The generalized Seiberg-Witten equations

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Abstract

We show a set of equations which generalizes the Seiberg-Witten equations

1 Recalls of differential geometry

The $Spin-C$-structures are reductions of a $SO(n).S^1$- fiber bundle to the group $Spin(n)\times_{\{1,-1\}}S^1$. For a four-manifold it exists always a $Spin-C$-structure for the tangent fiber bundle [F].

The DIrac operator is define over the $Spin-C$-structure with help of a connection A for the associated line bundle.

$$\mathcal{D}_A=\sum_i e_i.\nabla^A_{e_i}$$

with ∇^A the connection defined by the Levi-Civita connection and the connection A.

The self-dual part of the curvature (which is a 2-form) of the connection A is considered:

$$\Omega^+_A$$

A 2-form bound to a spinor ψ is also defined by [F]:

$$\omega(\psi)(X,Y)=<X.Y.\psi,\psi>+<X,Y>|\psi|^2$$

2 The Seiberg-Witten equations

The Seiberg-Witten equations are the following ones [F] [M]:
1)

$$\mathcal{D}_A(\psi)=0$$

2)

$$\Omega^+_A=-(1/4)\omega(\psi)$$

3 The generalization of the SW equations

We consider two spinors ψ, ϕ and we define [F] the coupled Seiberg-Witten equations $(A, A', f, g, \psi, \phi)$:
1)
$$\mathcal{D}_A(f\psi) = 0$$
2)
$$\mathcal{D}_{A'}(g\phi) = 0$$
3)
$$\Omega_A^+ = -(1/4)\omega(\psi)$$
4)
$$\Omega_{A'}^+ = -(1/4)\omega(\phi)$$
5)
$$(f^2)^* A = (g^2)^* A'$$
6)
$$fg =< \psi, \bar{\phi} >$$
A, A' are connections $f, g : M \to S^1$.

The gauge group acts:

$$(h, h').(A, A', f, \psi, \phi) = ((1/h^2)^* A, (1/h'^2)^* A', fh, , gh', h\psi, h'\phi)$$

Moreover, the situation can be generalized to n solutions of the Seiberg-Witten equations:
1)
$$\mathcal{D}_{A_i}(f_i\psi_i) = 0$$
2)
$$\Omega_{A_i}^+ = -(1/4)\omega(\psi_i)$$
3)
$$(f_i^2)^* A_i = B$$

4 The compacity of the generalized SW moduli spaces

Theorem 1 *Let (ψ, A) be a solutions of $\mathcal{D}_A\psi = 0, \Omega_A^+ = -(1/4)\omega(\psi)$ over a compact Riemann manifold (M, g) with scalar curvature R. Then at each point,*
$$|\psi(x)|^2 \leq -R_{min}$$
with $R_{min} = min\{R(m), m \in M\}$

The proof is given in [F] p135.

Definition 1 *We define:*

$$M_L = \{(\psi, \phi, A, A', f, g) \in \Gamma(S^+)^2.C(P)^2.Map(M, S^1) : \mathcal{D}_A\psi = \mathcal{D}_{A'}\phi = 0,$$
$$\Omega_A^+ = -(1/4)\omega(\psi), \Omega_{A'}^+ = -(1/4)\omega(\phi), (f^2)^* A = (g^2)^* A'\}/\mathcal{G}$$

Theorem 2 *M_L is compact.*

Proof : Let

$$F(L) = \{\omega \in \Lambda(M) : d\omega = 0, [\omega]_{DR} = c_1(L)\}$$

Since the curvature form is gauge invariant, we obtain a mapping:

$$P : M_L \to F(P), P[A, A', \psi, \phi, f, g] = \Omega_A = \Omega_{A'}$$

4.1 First step

$P(M_L) \to F(L)$ is a compact subset.

The proof is given in [F] P136-137.

4.2 Second step

Let be $(P_1, P_2) : M_L \to \mathcal{C}(P)^2/\mathcal{G}(P)^2$,

$$P_1(\psi, \phi, A, A', f, g) = A,$$

and

$$P_2(\psi, \phi, A, A', f, g) = A',$$

then $(P_1, P_2)(M_L) \subset \mathcal{C}(P)^2/\mathcal{G}(P)^2$ is a compact subset.

We use Weyl's theorem. The mapping $\mathcal{C}^2(P)/\mathcal{G}(P)^2 \to F(P), (A, A') \to \Omega_A = \Omega_{A'}$ is a fibration with compact fiber $Pic(M) = H^1(M, \mathbf{R})/H^1(M, \mathbf{Z})$. The following diagram commutes:

$$\begin{array}{ccc} M_L & \to^{(P_1,P_2)} & \mathcal{C}(P)^2/G(P)^2 \\ \downarrow^P & & \downarrow \\ F(L) & = & F(L) \end{array}$$

$(P_1, P_2)(M_L) \subset \mathcal{C}(P)^2/\mathcal{G}(P)^2$ is a compact subset.

4.3 Third step

Let be $(F/G) : M_L \to \mathcal{G}(P)$, $(F/G)(\psi, \phi, A, A', f, g) = f/g$, then $(F/G)(M_L) \subset \mathcal{G}(P)$ is a compact subset.

Consider the maps: $K, K' : M_L \to \Lambda^1(M)$,

$$K(A, A', \psi, \phi, f, g) = \frac{df}{f}$$

$$K'(A, A', \psi, \phi, f, g) = \frac{dg}{g}$$

then $(K - K')(M_L) \subset \Lambda^1(M)$ is compact. Indeed, $2\frac{df}{f} - 2\frac{dg}{g} = A' - A$ which is in a compact set. And the fiber is $\frac{df}{f} = \frac{dg}{g}$, $f/g = cst \in S^1$.

4.4 Fourth step: M_L is compact

$(P_1, P_2)^{-1}(A, A') \cap (F/G)^{-1}(f/g)$ consists of the solutions of

$$\mathcal{D}_A f\psi = \mathcal{D}_{A'} g\phi = 0, max(|\psi(x)|, |\phi(x)|) \leq -R_{min}$$

$$fg = < \psi, \bar{\phi} >$$

This is a bounded ball in a finite-dimensional vector space. The system is of finite dimension.

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