$

Really we are ready to give the series representation of zeraoulia function Over $[-1; 1]$ using error function approximation and pade approximant .

5 | SERIES REPRESENTATION OF ZERAOULIA FUNCTION OVER [-1;1] USING ERROR FUNCTION APPROXIMATION AND PADE APPROXIMANT:

Recall :

$$I_k = \int_{-1}^{1} x^{2k}\ [\operatorname{erf}(x)]^k\ dx \tag{5.1}$$

is 0 if k is odd. So, we need to focus on

$$I_{2k} = \int_{-1}^{1} x^{4k}\ [\operatorname{erf}(x)]^{2k}\ dx \tag{5.2}$$

which could be approximated, as we showed above in section 3 to get (4.8) using :

$$[\operatorname{erf}(x)]^2 \approx 1 - e^{-ax^2} \qquad \text{with} \qquad a = (1+\pi)^{2/3}\log^2(2) \tag{5.3}$$

making

$$I_{2k} = \int_{-1}^{1} x^{4k}\ \left(1 - e^{-ax^2}\right)^k\ dx \tag{5.4}$$

to be developed using the binomial expansion. So, in practice, we face the problem of

$$J_{n,k} = \int_{-1}^{1} x^{4k}\, e^{-nax^2}\, dx \tag{5.5}$$

The antiderivative

$$\int x^{4k}\, e^{-nax^2}\, dx = -\frac{1}{2} x^{4k+1} E_{\frac{1}{2}-2k}\left(anx^2\right) \tag{5.6}$$

where appears the exponential integral function. Using the bounds, this reduces to

$$J_{n,k} = -E_{\frac{1}{2}-2k}(an) \tag{5.7}$$

and leads to "reasonable" approximation as shown in the table below

$$\begin{pmatrix} k & \text{approximation} & \text{exact} \\ 1 & 0.22870436048 & 0.22959937502 \\ 2 & 0.08960938943 & 0.08997882179 \\ 3 & 0.04400808083 & 0.04418398568 \\ 4 & 0.02389675159 & 0.02398719298 \\ 5 & 0.01374034121 & 0.01378897319 \\ 6 & 0.00819869354 & 0.00822557475 \\ 7 & 0.00502074798 & 0.00503586007 \\ 8 & 0.00313428854 & 0.00314286515 \\ 9 & 0.00198581489 & 0.00199069974 \\ 10 & 0.00127304507 & 0.00127582211 \end{pmatrix}$$

Another approximation could be obtained using the simplest Padé approximant [08] of the error function

$$\mathrm{erf}(x) = \frac{2x}{\sqrt{\pi}\left(1 + \frac{x^2}{3}\right)} \tag{5.8}$$

which would lead to

$$I_{2k} = \int_{-1}^{1} x^{4k}\ [\mathrm{erf}(x)]^{2k}\ dx = \frac{2}{6k+1}\left(\frac{4}{\pi}\right)^k {}_2F_1\left(2k, \frac{6k+1}{2}; \frac{6k+3}{2}; -\frac{1}{3}\right) \tag{5.9}$$

slightly less accurate than the previous one. Continuing with Padé approximant

$$\mathrm{erf}(x) = \frac{\frac{2x}{\sqrt{\pi}} - \frac{x^3}{15\sqrt{\pi}}}{1 + \frac{3x^2}{10}} \tag{5.10}$$

we should get

$$I_{2k} = \int_{-1}^{1} x^{4k}\ [\mathrm{erf}(x)]^{2k}\ dx = \frac{2}{6k+1}\left(\frac{4}{\pi}\right)^k F_1\left(\frac{6k+1}{2}; -2k, 2k; \frac{6k+3}{2}; \frac{1}{30}, -\frac{3}{10}\right) \tag{5.11}$$

where appears the the Appell hypergeometric function of two variables. Finally we conclude the series representation as :

$$I(t) = \int_{-1}^{1} \exp(-x^2\,\mathrm{erf}(x))dx. \sim \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} I_{2k} \tag{5.12}$$

6 | APPROXIMATION OF ZERAOULIA FUNCTION BY MEANS OF A POLYNOMIAL.

Let f be the function :

$$f(x) = Zeraoulia\left(\frac{b+a}{2} + \frac{b-a}{2}x\right),\ -1 \le x \le 1 \tag{6.1}$$

We may approximate the function f on the interval $[-1,1]$ by using Chebyshev polynomials [05] of the first kind. To this end, we choose some positive integer n and we define the coefficients c_n by the formuas

$$c_j = \frac{2}{\pi}\int_{-1}^{1}\frac{T_j(x)}{\sqrt{1-x^2}}f(x)dx \text{ for } j = 0, 1, ..., n.$$

Then the polynomial

$$P_n(x) = \frac{1}{2}c_0 + \sum_{j=1}^{n} c_j T_j(x) \tag{6.2}$$

approximates $f(x)$ in the best possible way. Since

$$Zeraoulia(x) = f(\frac{a+b-2x}{a-b}) \text{ for } a \leq x \leq b \tag{6.3}$$

we see that the polynomial $Q_n(a,b,x) = P_n(\frac{a+b-2x}{a-b})$ is an approximant to Zeraoulia function on $[a,b]$.Calculations give :

$$\begin{aligned}
Q_{11}(0,3/2,x) \quad = \quad & 0.0137936039435x^{11} - \\
& 0.135129528505x^{10} + \\
& 0.548169602543x^{9} - \\
& 1.16161653976x^{8} + \\
& 1.31691631085x^{7} - \\
& 0.746480407376x^{6} + \\
& 0.338453415662x^{5} - \\
& 0.370071852413x^{4} + \\
& 0.0133517048763x^{3} - \\
& 0.00104123958376x^{2} + \\
& 1.00003172454x
\end{aligned}$$

and

$$
\begin{aligned}
Q_{11}(3/2,3,x) = {} & -0.0000675632422240x^{11} + \\
& 0.00188305739843x^{10} - \\
& 0.0239397852528x^{9} + \\
& 0.183255163671x^{8} - \\
& 0.937675010268x^{7} + \\
& 3.35913844398x^{6} - \\
& 8.55140470408x^{5} + \\
& 15.3046428836x^{4} - \\
& 18.4622672665x^{3} + \\
& 13.5920479951x^{2} - \\
& 4.69093970289x + \\
& 1.04191571066
\end{aligned}
$$

For both approximations the error is less than 10^{-6}. Indeed, numerical integration gives:

$$
\begin{aligned}
||Zera(x) - Q_{11}(0,3/2,x)|| &= \\
\sqrt{\int_0^{\frac{3}{2}} (Zera(x) - Q_{11}(0,3/2,x))^2 dx} &\approx 2.26 \times 10^{-7}
\end{aligned}
$$

and

$$
\begin{aligned}
||Zera(x) - Q_{11}(3/2,3,x)|| &= \\
\sqrt{\int_{\frac{3}{2}}^{3} (Zera(x) - Q_{11}(3/2,3,x))^2 dx} &\approx 3.66 \times 10^{-10}
\end{aligned}
$$

Thus, we may evaluate the Zeraoulia function with high accuracy on the interval [0,3]. For $x > 3$ we may use the following approximation formula in the terms of the error function :

$$
\text{Zeraoulia}(x) \approx \varphi(x) \stackrel{def}{=} \int_0^3 \exp(-t^2 \text{erf}(t))dt + \frac{\sqrt{\pi}}{2}(\text{erf}(x) - \text{erf}(3)), x \geq 3.
$$

The quadratic mean error on $[3, 100]$ is

$$
||Zera(x) - \varphi(x))|| = \sqrt{\int_3^{100} (Zera(x) - \varphi(x))^2 dx} \approx 2.02 \times 10^{-8}.
$$

7 | CONCLUSION:

We have studied A new probability distribution defined on $[0, +\infty)$.and we gave series representations for zeraoulia function using pade approximant ,Really We approximated the CDF for that distribution by means of Chebyshev polynomials and the error function. The methods we applied are suitable for approximating other CDF for probability distributions, since it's cdf are bounded and they take values from 0 to 1. And it is well known that Chèbyshev polynomials are the optimal ones for aproximating continuous functions. On the other hand, it is also possible to approximate such functions by means of rational Chèbyshev approximants. This technique may be used in future works.

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