Entanglement condition for W type multimode states

M. Karthick Selvan
karthick.selvan@yahoo.com

July 20, 2018

Abstract

We derive a class of inequality relations for detecting the multimode entanglement of non-Gaussian states of electromagnetic field, using a set of operators satisfying the Lie algebra of Pauli matrices. These relations are obtained using the fact that for separable states, the expectation value of tensor product of operators acting on the space of composite system is equal to the product of expectation values of individual operators acting on the space of subsystems and Schwarz inequality. The operators involved are quadratic in mode creation and annihilation operators and they can be experimentally measured. The derived inequality relation is proved to be a necessary and sufficient condition for W type entanglement.

Introduction

Quantum entanglement, a nonlocal trait of composite quantum systems, is of fundamental interest in quantum information theory. Both the discrete and continuous variable entanglement play a vital role in quantum information processing. However, continuous variable quantum information processing protocols can readily be implemented on optical systems. Quantum teleportation [1, 2], super dense coding [3, 4], quantum cryptography [5, 6] and quantum error corrections [7, 8] were demonstrated using Gaussian entangled states. Advanced quantum information processing protocols such as universal quantum computation are proposed using non-Gaussian entangled states [9, 10]. Despite many applications of continuous variable entangled systems, the basic problem of detection of continuous variable entanglement remains unexplored completely.

Many inseparability criteria for continuous variable systems were proposed in the form of inequality relations. Duan *et al* derived an inseparability criterion for bipartite Gaussian states using a pair of EPR type operators [11]. In the same issue of publication, Simon proved that the negativity under partial transposition (NPT) to be a necessary and sufficient condition for inseparability of bipartite Gaussian states [12]. Another inseparability criterion for bipartite Gaussian states was proposed by Mancini *et al* [13] and later it was shown to be related with the Duan *et al* condition by Agarwal and Biswas [14].

Further, in their work, Agarwal *et al* pointed out that the failure of inequality relations, which are based on second order correlation, to detect the entanglement of non-Gaussian states and proposed new inseparability criteria for bipartite non-Gaussian states, using the Heisenberg uncertainty relation of operators, satisfying $su(2)$ and $su(1,1)$ algebra, in conjunction with partial transposition [14]. Hillery and Zubairy obtained a condition in which the total variance of pair of operators satisfying $su(2)$ algebra, violate the lower bound set by separable states. A similar condition was obtained using $su(1,1)$ algebra operators and those two conditions were proposed as inseparability criteria, satisfying one of which implies the bipartite entanglement of non-Gaussian states [15]. However, Nha and Kim showed that the result of Agarwal *et al* to be a strongest criterion and provided an alternate expression of it, in terms of $su(2)$ algebra operators. Further an experimental scheme was proposed to measure those operators [16].

Bipartite inseparability conditions were generalized to multipartite continuous variable systems [15, 17]. However, these criteria detect only the entanglement of GHZ type states and biseparable states. They were not able to distinguish W type states from separable states. The entanglement of n-partite W type states are bipartite in nature, that is, any two arbitrary subsystems obtained by tracing out other $(n-2)$ subsystems are entangled. Different schemes were proposed to prepare n-mode W states and bipartite inseparablity criterion were used for $n(n-1)/2$ times to each pair of modes, in order to demonstrate the entanglement of W states [18, 19].

In this paper, we derive an inseparability criterion for multimode W type states. In contrary to previous n-mode inseparability conditions, where the operators under consideration have terms that are product of n mode creation and annihilation operators, we consider a set of three operators that are quadratic in mode creation and annihilation operators. These set of operators satisfy $su(2)$ algebra. Following the method used in [15] , we first derive an inseparability condition for three mode states and prove that to be a necessary and sufficient condition for W type entanglement. Further we provide a general form of inseparability criterion and generalize it to multimode W type states.

Inseparability Condition for three mode states

Let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be the annihilation operators of three modes of electromagnetic field. We define the following set of three operators.

$$\hat{L}_1 = \frac{1}{\sqrt{2}}\big[(\hat{a}+\hat{b})\hat{c}^\dagger + (\hat{a}+\hat{b})^\dagger\hat{c}\big] \tag{1}$$

$$\hat{L}_2 = \frac{i}{\sqrt{2}}\big[(\hat{a}+\hat{b})\hat{c}^\dagger - (\hat{a}+\hat{b})^\dagger\hat{c}\big] \tag{2}$$

$$\hat{L}_3 = \hat{N}_{a+b} - \hat{N}_c \tag{3}$$

where $\hat{N}_{a+b} = \dfrac{(\hat{a}+\hat{b})^\dagger(\hat{a}+\hat{b})}{2}$.

These three operators satisfy $su(2)$ algebra, that is, $[\hat{L}_p, \hat{L}_q] = 2i\epsilon_{pqr}\hat{L}_r$. Calculating the variances of $\hat{L}_1$ and $\hat{L}_2$ and adding them,

$$(\Delta\hat{L}_1)^2 + (\Delta\hat{L}_2)^2 = 2\big[\langle\hat{N}_{a+b}(\hat{N}_c+1)\rangle + \langle(\hat{N}_{a+b}+1)\hat{N}_c\rangle - |\langle(\hat{a}+\hat{b})\hat{c}^\dagger\rangle|^2\big] \tag{4}$$

If the state under consideration is a product state then the sum of variances become,

$$(\Delta\hat{L}_1)^2 + (\Delta\hat{L}_2)^2 = 2\big[\langle\hat{N}_{a+b}\rangle\langle\hat{N}_c+1\rangle + \langle\hat{N}_{a+b}+1\rangle\langle\hat{N}_c\rangle - |\langle(\hat{a}+\hat{b})\rangle\langle\hat{c}^\dagger\rangle|^2\big] \tag{5}$$

The Schwarz inequality implies that $|\langle\hat{a}\rangle|^2 \le \langle\hat{N}_a\rangle$, $|\langle\hat{b}\rangle|^2 \le \langle\hat{N}_b\rangle$ and $|\langle\hat{c}^\dagger\rangle|^2 \le \langle\hat{N}_c\rangle$.

So for a product state, we have the following inequality relation.

$$(\Delta\hat{L}_1)^2 + (\Delta\hat{L}_2)^2 \;\ge\; 2\big[\langle\hat{N}_{a+b}\rangle + \langle\hat{N}_c\rangle\big] \tag{6}$$

This inequality can be extended to any separable state as shown in [15]. Hence an inseparable state should violate this inequality. From eq.(4) and (6), a state is entangled if it satisfies the following condition.

$$|\langle(\hat{a}+\hat{b})\hat{c}^\dagger\rangle|^2 \;>\; 2\langle\hat{N}_{a+b}\hat{N}_c\rangle \tag{7}$$

There exist states that satisfy this inequality relation. To show this, we consider the Heisenberg uncertainty relation of $\hat{L}_1$ and $\hat{L}_2$.

$$(\Delta\hat{L}_1)(\Delta\hat{L}_2) \;\ge\; \frac{1}{2}|[\hat{L}_1, \hat{L}_2]| \tag{8}$$

This implies,

$$(\Delta\hat{L}_1)^2 + (\Delta\hat{L}_2)^2 \;\ge\; |\langle\hat{N}_{a+b} - \hat{N}_c\rangle| \tag{9}$$

This relation holds for any state. This lower bound of the sum of variances of $\hat{L}_1$ and $\hat{L}_2$ is smaller than the lower bound set by separable states. Hence there are states that satisfy eq. (9) and violate eq. (6). Such states are entangled and they obey eq. (7).

The inequality relation represented by eq. (7) is a necessary and sufficient condition for W type entanglement. To prove this, we use the argument of Hillery *et al* [20]. Let $\hat{\rho}$ be the density matrix representing the state of three modes of electromagnetic field. If the reduced density matrix $\hat{\rho}_{ac}$ satisfies the bipartite inseparability condition,

$$|\langle\hat{a}\hat{c}^\dagger\rangle_{\hat{\rho}_{ac}}|^2 \;>\; \langle\hat{N}_a\hat{N}_b\rangle_{\hat{\rho}_{ac}}, \tag{10}$$

then $\hat{\rho}$ cannot be written as either $\sum_j p_j\hat{\rho}_{a_j}\otimes\hat{\rho}_{bc_j}$ or $\sum_j p_j\hat{\rho}_{ab_j}\otimes\hat{\rho}_{c_j}$. Similarly, if the reduced density matrix $\hat{\rho}_{bc}$ satisfies the condition,

$$|\langle\hat{b}\hat{c}^\dagger\rangle_{\hat{\rho}_{bc}}|^2 \;>\; \langle\hat{N}_b\hat{N}_c\rangle_{\hat{\rho}_{bc}}, \tag{11}$$

then $\hat{\rho}$ is not of the forms $\sum_j p_j\hat{\rho}_{ab_j}\otimes\hat{\rho}_{c_j}$ and $\sum_j p_j\hat{\rho}_{ac_j}\otimes\hat{\rho}_{b_j}$.

Thus if the reduced density matrices, $\hat{\rho}_{ac}$ and $\hat{\rho}_{bc}$ satisfy eq. (10) and (11) respectively, then $\hat{\rho}$ represents a genuinely entangled W type state [20] and it satisfies the following condition,

$$|\langle\hat{a}\hat{c}^\dagger\rangle_{\hat{\rho}}|^2 + |\langle\hat{b}\hat{c}^\dagger\rangle_{\hat{\rho}}|^2 \;>\; \langle\hat{N}_a\hat{N}_c\rangle_{\hat{\rho}} + \langle\hat{N}_b\hat{N}_c\rangle_{\hat{\rho}} \tag{12}$$

Since, for separable states, $\langle\hat{a}\hat{c}^\dagger\rangle\langle\hat{b}^\dagger\hat{c}\rangle \;\le\; \langle\hat{a}\hat{b}^\dagger\hat{N}_c\rangle$ and $\langle\hat{b}\hat{c}^\dagger\rangle\langle\hat{a}^\dagger\hat{c}\rangle \;\le\; \langle\hat{b}\hat{a}^\dagger\hat{N}_c\rangle$, eq. (7) is indeed a necessary and sufficient condition for W type entanglement.

There exist a family of inseparability criteria. The most general form of inseparability criterion can be written as

$$|\langle(\hat{a}+\hat{b})^m(\hat{c}^\dagger)^n\rangle|^2 \;>\; \langle[(\hat{a}+\hat{b})^\dagger]^m[(\hat{a}+\hat{b})]^m(\hat{c}^\dagger)^n(\hat{c})^n\rangle \tag{13}$$

where m and n are integers.

The proof of the statement follows from the fact that, for any separable state, the following inequality holds.

$$|\hat{A}\hat{B}^\dagger| \;\leq\; (\langle\hat{A}^\dagger\hat{A}\hat{B}^\dagger\hat{B}\rangle)^{1/2} \tag{14}$$

Details of the proof can be seen in [15]. With $\hat{A} = (\hat{a}+\hat{b})^m$ and $\hat{B} = \hat{c}^n$, it can be verified that eq. (12) is the general form of entanglement condition.

Generalization to multimode states

The inseparability criterion generalized to n-mode ($n \;>\; 3$) states can be written as follows.

$$|\langle(\hat{a}_1+\hat{a}_2+.....+\hat{a}_{n-1})\hat{a}_n^\dagger\rangle|^2 \;>\; (n-1)\langle\hat{N}_{a_1+a_2+.....+a_{n-1}}\hat{N}_{a_n}\rangle \tag{15}$$

where $\hat{N}_{a_1+a_2+.....+a_{n-1}} = \dfrac{(\hat{a}_1+\hat{a}_2+.....+\hat{a}_{n-1})^\dagger(\hat{a}_1+\hat{a}_2+.....+\hat{a}_{n-1})}{(n-1)}$.

The most general form of entanglement condition for n-mode states can be written as

$$|\langle(\hat{a}_1+\hat{a}_2+.....+\hat{a}_{n-1})^p(\hat{a}_n^\dagger)^q\rangle|^2 \;>\; \langle[(\hat{a}_1+\hat{a}_2+.....+\hat{a}_{n-1})^\dagger]^p[(\hat{a}_1+\hat{a}_2+.....+\hat{a}_{n-1})]^p(\hat{a}_n^\dagger)^q(\hat{a}_n)^q\rangle \tag{16}$$

where p and q are integers.

Conclusion

In this paper, we have derived an inequality relation for three mode states of electromagnetic field, using a set of $su(2)$ algebra operators that are quadratic in mode creation and annihilation operators. This relation is shown to be an inseparability criterion for three-mode states by proving that a state satisfying this inequality relation to be entangled. However violation of this condition does not mean that the state is separable. It is further proved that this inseparability criterion to be a necessary and sufficient condition to detect W type entanglement. A general form is provided for a family of such inseparable conditions and they are generalized to multimode states. Thus this paper circumvents the difficulty of testing $n(n-1)/2$ biparatite entanglement conditions for each pair of modes to detect the n-mode W type entanglement, by proposing a necessary and sufficient single inseparaiblity condition that can be experimentally tested.

References

[1] L. Vaidman, *Phys. Rev. A* **49**, 1473 (1994)

[2] S.F. Pereira, Z.Y. Ou and H.J. Kimble, *Phys. Rev. A* **62**, 042311 (2000)

[3] M. Ban, *J. Opt. B: Quantum Semiclass. Opt.* **1**, 9 (1999)

[4] S.L. Braunstein and H.J. Kimble, *Phys. Rev. A* **61**, 042302 (2000)

[5] D. Gottesman and J. Preskill, *Phys. Rev. A* **63**, 022309 (2001)

[6] T. Tyc and B.C. Sanders, *Phys. Rev. A* **65**, 042310 (2002)

[7] S.L. Braunstein, *Phys. Rev. Lett.* **80**, 4084 (1998)

[8] S. Lloyd and J-J. E. Slotine, *Phys. Rev. Lett.* **80**, 4088 (1998)

[9] S.D. Barlett and B.C. Sanders, *Phys. Rev. A* **65**, 042304 (2002)

[10] S. Ghose and B.C. Sanders, *J. Mod. Opt.* **54**, 855 (2007)

[11] L.M. Duan, G. Giedke, J.I. Cirac and P. Zoller, *Phys. Rev. Lett.* **84**, 2722 (2000)

[12] R. Simon, *Phys. Rev. Lett.* **84**, 2726 (2000)

[13] S. Mancini, V. Giovannetti, D. Vitali and P. Tombesi *Phys. Rev. Lett.* **88**, 120401 (2002)

[14] G.S. Agarwal and A. Biswas, *New J. Phys.* **7**, 211 (2005)

[15] M. Hillery and M.S. Zubairy, *Phys. Rev. Lett.* **96**, 050503 (2006)

[16] H. Nha and J. Kim, *Phys. Rev. A* **74**, 012317 (2006)

[17] Z-G. Li, S-M. Fei, Z-X. Wang and K. Wu, *Phys. Rev. A* **75**, 012311 (2007)

[18] H. Nha and J. Kim, *Phys. Rev. A* **75**, 012326 (2007)

[19] K. Selvan, *Eur. Phys. J. D* **72** 71, (2018)

[20] M. Hillery and M.S. Zubairy, *Phys. Rev. A* **74** 032333 (2006)