Differential equations for 2nd-order curves
Dynamic equations for 2nd-order curves

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Abstract*: The motion of a point along an ellipse under the action of a generalized force is investigated.*
Result: differential equation of second-order curves with respect to the focus, differential equation of curves of the second order with respect to the center, general differential equation of second order curves. Several examples of the application of these equations are proposed.

keywords: *ellipse, angular acceleration, eccentricity, differential equation.*

Introduction

“To solve the mass point motion problem we need differential equations for the motion. The way we derive these equations doesn’t matter”: [1,§11,п.3].

Let us consider two variants of motion of a point along a second-order curve under the action of a generalized force. In the first variant, the ellipse, around the left focus, Fig. 1. In the second variant, the ellipse, around the center, Fig. 2.

Option 1.

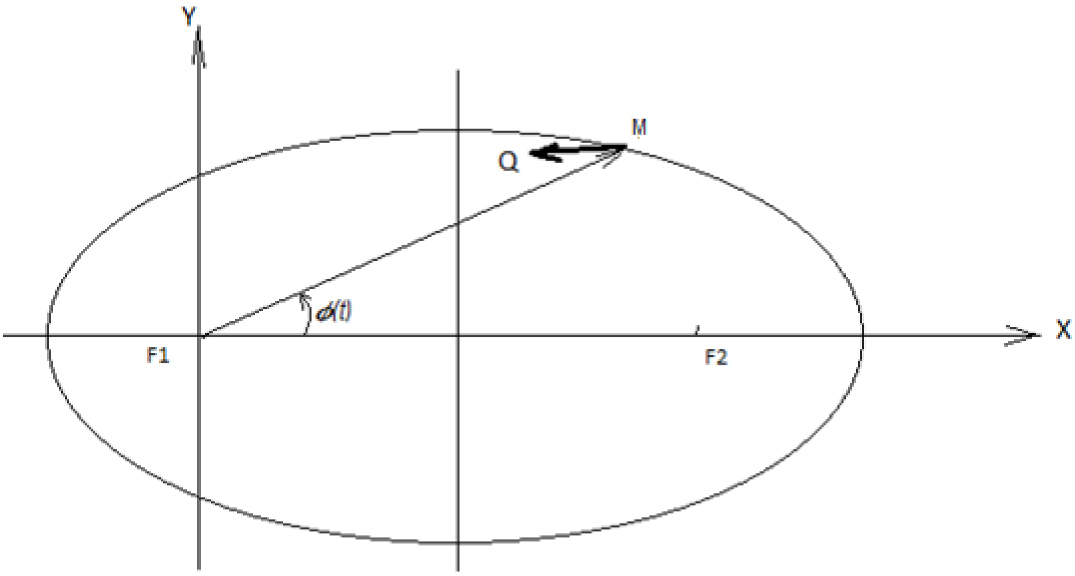

Figure 1.The mass point movement around the left focus.

M - material point.
Q – the force acting on the point.
F1- left focus.
F2 - right focus.
$\varphi(t)$ - angle between X axis and the line connecting left focus and the point.

Let us place the left center into the origin of coordinates.

$p = \frac{b^2}{a}$ semi-latus rectum, a - semi-major axis, b – semi-minor axis.

$e = \sqrt{1 - \frac{b^2}{a^2}}$ - eccentricity

$$m\ddot{x} = -Qcos(\varphi(t)) \quad (1)$$
$$m\ddot{y} = -Qsin(\varphi(t)) \quad (2)$$

From equation (1) we can get

$$Q = \frac{-m\ddot{x}}{cos(\varphi(t))} \quad (3)$$

Let us substitute equation (3) into equation (2)

$$\ddot{y} = \frac{\ddot{x}}{cos(\varphi(t))} sin(\varphi(t)) \quad (4)$$

The point coordinates can be represented as the function of angle of deflection φ(t) and radius r(t).

$$x = r(\varphi(t)) \cdot cos(\varphi(t)) \quad (5)$$
$$y = r(\varphi(t)) \cdot sin(\varphi(t)) \quad (6)$$
$$r(\varphi(t)) = \frac{p}{1-e*cos(\varphi(t))} \quad (7)$$

Let us calculate the first and second time derivative From equations (5), (6), (7). Let second time derivative us substitute equation (4) and move everything to the left side.

$$\ddot{\varphi} = \frac{2*e*\sin(\varphi)*\dot{\varphi}^2}{1-e*\cos(\varphi)} \quad (8)$$

Equation (8) is differential equation of second-order curves with respect to the focus. Different values of the eccentricity will lead into a different shape of the curve.

Option 2.

Here the beginning of coordinates in the center of an ellipse, figure 2.

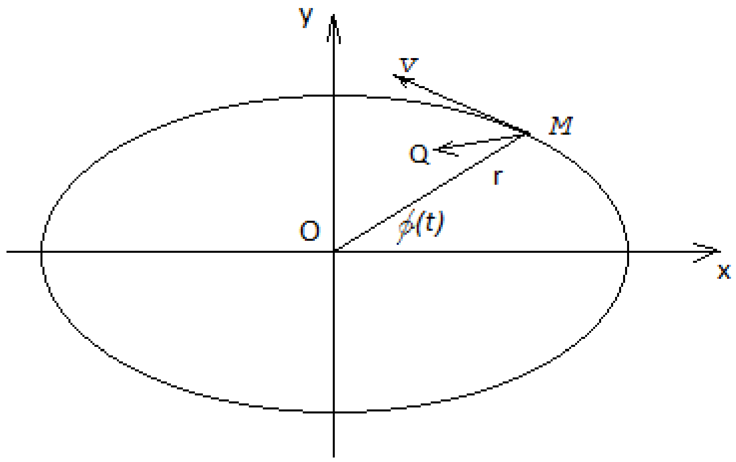

Figure 1.The mass point movement around the center.

M - material point.
Q – the force acting on the point.
O - center.
v - linear velocity of a point
φ(t) - angle between X axis and the line connecting center and the point.

Let us repeat the reasoning of option 1.

$$m\ddot{x} = -Qcos\left(\varphi(t)\right) \quad (9)$$

$$m\ddot{y} = -Qsin\left(\varphi(t)\right) \quad (10)$$

From equation (9) we can get

$$Q = \frac{-m\ddot{x}}{cos(\varphi(t))} \quad (11)$$

Let us substitute equation (11) into equation (10)

$$\ddot{y} = \frac{\ddot{x}}{cos(\varphi(t))} sin\left(\varphi(t)\right) \quad (12)$$

The point coordinates can be represented as the function of angle of deflection φ(t) and radius r(t).

$$x = r(\varphi(t)) \cdot cos\left(\varphi(t)\right) \quad (13)$$

$$y = r(\varphi(t)) \cdot sin\left(\varphi(t)\right) \quad (14)$$

$$r\left(\varphi(t)\right) = \frac{bcos\left(\varphi(t)\right)}{\sqrt{1-e^2cos^2\varphi(t)}} \quad (15)$$

Let us calculate the first and second time derivative From equations (13), (14), (15). Let second time derivative us substitute equation (4) and move everything to the left side.

$$\ddot{\varphi} = \frac{2*e^2*cos(\varphi)*sin(\varphi)*\dot{\varphi}^2}{1-e^2*cos(\varphi)^2} \quad (16)$$

Equation (8) is differential equation of second-order curves with respect to the center. Different values of the eccentricity will lead into a different shape of the curve.

The constant sectorial velocity is a property of equations (8), (16). Programs TygeBraheKepler2_focal.exe, TygeBraheKepler2_center.exe calculate the motion parameters and show the equality of areas of sectors with equal time intervals. Programs can be found http://fayloobmennik.cloud/7241309

Equation (8) allows us to model orbits using Kepler's laws. Some video samples can be found here http://www.fayloobmennik.net/4823487
Some executables samples can be found http://www.fayloobmennik.net/4909818

Equation (16) is applicable for modeling the streamlines of fluid and gas particles [8].

General differential equation of second order curves.

We select the values of the radius (7), (15) in equations (8), (16)

$$\ddot{\varphi} = \frac{2*e*\sin(\varphi)*\dot{\varphi}^2}{1-e*\cos(\varphi)} = 2*p*e*\sin(\varphi)*\dot{\varphi}^2$$

$$\ddot{\varphi} = \frac{2*e^2*cos(\varphi)*sin(\varphi)*\dot{\varphi}^2}{1-e^2*cos(\varphi)^2} = \frac{2*e^2*\sin(\varphi)*\dot{\varphi}^2}{b^2}$$

We obtained a general differential equation of second-order curves

$$\ddot{\varphi} = 2 * f\left(r\big(\varphi(t)\big)\right) * \sin(\varphi) * \dot{\varphi}^2 \qquad (17)$$

Follows from the equation (17) that it is possible to calculate the movement of a point on the given ellipse concerning any point of the plane, figure 3. Let us assume that the motion of a point along an ellipse is relative to the point $O_1(x_1, y_1)$. If the ellipse parameters are known and the angle φ_1(t) is known, then the radius r and the angle φ(t) with respect to the center. After the corresponding substitutions we will receive a formula (16).

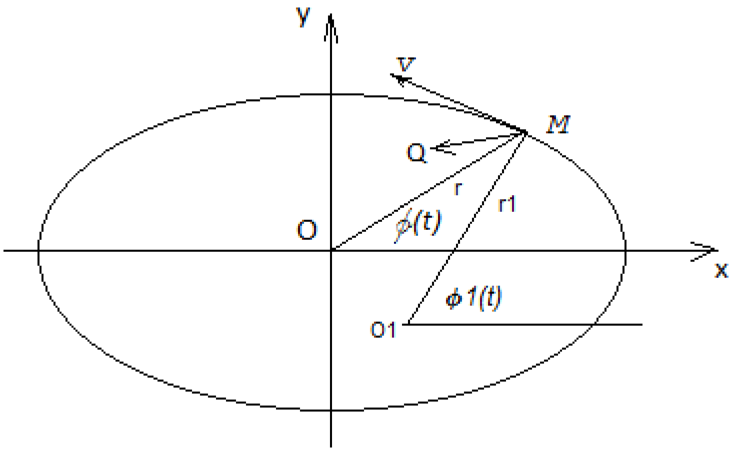

Figure 3. The movement of the point on the ellipse relative to an arbitrarily given point.

Examples of calculating orbits by the equation (14) in the article – "The Algorithm Simulation of the Orbits of Objects Differential Equation Second-Order Curves "

The equation (16) in article – "The motion of a fluid with constant and variable volume".

External links

[1] Sivukhin D. V. General course of physics, Electricity, Moscow,Russia, Nauka, 1996