THE UPPER BOUND OF COMPOSITION SERIES

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Abstract. In this paper we prove that among all finite groups of order $n \in \mathbb{N}$ with $n \geq 2$ where $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$, the abelian group with elementary abelian sylow subgroups, has the highest number of composition series and it has $\prod_{i=1}^{r} \left(\prod_{j=1}^{\alpha_i} \frac{p_i^j - 1}{p_i - 1} \right) \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$ distinct composition series. We also prove that among all finite groups of order $\leq n$, $n \in \mathbb{N}$ with $n \geq 4$, the elementary abelian group of order 2^{α} where $\alpha = [\log_2 n]$ has the highest number of composition series.

1. Introduction. A composition series, that is a series of subgroups each normal in the previous such that corresponding factor groups are simple. Any finite group has a composition series. The famous Jordan-Holder theorem proves that, the composition factors in a composition series are unique up to isomorphism. Sometimes a group of small order has a huge number of distinct composition series. For example an elementary abelian group of order 64 has 615195 distinct composition series. In [6] there is an algorithm in GAP to find the distinct composition series of any group of finite order.The aim of this paper is to provide an upper bound of the number of distinct composition series of any group of finite order. The approach of this is combinatorial and the method is elementary. All the groups considered in this paper are of finite order.

2. Some Basic Results On Composition Series. We start with some basic results. As they are known, they are given without proof.

Theorem. *2.1 Among all finite groups of order n , the abelian group with elenentary abelian sylow subgroups , has the highest number of distinct composition series.*

3. Some New Results On Composition Series. In this section, we study some new results on composition series of finite groups.

Definition. 3.1 We define C_G as the set of all distinct composition series of the group G.

Theorem. *3.2 Let $n \geq 2$ be a positive integer where $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$. Then* $| C_{Z_n} | = \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$.

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Proof. We know that for each positive divisor d of n there exists a unique subgroup of Z_n of order n.

Let S be the set of sequence of primes from the set $\{p_1, ..., p_r\}$ of length $\alpha_1 + \alpha_2 + ... + \alpha_r$ where p_i occurs α_i times.

Then $\mid S \mid = \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$.

Now C_{Z_n} is the set of distinct composition series of Z_n.

Now define a function

$f : S \to C_{Z_n}$ by $f\left(\beta_1\beta_2...\beta_{\alpha_1+\alpha_2+...+\alpha_{r-1}}\beta_{\alpha_1+\alpha_2+...+\alpha_r}\right) = \{e\} \trianglelefteq Z_{\beta_1} \trianglelefteq Z_{\beta_1\beta_2} \trianglelefteq ... \trianglelefteq Z_{\beta_1\beta_2...\beta_{\alpha_1+\alpha_2+..+\alpha_r}} = Z_n$,

where $\beta_1, \beta_2, ..., \beta_{\alpha_1+\alpha_2+...+\alpha_r}$ are primes such that $\beta_i = p_i$ has exactly α_i solutions for $1 \leq i \leq \alpha_1 + \alpha_2 + ... + \alpha_r$.

So $\beta_1\beta_2...\beta_{\alpha_1+\alpha_2+...+\alpha_{r-1}}\beta_{\alpha_1+\alpha_2+...+\alpha_r} \in S$.

Now $\{e\} \trianglelefteq Z_{\beta_1} \trianglelefteq Z_{\beta_1\beta_2} \trianglelefteq ... \trianglelefteq Z_{\beta_1\beta_2...\beta_{\alpha_1+\alpha_2+...+\alpha_r}} = Z_n$ i.e.$\{e\} \trianglelefteq Z_{\theta_1} \trianglelefteq Z_{\theta_2} \trianglelefteq ... \trianglelefteq Z_{\theta_{\alpha_1+\alpha_2+...+\alpha_r}} = Z_n$ is a composition series of Z_n where $\theta_i = \prod_{j=1}^{i} \beta_j$, $1 \leq i \leq \alpha_1 + \alpha_2 + ... + \alpha_r$.

Then $f : S \to C_{Z_n}$ is an injective mapping by its construction. Now we will prove that it is surjective also.

Let $\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq ... \trianglelefteq G_{\alpha_1+\alpha_2+...+\alpha_r} = Z_n$ be a composition series of Z_n.

Since for each divisor d of n, there exists a unique subgroup of order d of Z_n and any subgroup of Z_n is cyclic

then $\frac{|G_i|}{|G_{i-1}|}$ is a prime number.

Define $g : C_{Z_n} \to S$ by $g\left(C_{Z_n}\right) = q_1q_2...q_{\alpha_1+\alpha_2+...+\alpha_r}$, where $q_i = \frac{|G_i|}{|G_{i-1}|}$, for $1 \leq i \leq \alpha_1 + \alpha_2 + ... + \alpha_r$.

Then each q_i is a prime for $1 \leq i \leq \alpha + \alpha_2 + ... + \alpha_r$.

So $q_1q_2...q_{\alpha_1+\alpha_2+...+\alpha_r}$ is a sequence of primes such that $q_i = p_i$ has exactly α_i solutions for $1 \leq i \leq \alpha_1 + \alpha_2 + ... + \alpha_r$.

So, $q_1q_2...q_{\alpha_1+\alpha_2+...+\alpha_r} \in S$ and therefore $f^{-1} = g$ and hence f is surjective and hence bijective also.

Therefore $\mid S \mid = \mid C_{Z_n} \mid = \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$ i.e. Z_n has $\frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$ distinct composition series. □

Example. 3.3 Let us try to understand the above theorem for Z_{360}.

Solution. NOTE : $360 = 2^3 3^2 5^1$. Now $\frac{(3+2+1)!}{3!2!1!} = 60$ distinct sequence of primes can be formed of length 6 such that each sequence contains 3 times prime 2,2 times prime 3 and 1 time prime 5. Some of them are 232253,323522,523322.

Now we will construct a composition series corresponding to these numbers.

(1) The composition series corresponding to 232253 will be

$\{e\} \trianglelefteq Z_2 \trianglelefteq Z_6 \trianglelefteq Z_{12} \trianglelefteq Z_{24} \trianglelefteq Z_{120} \trianglelefteq Z_{360}$.

(2) The composition series corresponding to 323522 will be

$\{e\} \trianglelefteq Z_3 \trianglelefteq Z_6 \trianglelefteq Z_{18} \trianglelefteq Z_{90} \trianglelefteq Z_{180} \trianglelefteq Z_{360}$.

(3) The composition series corresponding to 523322 will be
$\{e\} \trianglelefteq Z_5 \trianglelefteq Z_{10} \trianglelefteq Z_{30} \trianglelefteq Z_{90} \trianglelefteq Z_{180} \trianglelefteq Z_{360}$.

Theorem. *3.4 Let G be an abelian group of order n where $n \geq 2$ be a natural number such that $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$. Then $\mid C_G \mid = \prod_{i=1}^{r} t_i \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$ where t_i are the numbers of distinct composition series of the sylow p_i- subgroups of G.*

Proof. Note that in finite abelian group there exists at least one subgroup of order h for each divisor h of the order of the group n and also each composition factor of an abelian group is group of prime order. The order of composition factors of an fintite abelian group is a prime number and using them we can make a sequence of primes which belongs to S (which is described in the THEOREM 3.2).

Therefore the number of distinct composition series of G is a multiple of $\frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$.

Now let $\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq ... \trianglelefteq G_m \trianglelefteq G_{m+1} \trianglelefteq ... \trianglelefteq G_{\alpha_1+\alpha_2+...+\alpha_r} = G$ be a composition series of G where j be the least positive integer such that $p_1^{s_1}$ divide $\mid G_j \mid$and k be the least positive integer such that $p_1^{s_1+1}$ divide $\mid G_k \mid$ when $1 \leq j < k \leq \alpha_1 + \alpha_2 + ... + \alpha_r$, $1 \leq s < s+1 \leq \alpha_1$.

Therefore $\mid G_j \mid = p_1^{s_1} q$ where $p_1^{s_1}, q \in \mathbb{N}$ with gcd $(p_1^{s_1}, q) = 1$. $\mid G_k \mid = p_1^{s_1+1} u$ where $p_1^{s_1+1}, u \in \mathbb{N}$ with gcd $\left(p_1^{s_1+1}, u\right) = 1$.

Let $G_j = A_j B_j$ where A_j and B_j are two subgroups of G of order p_1^s and q respectively. Let $G_k = C_k D_k$ where C_k and D_k are two subgroups of G of order $p_1^{s_1+1}$ and u respectively.

We will prove that $A_j \subseteq C_k$; if not then, $\exists$ some $x \in A_j$ such that $x \notin C_k$. $x \in A_j \Rightarrow x = p_1^{v_1}$. Now $x \in A_j \Longrightarrow x \in G_{j,}\ x \in G_j \Longrightarrow x \in G_{j+1}$ as G_j is a normal subgroup of G_{j+1}. $x \in G_{j+1} \Longrightarrow x \in G_{j+2}$ as G_{j+1} is a normal subgroup of G_{j+2}.

So proceeding in this way we can prove that $x \in G_k$. Now $G_k = C_k D_k$ where $C_k \cap D_k = \{e\}$. $x \in G_k \Longrightarrow x \in C_k D_k$, i.e. $x = cd$ where $c \in C_k, d \in D_k$. Therefore $x = p_1^{c_1} d, p_1^{c_1} \in C_k$. So $p_1^{v_1} = p_1^{c_1} d$ i.e. $p_1^{v_1 - c_1} = d$, As gcd $(p_1^{\alpha_1}, u) = 1$ so $P_1 \cap D_k = \{e\}$ but $p_1^{v_1 - c_1} \in P_1 \cap D_k$ i.e. $p_1^{v_1 - c_1} = e$,$p_1^{v_1} = p_1^{c_1}$ hence $x\,(= p_1^{c_1}) \in C_k$, a contradiction.

Therefore in the composition series of sylow p_1- group, A_j is a subgroup of C_k. So if the sylow p_1- group has t_1 distinct composition series then we have $t_1 \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$ such possible composition series due to sylow p_1- subgroup in G.

In a similar way for the other sylow p_2 - group, ..., sylow p_r- group we have $\prod_{i=1}^{r} t_i \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$ distinct composition series for G. □

Theorem. *3.5 In an elementary abelian group of order p^n, there are $\prod_{k=1}^{n} \left(\frac{p^k - 1}{p-1}\right) = \frac{\prod_{k=1}^{n}(p^k - 1)}{(p-1)^n}$ distinct composition series.*

Proof. Let $\{e\} \trianglelefteq p \trianglelefteq p^2 \trianglelefteq ... \trianglelefteq p^{k-1} \trianglelefteq p^k \trianglelefteq ... \trianglelefteq p^n$ be a composition series for the elementary abelian group of order p^n.

To find the number of distinct composition series, we will find the possible choice from p^{k-1} to p^k at first.

Let there be C_{k-1} choice available for p^{k-1} to p^k.

Now in an elementary abelian group of order p^n

there are $\frac{\prod_{i=0}^{k-1}((p^n-1)-(p^i-1))}{\prod_{i=0}^{k-1}(p^k-p^i)} = A$(say) distinct subgroups of order p^k each

of which is an elementary abelian (By [2], page - 65).

In an elementary abelian group of order p^n ,

also there are $\frac{\prod_{i=0}^{k-2}((p^n-1)-(p^i-1))}{\prod_{i=0}^{k-2}(p^k-p^i)} = B$(say) distinct subgroups of order p^{k-1} each

of which is an elementary abelian(By [2], page-65).

Again in an elementary abelian group of order p^k

there are $\frac{\prod_{i=0}^{k-2}((p^k-1)-(p^i-1))}{\prod_{i=0}^{k-2}(p^k-p^i)} = D$(say) distinct subgroups of order p^{k-1} each

of which is elementary abelian (By [2],page-65).

Now there is an identical algebraic structure of any two subgroups of same order in an elementary abelian group

i.e. in elementary abelian group of order p^n , any two distinct subgroups of order p^k have the same number of subgroup order p^{k-1}.

Then $C_{k-1} = \frac{A.B}{D} = \frac{p^{n+1-k}-1}{p-1}$.

As a result we have $\prod_{k=1}^{n-1}\left(\frac{p^{n+1-k}-1}{p-1}\right)$distinct possible choice for the composition series of an elementary abelian group of order p^n (as $k \geq 1$) which is same as $\prod_{k=1}^{n}\left(\frac{p^{n+1-k}-1}{p-1}\right) = \prod_{k=1}^{n}\left(\frac{p^k-1}{p-1}\right) = \frac{\prod_{k=1}^{n}(p^k-1)}{(p-1)^n}$. □

Corollary. *3.6 Let G be an abelian group with elementary abelian sylow subgroups of order $n \geq 2$ where $n = p_1^{\alpha_1}p_2^{\alpha_2}...p_r^{\alpha_r}$,*

then $| C_G | = \prod_{i=1}^{r}\left(\prod_{j=1}^{\alpha_i}\frac{p_i^j-1}{p_i-1}\right)\frac{(\sum_{i=1}^{r}\alpha_i)!}{\prod_{i=1}^{r}\alpha_i!}$

Proof. Which is an obvious result followed from THEOREM 3.4 and THEOREM 3.5. □

Corollary. *3.7 Let G be a finite group of order n where $n = p_1^{\alpha_1}p_2^{\alpha_2}...p_r^{\alpha_r}$,*

then $| C_G | \leq \prod_{i=1}^{r}\left(\prod_{j=1}^{\alpha_i}\frac{p_i^j-1}{p_i-1}\right)\frac{(\sum_{i=1}^{r}\alpha_i)!}{\prod_{i=1}^{r}\alpha_i!}$ *and the equality is hold if and only if G is an abelian group with elementary abelian sylow subgroups.*

Proof. Which is an obvious result followed

from THEOREM 2.1 and COROLLARY 3.6. □

Lemma. *3.8 Let* $X = \prod_{i=\alpha_1+1}^{\alpha_1+k}(2^i-1)\alpha_1!\alpha_r!$ *and* $Y = \prod_{j=1}^{\alpha_r}(\frac{p^j-1}{p-1})(\alpha_1+\alpha_r)!$ *;where $p \geq 5$ is a prime and $\alpha_1 \geq 0$ is an integer ; $\alpha_r \in \mathbb{N}$, such that $t = p^{\alpha_r}$*

and $[\log_2 t] = k$ *. Also* $\alpha_r \geq 2$ *whenever* $p = 3$*. Then* $\frac{X}{Y}$ *is a monotone increasing function of* α_1*.*

Proof. Let $f(\alpha_1) = \frac{\prod_{i=1}^{\alpha_1+k}(2^i-1)}{\prod_{i=1}^{\alpha_1}(2^i-1)\prod_{j=1}^{\alpha_r}(\frac{p^j-1}{p-1})\frac{(\alpha_1+\alpha_r)!}{\alpha_1!\alpha_r!}}$

then $f(\alpha_1+1) = \frac{\prod_{i=1}^{\alpha_1+k+1}(2^i-1)}{\prod_{i=1}^{\alpha_1+1}(2^i-1)\prod_{j=1}^{\alpha_r}(\frac{p^j-1}{p-1})\frac{(\alpha_1+\alpha_r+1)!}{(\alpha_1+1)!\alpha_r!}}$.

Therefore $\frac{f(\alpha_1+1)}{f(\alpha_1)} = \frac{2^{\alpha_1+k+1}-1}{2^{\alpha_1+1}-1}\frac{\alpha_1+1}{\alpha_1+\alpha_r+1}$.

In order to prove $f(\alpha_1+1) > f(\alpha_1)$ we have to prove that $\frac{2^{\alpha_1+k+1}-1}{2^{\alpha_1+1}-1} > \frac{\alpha_1+\alpha_r+1}{\alpha_1+1}$.

As $\frac{2^{\alpha_1+1+k}-1}{2^{\alpha_1+1}-1} > 2^k$ and $\alpha_r + 1 \geq \frac{\alpha_1+1+\alpha_r}{\alpha_1+1}$.

So it is sufficient enough to prove $2^k > \alpha_r + 1 \iff k > \log_2(\alpha_r+1)$.

Now $\log_2 = k + f$ where $0 < f < 1$. So $k = \log_2 - f$.

Therefore we have to prove that $\log_2 t - f > \log_2(\alpha_r+1) \iff \log_2(\frac{t}{\alpha_r+1}) > f$.

As $0 < f < 1$ it is sufficient enough to prove $\log_2(\frac{t}{\alpha_r+1}) > 1$ as

$\log x$ is a monotone increasing function for $x > 0$.i.e.

$\frac{t}{\alpha_r+1} > 2 \iff p^{\alpha_r} > 2\alpha_r + 2$ for all prime $p \geq 5$ with $\alpha_r \geq 1$ and $p = 3$ with $\alpha_r \geq 2$, which can easily be proved using mathematical induction.

Hence $\frac{X}{Y}$ is a monotone increasing function of α_1. □

Theorem. *3.9 Among all p groups of order* $\leq n$ *, where* $n \geq 4$ *is a positive integer ; the elementary abelian group of order* 2^α *where* $\alpha = [\log_2 n]$ *, has the highest number of composition series.*

Proof. We know that among all finite $p-$ groups of equal order the elementary abelian group has the highest number of composition series.

Let G_2 be an elementary abelian group of order 2^α where $\alpha = [\log_2 n]$.

Let G_p be an elementary abelian p group of order p^β where $\beta = \left[\log_p n\right]$ with $p \geq 3$ be a prime and $n \geq 4$.

Now we know that if $\left[\log_q r\right] = t$ then $\left[\log_q qr\right] = t+1$ for all prime $q \geq 2$ and $r, t \in \mathbb{N}$ with $q \leq r$.

Obviously 2 divides $p - 1$.

Therefore we get $2^{[\log_2 n]} - 1 > \frac{p^{[\log_p n]}-1}{p-1}$ whenever $n \geq 4$...(A).

$\mid C_{G_2} \mid = \prod_{i=1}^{[\log_2 n]}\left(2^i - 1\right)$, $\mid C_{G_p} \mid = \prod_{i=1}^{\left[\log_p n\right]}\left(\frac{p^i-1}{p-1}\right)$.

From (A) it implies that $\prod_{i=1}^{[\log_2 n]}\left(2^i - 1\right) > \prod_{i=1}^{\left[\log_p n\right]}\left(\frac{p^i-1}{p-1}\right)$ i.e.$\mid C_{G_2} \mid > \mid C_{G_p} \mid$.

Therefore among all p groups of order $\leq n$, the elementary abelian group of order 2^α has the highest number of composition series where $\alpha = [\log_2 n]$. □

Theorem. *3.10 Let* $n \geq 4$ *be a positive integer. Among all finite groups of order* $\leq n$*, the elementary abelian group of order* 2^α *has the highest number of composition series where* $\alpha = [\log_2 n]$*.*

Proof. $n \geq 4$ is a natural number and $\alpha = [\log_2 n]$ and let G be the elementary abelian group of order 2^α then $\mid C_G \mid= \prod_{i=1}^{[\log_2 n]} \left(2^i - 1\right)$.

As we know that among all finite groups of order $n \geq 2$, the abelian group with elementary abelian sylow subgroups has the highest number of composition series.

So in order to prove our claim we have to prove that
$\prod_{i=1}^{[\log_2 n]} \left(2^i - 1\right) \geq \prod_{i=1}^{q} \left(\prod_{j=1}^{\beta_i} \frac{p_i^j - 1}{p_i - 1}\right) \frac{\left(\sum_{i=1}^{q} \beta_i\right)!}{\prod_{i=1}^{q} \beta_i!}$

where each p_i is a prime and $\beta_i, q \in \mathbb{N}$ such that $\prod_{i=1}^{q} p_i^{\beta_i} \leq n$ and the equality is hold if and only if $q = 1, p_1 = 2$ and $\beta_1 = [\log_2 n]$.

The inequality will be proved in the following steps.

STEP 1 : Let H be an abelian group of order m with elementary abelian sylow subgroupssuch that $4 \leq m \leq n$ where $m = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$

also $\mid C_H \mid$denotes the number of distnct composition series of H.

Let p_r be a prime such that $p_r \neq 2$ and $p_r^{\alpha_r} = t$, $[\log_2 t] = k$. Then $2^k < p_r^{\alpha_r}$.

Let H' be an abelian group of order $\frac{|H|.2^k}{p_r^{\alpha_r}}$ with elementary abelian sylow subgroups.

Then $\mid H' \mid<\mid H \mid$. We will prove that $\mid C_{H'} \mid>\mid C_H\mid$.

STEP 2 : If $\mid H' \mid= 2^k p_1^{\alpha_1} ... p_{r-1}^{\alpha_{r-1}}$ i.e. 2 is not a divisor of m then we have $k > \alpha_r$

and $\prod_{i=1}^{k} \left(2^k - 1\right) > \prod_{i=1}^{\alpha_r} \left(\frac{p_r^i - 1}{p_r - 1}\right)$.

Therefore, obviously $\mid C_{H'} \mid>\mid C_H \mid$.

STEP 3 : If $\mid H \mid= 2^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$ then $\mid H' \mid= 2^{\alpha_1 + k} p_2^{\alpha_2} ... p_{r-1}^{\alpha_{r-1}}$.

In this case we will prove that $\mid C_{H'} \mid>\mid C_H \mid$.

$\mid C_H \mid= \prod_{i=1}^{\alpha_1} \left(2^i - 1\right) \prod_{i=1}^{\alpha_r} \left(\frac{p_r^{\alpha_r} - 1}{p_r - 1}\right) \prod_{i=2}^{r-1} \left(\prod_{j=1}^{\alpha_i} \frac{p_i^j - 1}{p_i - 1}\right) \frac{\left(\sum_{i=1}^{r} \alpha_i\right)!}{\prod_{i=1}^{r} \alpha_i!}$.

$\mid C_{H'} \mid= \prod_{i=1}^{\alpha_1 + k} \left(2^i - 1\right) \prod_{i=2}^{r-1} \left(\prod_{j=1}^{\alpha_i} \frac{p_i^j - 1}{p_i - 1}\right) \frac{\left(\sum_{i=1}^{r-1} \alpha_i + k\right)!}{\prod_{i=2}^{r-1} \alpha_i!(\alpha_1 + k)!}$.

Therefore, $\frac{|C_{H'}|}{|C_H|} = \frac{\prod_{i=\alpha_1+1}^{\alpha_1+k} \left(2^i - 1\right)(k+s)!\alpha_1!\alpha_r!}{\prod_{j=1}^{\alpha_r} \left(\frac{p_r^j - 1}{p_r - 1}\right)(\alpha_1+k)!(\alpha_r+s)!}$ where $s = \sum_{i=1}^{r-1} \alpha_i$.

So in order to prove our claim we have to prove that
$\prod_{i=\alpha_1+1}^{\alpha_1+k} \left(2^i - 1\right)(k + s)!\alpha_1!\alpha_r! >$
$\prod_{j=1}^{\alpha_r} \left(\frac{p_r^j - 1}{p_r - 1}\right)(\alpha_1 + k)!\,(\alpha_r + s)!$ ... (A) .

NOTE : $s = \sum_{i=1}^{r-1} \alpha_i \geq \alpha_1, \Longrightarrow s = \alpha_1 + a$

where $a \geq 0$, $k > \alpha_r \Longrightarrow k = \alpha_r + b$,$b > 0$; $a, b \in \mathbb{Z}$.

Then inequality (A) reduces to
$\prod_{i=\alpha_1+1}^{\alpha_1+k} \left(2^i - 1\right)(\alpha_1 + \alpha_r + a + b)!\alpha_1!\alpha_r! >$
$\prod_{j=1}^{\alpha_r} \left(\frac{p_r^j - 1}{p_r - 1}\right)(\alpha_1 + \alpha_r + a)!\,(\alpha_1 + \alpha_r + b)!$ ... (B)

[Here we are ignoring the case $p_r = 3$ with $\alpha_r = 1$, as $m \geq 4$ at least one other case must exist.]

NOTE : $\frac{(\alpha_1+\alpha_r+a+b)!\alpha_1!\alpha_r!}{(\alpha_1+\alpha_r+a)!(\alpha_1+\alpha_r+b)!}$ is a monotone increasing function of a for fixed α_1, α_r, b.

So it is sufficient enough to prove the inequality for $a = 0$

i.e. $\prod_{i=\alpha_1+1}^{\alpha_1+k}\left(2^i - 1\right)\alpha_1!\alpha_r! > \prod_{j=1}^{\alpha_r}\left(\frac{p_r^j-1}{p_r-1}\right)(\alpha_1+\alpha_r)!$ ... (C) .

Let $X = \prod_{i=\alpha_1+1}^{\alpha_r+k}(2^i - 1)\alpha_1!\alpha_r!$ and $Y = \prod_{j=1}^{\alpha_r}(\frac{p_r^j-1}{p_r-1})(\alpha_1+\alpha_r)!$.

Then according to LEMMA 3.8 $\frac{X}{Y}$ is a monotone increasing function of α_1 whenever p_r and α_r are fixed.

So it is sufficiently enough to prove the inequality (C) for $\alpha_1 = 0$

i.e.$\prod_{i=1}^{k}\left(2^i - 1\right) > \prod_{j=1}^{\alpha_r}\left(\frac{p_r^j-1}{p_r-1}\right)$ which is following form THEOREM 3.9.

As a result we prove that $| C_{H'} | > | C_H |$.

Now we will repeat the same process in the group H' and get a group H'' such that $| C_{H''} | > | C_{H'} |$.

Since $| H |$is finite then $| H |$has only finitely many prime divisors .

So after a finite number of steps we get an elementary abelian group of order $2^\eta$where $\eta \leq [\log_2 n]$.

STEP 4 : Let $| H | = 2^{\alpha_1}3^1$ for any $\alpha_1 \in \mathbb{N}$.

Then $| C_H | = \prod_{i=1}^{\alpha_1}\left(2^i - 1\right)(\alpha_1 + 1)$.

Then we can find an elementary abelian group H' such that$| H' | = 2^{\alpha_1+1}$ and $| C_{H'} | = \prod_{i=1}^{\alpha_1+1}\left(2^i - 1\right)$

Obviously $| C_{H'} | > | C_H |$ as $2^{\alpha_1+1} > \alpha_1 + 2$ for each $\alpha_1 \in \mathbb{N}$.

Among all 2 groups of order $\leq n$,G has the highest number of composition series which proves the theorem. □

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5. APPENDIX.

Lemma. *5.1Let G be a finite solvable group and M be a maximal normal subgroup of G , then G/M is a cyclic group of prime order and hence abelian.*

Proof. Let G be a finite solvable group and M be a maximal normal subgroup of G. As G is solvable, both M and G/M are solvable.

Now G/M is simple as well as solvable group hence it is abelian also (a non abelian simple group cannot be solvable). Hence a simple as well as abelian group must be a cyclic group of prime order. □

CORRESPONDENCE THEOREM.

Theorem. *5.2 Given $G \rhd N$, denote the the family of subgroups of G that contain N by $sub(G; N)$ and the family of subgroups of G/N by $sub(G/N)$. Then there is a bijection*

$\phi : sub(G; N) \longrightarrow sub(G/N)$, $H \mapsto H/N$ which preserves all subgroup lattic and normality relationships.

Proof. It is a very well known theorem and can be found any standared book on Group Theory. □

Theorem. *5.3 Among all finite solvable groups of order n , the abelian group with elementary abelian sylow subgroups has the highest number of composition series.*

Proof. Let K be the abelian group of order n with elementary abelian sylow subgroups. It is sufficient enough to prove, among all finite solvable groups of order n ,

K has the highest number of maximal normal subgroup. Among all finite $p-$groups, the elementary abelian group has the highest number of maximal normal subgroups.(By [2], page-65).

Now any finite abelian group (also for nilpotent) is the direct product of its sylow subgroups and any maximal normal subgroup has prime index.

As a result, among all finite abelian groups (also for nilpotent) of ordrer n , K has the highest number of maximal normal subgroups.

Now let G be a solvable (not necessarily abelian) group of order n. Let G has m distinct maximal normal subgroups namely $H_1, H_2, ..., H_m$.

Let C' be the commutator subgroup of G. Then $C' \trianglelefteq H_i$ for $1 \leq i \leq m$. (As G/H_i is abelian from the lemma) .

Let $H = \cap_{i=1}^{m} H_i$. Then H is a normal subgroup of G such that $C' \trianglelefteq H$. Therefore G/H is abelian . Now by Correspondence Theorem , there exists a bijection

$\phi : sub(G; H) \longrightarrow sub(G/H)$, which preserves all subgroup lattic and normality relationship. Therefore G/H is an abelian group with exactly m maximal normal subgroups.

But it is already proved that among all finite abelian groups of order n, K has the highest number of maximal normal subgroup, which proves the theorem. □

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