Which Rows in Pascal's Triangle Sum to Perfect Numbers?

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Abstract: In this note we show which rows in Pascal's Triangle sum to Perfect Numbers. We end this note with an algorithm allowing us to trivially calculate all the proper divisors for any Perfect Number greater than 6.

Starting with Pascal's triangle we find the horizontal numbers sum to powers of 2

1	2^0
1 1	2^1
1 2 1	2^2
1 3 3 1	2^3
1 4 6 4 1	2^4

Next, a perfect number is a number that is equal to the sum of its proper divisors, but excluding the number itself. For example, 6 is a perfect number since $6 = 1 + 2 + 3$. With p some prime Euclid proved that $2^{p-1}(2^p - 1)$ is a perfect number when $2^p - 1$ is a Mersenne prime, and Euler proved that all even perfect numbers have this form for some p. It is unknown if there are infinitely many even perfect numbers or any *odd* perfect numbers.

To trivially see which rows in Pascal's triangle sum to perfect numbers we have the following:

$$2^1(2^2 - 1) = 6 = 2^1 + 2^2$$

$$2^2(2^3 - 1) = 28 = 2^2 + 2^3 + 2^4$$

$$2^4(2^5 - 1) = 496 = 2^4 + 2^5 + 2^6 + 2^7 + 2^8$$

$$2^6(2^7 - 1) = 8128 = 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} + 2^{11} + 2^{12}$$

Theorem: For every perfect number $2^{p-1}(2^p - 1)$

$$2^{p-1}(2^p - 1) = 2^{p-1} + 2^p + 2^{p+1} + \cdots + 2^{2(p-1)}$$

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Proof:

We use the fact that

$$1 + 2^1 + 2^2 + 2^3 + \dots + 2^m = 2^{m+1} - 1$$

Therefore,

$$\left(1 + 2^1 + 2^2 + 2^3 + \dots + 2^{2(p-1)}\right) - (1 + 2^1 + 2^2 + 2^3 + \dots + 2^{p-2})$$
$$= 2^{2(p-1)+1} - 1 - \left(2^{(p-2)+1} - 1\right) = 2^{2p-1} - 2^{p-1} = 2^p(2^{p-1}) - 2^{p-1}$$
$$= 2^{p-1}(2^p - 1)$$

□

The theorem allows us to easily and systematically calculate all the proper divisors for any perfect number greater than 6.

Theorem: For $p > 2$ the proper divisors for any perfect number $2^{p-1}(2^p - 1)$ are given by

$$2^{p-1}(2^p - 1) = 2^{p-1} + \left(1 + (2^p - 1)\right) + \left(2 + (2^{p+1} - 2)\right) + \left(2^2 + (2^{p+2} - 2^2)\right)$$
$$+ \left(2^3 + (2^{p+3} - 2^3)\right) + \cdots + \left(2^{p-2} + \left(2^{2(p-1)} - 2^{p-2}\right)\right)$$

Proof: Arrange the terms in ascending order and Euclid's proof applies.

□

For example,

$$2^4(2^5 - 1) = 496 = 2^4 + 2^5 + 2^6 + 2^7 + 2^8$$
$$= 16 + (1 + 31) + (2 + 62) + (4 + 124) + (8 + 248)$$
$$= 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$$

Finally, there are two symmetric ways of adding the numbers in the first three rows of any perfect number greater than 6 showing the proper divisors contained within those three rows. For example,

$$2^2(2^3 - 1) = 28 = 2^2 + 2^3 + 2^4$$
$$= 4 + (1 + \mathbf{7}) + (2 + 14)$$
$$= 1 + 2 + 4 + 7 + 14$$
$$= 1 + (1 + 1) + (1 + 2 + 1) + (\mathbf{3} + \mathbf{3} + \mathbf{1}) + (4 + 6 + 4)$$
$$= 1 + (1 + 1) + (1 + 2 + 1) + (\mathbf{1} + \mathbf{3} + \mathbf{3}) + (4 + 6 + 4)$$

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