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THE ISOTROPIC CONSTANT DEPENDS ONLY ON THE DIMENSION OF THE DOMAIN

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ABSTRACT. In this preprint we will prove that the isotropic constant depends only on the dimension of the domain. Thus we prove a result that implies the sharp isotropic constant conjecture.

1. INTRODUCTION

All the the problems that we investigate in this preprint have their roots in classical mechanics. However, our tools are from the asymptotic convex geometry. We will prove the following theorem.

Theorem 1. *Let the domain $D \subset \mathbf{R}^n$ be isotropic with respect to the probability density function $f/|D|$. Then*

$$\frac{1}{|D|}\int_D ||x||^2 f(x)dx = \frac{n}{n+p}|B_n|^{-2/n},$$

where B_n is the euclidean n-dimensional unit ball.

It is perhaps counter intuitive that the integral

$$L_n^2 = \frac{1}{|D|}\int_D ||x||^2 f(x)dx$$

is not more sensitive to the domain D. Lately, it has been proved by Klartag that in addition of the Sharp hyperplane conjecture even the celebrated Mahler conjecture follows from our results. For the connections of our results to the sharp hyperplane conjecture and the Mahler-conjecture see [4].

2. DEFINITIONS AND KNOWN RESULTS

A domain D is isotropic with respect to a probability density function $f/|D|$, if

$$\frac{1}{|D|}\int_D < y, x >^2 f(x)dx = \alpha^2 ||y||^2,$$

for some constant $\alpha > 0$ and all $y \in \mathbb{R}^n$. The following integral can be calculated in spherical coordinates.

$$\int_{B_n} ||x||^2 dx = \frac{n}{n+2}|B_n|, \tag{1}$$

where B_n is the unit ball.

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3. The proof of the main theorem

Let D be an isotropic domain with respect to a probability density function $f/|D|$. Because centralized balls are isotropic then via scaling there exists a ball L with the same isotropic constant α. So the real question is the volume of that ball L. Now, we integrate the following in spherical coordinates and obtain

$$\frac{1}{|D||L|}\int_D ||x||^2 f(x)dx = \frac{1}{|L|}\int_L ||x||^2 dx = \frac{1}{|L|}\frac{nR^{n+2}}{n+2}|B_n| = \frac{n}{n+2}R^2, \tag{2}$$

where R is the radius of the inertia ball L and we used scaling in order to rewrite the expression. Calculating the same integral in an other way we get

$$\frac{1}{|L|}\int_L ||x||^2 dx = \frac{nR_n^{n+2}}{n+2}|B_n| = \frac{n}{n+2}R_n^2, \tag{3}$$

where R_n is the radius of the ball of the unit volume. Thus, from equations (2) and (3) it follows that

$$R = R_n. \tag{4}$$

It follows from (4) that the inertia ball L is the centralized ball of the unit volume and

$$|L| = 1 = |B_n|R_n^n = |B_n|R^n.$$

Now, from (4) it follows that

$$R^2 = R_n^2 = |B_n|^{-2/n}.$$

Thus, from above and equation (2) we get

$$\frac{1}{|D|}\int_D ||x||^2 f(x)dx = \frac{n}{n+2}|B_n|^{-2/n},$$

and we have proved our main theorem 1.

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