

## Which rows in Pascal's Triangle sum to Perfect Numbers?

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**Abstract:** In this note we show which rows in Pascal's Triangle sum to Perfect Numbers.

Starting with Pascal's Triangle we find the horizontal numbers sum to powers of 2

$$\begin{array}{cccccc}
 & & & & & 1 & & & & & & & & & & 2^0 \\
 & & & & & & & & & & & 1 & & & & 2^1 \\
 & & & & & 1 & & 1 & & & & & & & & 2^2 \\
 & & & & 1 & & 2 & & 1 & & & & & & & 2^3 \\
 & & 1 & & 3 & & 3 & & 1 & & & & & & & 2^4 \\
 1 & & 4 & & 6 & & 4 & & 1 & & & & & & & 2^4
 \end{array}$$

Next, a perfect number is a number that is equal to the sum of its proper divisors, but excluding the number itself. For example, 6 is a perfect number since  $6 = 1 + 2 + 3$ . With  $p$  some prime Euclid proved that  $2^{p-1}(2^p - 1)$  is a perfect number when  $2^p - 1$  is a Mersenne prime, and Euler proved that all even perfect numbers have this form for some  $p$ . It is unknown if there are infinitely many even perfect numbers or any *odd* perfect numbers.

To see which rows in Pascal's triangle sum to perfect numbers we have the following:

$$2^1(2^2 - 1) = 6 = 2^1 + 2^2$$

$$2^2(2^3 - 1) = 28 = 2^2 + 2^3 + 2^4$$

$$2^4(2^5 - 1) = 496 = 2^4 + 2^5 + 2^6 + 2^7 + 2^8$$

$$2^6(2^7 - 1) = 8128 = 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} + 2^{11} + 2^{12}$$

*Theorem:*

$$2^{p-1}(2^p - 1) = 2^{p-1} + 2^p + 2^{p+1} + \dots + 2^{2(p-1)}$$

*Proof:*

We use the fact that

$$1 + 2^1 + 2^2 + 2^3 + \dots + 2^m = 2^{m+1} - 1$$

Therefore,

$$\begin{aligned} & (1 + 2^1 + 2^2 + 2^3 + \dots + 2^{2(p-1)}) - (1 + 2^1 + 2^2 + 2^3 + \dots + 2^{p-2}) \\ &= 2^{2(p-1)+1} - 1 - (2^{(p-2)+1} - 1) = 2^{2p-1} - 2^{p-1} = 2^p(2^{p-1}) - 2^{p-1} \\ &= 2^{p-1}(2^p - 1) \end{aligned}$$

□

We end this note with another question. Is there a systematic and unique way to add the numbers from the relevant rows for a particular perfect number in Pascal's triangle showing the proper divisors of that perfect number? For example, from rows 1 and 2 in Pascal's triangle we have  $6 = 1 + 2 + (1 + 1 + 1)$ .