On the represenation of positive integers by the sum of prime numbers

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Abstract

The main objective of this short note is prove that some statements concerning the represenation of positive integers by the sum of prime numbers are equivalent to some true trivial cases. This implies that these statements are also true. The analysis is based on a new prime formula and some trigonometric expressions.

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1 Introduction

To be completed.

2 Preliminary results

Let $(p_m)_{m\in\mathbb{N}}$ denotes the sequence of odd primes. Let $x \in \mathbb{R}$, then $\lfloor x \rfloor$ denotes the floor function, i.e., the largest integer not greater than x. We have

$$\lfloor x \rfloor = j \Leftrightarrow j \leq x < j + 1$$

and for all $j \in \mathbb{N}$, we have

$$\lfloor x + j \rfloor = \lfloor x \rfloor + j \tag{1}$$

Let $r \geq 4$ be any positive integer and define

$$\alpha_r = \sum_{m=1}^{+\infty} p_m r^{-m^2} \tag{2}$$

Equality (2) was proposed in [1] as a series defined by primes. We have proved the following result:

Lemma 1 *The series (2) defining α_r is convergent.*

Proof. We have $p_m \leq r^m$ because by induction: $p_m = 2 < 4 \leq r^1 = r$. Assume that $p_m \leq r^m$, then by the Bertrand's Postulate, we have $p_{m+1} < 2 \times p_m < 2 \times r^m < r \times r^m = r^{m+1}$. So the series defining α_r is convergent since via the Cauchy root test we have:

$$\sqrt[m]{\frac{p_m}{r^{m^2}}} < \sqrt[m]{\frac{r^m}{r^{m^2}}} = \frac{1}{r^{m-1}} \leq \frac{1}{4^{m-1}} \leq \frac{1}{4} < 1.$$

■

Let

$$\delta_d = \sum_{m=1}^{d} p_m r^{-m^2}, d = 1, 2, ...$$

be the partial sum of the convergent series defining α_r. Then we have

Lemma 2 *The following inequalities*

$$\begin{cases} 0 < r^{d^2} \displaystyle\sum_{m=1+d}^{\infty} p_m r^{-m^2} < 1 \\ \\ 0 < r^{d^2} \displaystyle\sum_{m=1+d}^{n-1} p_m r^{-m^2} < 1 \end{cases} \tag{3}$$

holds true for all indices $d \geq 1$ and n verifying $n \geq d + 2$.

Proof. To prove the first statement of (3), let $m = d + 1 + l$, then

$$\begin{aligned} 0 &< r^{d^2} \sum_{m=1+d}^{\infty} p_m r^{-m^2} < \sum_{m=1+d}^{\infty} r^{d^2+m-m^2} = \sum_{l=0}^{\infty} r^{-\left(l+d+2ld+l^2\right)} \\ &\leq \sum_{l=0}^{\infty} r^{-(l+d+2ld)} = \frac{r^{2d+1}}{r^d\left(r^{2d+1}-1\right)} < 1. \end{aligned} \tag{4}$$

because $p_m \leq r^m$ and $d^2 + m - m^2 = -\left(l + d + 2ld + l^2\right) \leq -\left(l + d + 2ld\right)$ and $r^d\left(r^{2d+1}-1\right) - r^{2d+1} = r^d\left(r^{d+1}\left(r^d-1\right)-1\right) > 0$, so, $\frac{r^{2d+1}}{r^d\left(r^{2d+1}-1\right)} < 1$.
The second statement of (3) follows from the first one since

$$0 < r^{d^2} \sum_{m=1+d}^{n-1} p_m r^{-m^2} < r^{d^2} \sum_{m=1+d}^{\infty} p_m r^{-m^2} < 1$$

The condition $n \geq d + 2$ on the indices d and n implies that the sum $\sum\limits_{m=1+d}^{n-1}$ is well defined and not zero since it contain at least one term.

From (4) we conclud that

$$\begin{cases} \left\lfloor r^{d^2}\alpha_r \right\rfloor = r^{d^2}\delta_d, d = 1, 2, ... \\ \delta_d = \delta_{d-1} + \frac{p_d}{r^{d^2}}, d = 1, 2, ... \end{cases} \tag{5}$$

Indeed, the first equation of (5) is verified since by (1) we get

$$\left\lfloor r^{d^2}\alpha_r \right\rfloor = \left\lfloor r^{d^2} \sum_{m=1}^{\infty} p_m r^{-m^2} \right\rfloor = r^{d^2}\delta_d + \left\lfloor r^{d^2} \sum_{m=1+d}^{\infty} p_m r^{-m^2} \right\rfloor = r^{d^2}\delta_d$$

since $r^{d^2}\delta_d \in \mathbb{N}$ and $0 < r^{d^2} \sum\limits_{m=1+d}^{\infty} p_m r^{-m^2} < 1$ by (4). ∎

Define the sequence $(H_d)_{d\geq 2}$ by:

$$H_d = \left\lfloor r^{d^2}\left(\alpha_r - \delta_{d-1}\right) \right\rfloor \tag{6}$$

Lemma 3 *For all indice $d \geq 2$, we have*

$$H_d = p_d$$

Proof. We have $H_d \in \mathbb{N}$ because it is the integer part of the positive real number $r^{d^2}(\alpha_r - \delta_{d-1})$ since for all $d \geq 2$ we have $\alpha_r > \delta_{d-1}$. It is clear that $H_d = p_d$ becuase

$$\begin{aligned} H_d &= \left\lfloor r^{d^2}\alpha_r \right\rfloor - r^{d^2}\delta_{d-1} = r^{d^2}\delta_d - r^{d^2}\delta_{d-1} \\ &= r^{d^2}\left(\delta_{d-1} + \frac{p_d}{r^{d^2}}\right) - r^{d^2}\delta_{d-1} = p_d \end{aligned}$$

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Thus, the sequence (6) is a prime number but written in another equivalent and appropriate form. The new prime formula (6) will be used in the proof of Theorem 7 below.

Lemma 4 *For all $d \geq 2$, the real number $r^{d^2}(\alpha_r - \delta_{d-1})$ is not an integer.*

Proof. We have

$$\begin{aligned} r^{d^2}(\alpha_r - \delta_{d-1}) &= r^{d^2}\left(\sum_{m=1}^{+\infty} p_m r^{-m^2} - \sum_{m=1}^{d-1} p_m r^{-m^2}\right) \\ &= r^{d^2}\left(\sum_{m=d}^{+\infty} p_m r^{-m^2}\right) \\ &= r^{d^2}\left(p_d r^{-d^2} + \sum_{m=1+d}^{+\infty} p_m r^{-m^2}\right) \\ &= p_d + r^{d^2}\sum_{m=1+d}^{+\infty} p_m r^{-m^2} \end{aligned}$$

is not an intgere since $0 < r^{d^2}\sum_{m=1+d}^{\infty} p_m r^{-m^2} < 1$ by Lemma 2. ■

Lemma 5 *For all non-integer x we have*

$$\lfloor x \rfloor = -\frac{1}{2} + x + \frac{\arctan(\cot \pi x)}{\pi} \tag{7}$$

Proof. The cotangent function has period π, then the cotangent of πx has period 1. By definition, the arc tangent function is defined from $\mathbb{R}$ to$\left]\frac{-\pi}{2}, \frac{\pi}{2}\right[$. Hence, for all x not an integer we have:

$$\frac{\arctan(\cot \pi x)}{\pi} = \frac{\arctan\left(\tan\left(\frac{\pi}{2} - \pi x\right)\right)}{\pi} = \frac{1}{2} - x \bmod 1$$

But, we have

$$-\frac{1}{2}+\frac{1}{2}+x-x \bmod 1 = x-\{x\} = \lfloor x \rfloor$$

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Lemma 6 *For all real x and y we have*

$$\arctan x + \arctan y = \begin{cases} \arctan\left(\frac{x+y}{1-xy}\right), \ \textit{if } xy<1 \\ \textit{or} \\ \pi + \arctan\left(\frac{x+y}{1-xy}\right), \ \textit{if } x>0, y>0, xy>1 \\ \textit{or} \\ -\pi + \arctan\left(\frac{x+y}{1-xy}\right), \ \textit{if } x<0, y<0, xy>1 \\ \textit{or} \\ \frac{\pi}{2}, \ \textit{if } x>0, xy=1 \\ \textit{or} \\ \frac{-\pi}{2}, \ \textit{if } x<0, xy=1 \end{cases} \tag{8}$$

Proof. Let $-\frac{\pi}{2} < \gamma = \arctan u < \frac{\pi}{2}$, $-\frac{\pi}{2} < \alpha = \arctan x < \frac{\pi}{2}, -\frac{\pi}{2} < \beta = \arctan y < \frac{\pi}{2}$, then we have $\tan\gamma = \tan(\alpha+\beta) = \frac{x+y}{1-xy} = u$. If $xy > 1$, i.e., x and y have the same sign, then we have the two cases:

(a) $x > 0, y > 0 \to u < 0 \to \gamma < 0 \to \alpha+\beta = \gamma+\pi$, i.e., $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) + \pi$

(b) $x < 0, y < 0 \to u > 0 \to \gamma > 0 \to \alpha+\beta = \gamma-\pi$, i.e., $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) - \pi$

For the last two formulas: If $x > 0$, then $0 < \arctan x < \frac{\pi}{2}$, and if $t = \arctan x$, then $\cot t = \frac{1}{\tan t} = \frac{1}{x}$, so that $\arctan\frac{1}{x} = \frac{\pi}{2} - t$. If $x > 0$, then $\arctan x + \arctan\frac{1}{x} = \frac{\pi}{2}$. As arctan is an odd function, if $x < 0$, then $\arctan x + \arctan\frac{1}{x} = -\frac{\pi}{2}$. ■

Let us consider the following conjecture:

Conjecture 7 *For all positive integer $k \geq 3$, there exist two primes $p_{\lambda_k} \geq 3, p_{\mu_k} \geq 3$ such that*

$$p_{\lambda_k} + p_{\mu_k} = 2k \tag{9}$$

By formula (6), we have

$$\begin{cases} p_{\lambda_k} = \left\lfloor r^{\lambda_k^2}\left(\alpha_r - \delta_{\lambda_k - 1}\right)\right\rfloor \\ \\ p_{\mu_k} = \left\lfloor r^{\mu_k^2}\left(\alpha_r - \delta_{\mu_k - 1}\right)\right\rfloor \end{cases} \tag{10}$$

Define

$$\begin{cases} x_k = r^{\lambda_k^2}\left(\alpha_r - \delta_{\lambda_k - 1}\right) = p_{\lambda_k} + r^{\lambda_k^2} \sum_{m=1+\lambda_k}^{\infty} p_m r^{-m^2} = p_{\lambda_k} + z_k \\ \\ y_k = r^{\mu_k^2}\left(\alpha_r - \delta_{\mu_k - 1}\right) = p_{\mu_k} + r^{\mu_k^2} \sum_{m=1+\mu_k}^{\infty} p_m r^{-m^2} = p_{\mu_k} + t_k \end{cases} \tag{11}$$

We have proved the following result:

Theorem 8 *The Conjecture 7 is equivalent to the true trivial case: For all $k \geq 3$, there exist two positive integers $\lambda_k \geq 2, \mu_k \geq 2$ such that*

$$\begin{cases} 0 < t_k + z_k < \frac{1}{2} \text{ or } z_k + t_k = \frac{1}{2} \text{ or } \frac{1}{2} < z_k + t_k < \frac{3}{2} \\ \text{or} \\ z_k + t_k = \frac{3}{2} \text{ or } \frac{3}{2} < t_k + z_k < 2 \end{cases}$$

Proof. The fact that x_k and y_k are not integers by Lemma 4 and from (7) and (9) we get

$$\begin{aligned} p_{\lambda_k} + p_{\mu_k} &= 2k \\ &\Leftrightarrow \lfloor x_k \rfloor + \lfloor y_k \rfloor = 2k \\ &\Leftrightarrow -\frac{1}{2} + x_k + \frac{\arctan\left(\cot \pi x_k\right)}{\pi} - \frac{1}{2} + y_k + \frac{\arctan\left(\cot \pi y_k\right)}{\pi} = 2k \\ &\Leftrightarrow \arctan\left(\cot \pi x_k\right) + \arctan\left(\cot \pi y_k\right) = \pi\left(2k - x_k - y_k + 1\right) \end{aligned}$$

Here x_k and y_k depends on the positive integers indices $\lambda_k \geq 2, \mu_k \geq 2$. Hence, the Conjecture 7 is equivalent to the case: For all $k \geq 3$, there exist two positive integers $\lambda_k \geq 2, \mu_k \geq 2$ such that

$$\arctan\left(\cot \pi x_k\right) + \arctan\left(\cot \pi y_k\right) = \pi\left(2k - x_k - y_k + 1\right) \tag{12}$$

By definition, the arc tangent function is defined from $\mathbb{R}$ to $\left]\frac{-\pi}{2}, \frac{\pi}{2}\right[$. We have

$$\arctan u_k + \arctan v_k = \begin{cases} \arctan\left(\frac{u_k+v_k}{1-u_kv_k}\right), \text{ if } u_kv_k < 1 \\ or \\ \pi + \arctan\left(\frac{u_k+v_k}{1-u_kv_k}\right), \text{ if } u_k > 0 \text{ and } v_k > 0 \text{ and } u_kv_k > 1 \\ or \\ -\pi + \arctan\left(\frac{u_k+v_k}{1-u_kv_k}\right), \text{ if } u_k < 0 \text{ and } v_k < 0 \text{ and } u_kv_k > 1 \\ or \\ \frac{\pi}{2}, \text{ if } u_k > 0 \text{ and } v_k = \frac{1}{u_k} \\ or \\ \frac{-\pi}{2}, \text{ if } u_k < 0 \text{ and} v_k = \frac{1}{u_k} \end{cases}$$

Let $u_k = \cot \pi x_k, v_k = \cot \pi y_k$, then we get

$$\begin{cases} \arctan\left(\frac{u_k+v_k}{1-u_kv_k}\right) = \pi\left(2k - x_k - y_k + 1\right), \text{ if } u_kv_k < 1 \\ or \\ \pi + \arctan\left(\frac{u_k+v_k}{1-u_kv_k}\right) = \pi\left(2k - x_k - y_k + 1\right), \text{ if } u_k > 0 \text{ and } v_k > 0 \text{ and } u_kv_k > 1 \\ or \\ -\pi + \arctan\left(\frac{u_k+v_k}{1-u_kv_k}\right) = \pi\left(2k - x_k - y_k + 1\right), \text{ if } u_k < 0 \text{ and } v_k < 0 \text{ and } u_kv_k > 1 \\ or \\ \frac{\pi}{2} = \pi\left(2k - x_k - y_k + 1\right), \text{ if } u_k > 0 \text{ and } v_k = \frac{1}{u_k} \\ or \\ \frac{-\pi}{2} = \pi\left(2k - x_k - y_k + 1\right), \text{ if } u_k < 0 \text{ and} v_k = \frac{1}{u_k} \end{cases}$$

that is,

$$\begin{cases} \arctan\left(\frac{u_k+v_k}{1-u_kv_k}\right) = \pi\left(2k - x_k - y_k + 1\right), \text{ if } u_kv_k < 1 \\ or \\ \arctan\left(\frac{u_k+v_k}{1-u_kv_k}\right) = \pi\left(2k - x_k - y_k\right), \text{ if } u_k > 0 \text{ and } v_k > 0 \text{ and } u_kv_k > 1 \\ or \\ \arctan\left(\frac{u_k+v_k}{1-u_kv_k}\right) = \pi\left(2k - x_k - y_k + 2\right), \text{ if } u_k < 0 \text{ and } v_k < 0 \text{ and } u_kv_k > 1 \\ or \\ 4k - 2x_k - 2y_k + 1 = 0, \text{ if } u_k > 0 \text{ and } v_k = \frac{1}{u_k} \\ or \\ 4k - 2x_k - 2y_k + 3 = 0, \text{ if } u_k < 0 \text{ and } v_k = \frac{1}{u_k} \end{cases} \tag{13}$$

We have

$$\begin{aligned}\arctan\left(\frac{u_k+v_k}{1-u_kv_k}\right) &= \arctan\left(\frac{(\cot\pi x_k)+(\cot\pi y_k)}{1-(\cot\pi x_k)(\cot\pi y_k)}\right)\\ &= \arctan\left(\tan\left(-\pi x_k-\pi y_k\right)\right) = -\pi x_k-\pi y_k+j_k\pi\end{aligned}$$

for some integer j_k. Thus, (13) becomes

$$\left\{\begin{array}{c} -\pi x_k-\pi y_k+j_k\pi=\pi\left(2k-x_k-y_k+1\right)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\ \text{ and } u_kv_k<1\\ or\\ -\pi x_k-\pi y_k+j_k\pi=\pi\left(2k-x_k-y_k\right)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\ \text{ and}\\ u_k>0 \text{ and } v_k>0 \text{ and } u_kv_k>1\\ or\\ -\pi x_k-\pi y_k+j_k\pi=\pi\left(2k-x_k-y_k+2\right)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\ \text{ and}\\ u_k<0 \text{ and } v_k<0 \text{ and } u_kv_k>1\\ or\\ 4k-2x_k-2y_k+1=0,\ \text{if } u_k>0 \text{ and } v_k=\frac{1}{u_k}\\ or\\ 4k-2x_k-2y_k+3=0,\ \text{if } u_k<0 \text{ and } v_k=\frac{1}{u_k}\end{array}\right. \tag{14}$$

So, finding an integer j_k in the first three equations of (14) implies that they are verified. The fact that $\arctan(x)\in\left]\frac{-\pi}{2},\frac{\pi}{2}\right[$ implies that in the first three cases of (14) we must assume that the corresponding quantitty is located in the interval $\left]\frac{-\pi}{2},\frac{\pi}{2}\right[$:

$$\left\{\begin{array}{c} j_k=2k+1 \text{ and } \pi\left(2k-x_k-y_k+1\right)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\ \text{ and } u_kv_k<1\\ or\\ j_k=2k \text{ and } \pi\left(2k-x_k-y_k\right)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\ \text{ and}\\ u_k>0 \text{ and } v_k>0 \text{ and } u_kv_k>1\\ or\\ j_k=2k+2 \text{ and } \pi\left(2k-x_k-y_k+2\right)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\ \text{ and}\\ u_k<0 \text{ and } v_k<0 \text{ and } u_kv_k>1\\ or\\ 4k-2x_k-2y_k+1=0,\ \text{if } u_k>0 \text{ and } v_k=\frac{1}{u_k}\\ or\\ 4k-2x_k-2y_k+3=0,\ \text{if } u_k<0 \text{ and } v_k=\frac{1}{u_k}\end{array}\right. \tag{15}$$

We have

$$\left\{\begin{array}{l} u_k=\cot\pi x_k=\cot\pi\left(p_{\lambda_k}+z_k\right)=\cot\pi z_k\\ \\ v_k=\cot\pi y_k=\cot\pi\left(p_{\mu_k}+t_k\right)=\cot\pi t_k\end{array}\right.$$

since the cotangent function has period π. Thus, (15) becomes

$$\begin{cases} j_k = 2k+1 \text{ and } \pi(2k - x_k - y_k + 1) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ and } (\cot \pi z_k)(\cot \pi t_k) < 1 \\ \textit{or} \\ j_k = 2k \text{ and } \pi(2k - x_k - y_k) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ and } \cot \pi z_k > 0 \text{ and} \\ \cot \pi t_k > 0 \text{ and } (\cot \pi z_k)(\cot \pi t_k) > 1 \\ \textit{or} \\ j_k = 2k+2 \text{ and } \pi(2k - x_k - y_k + 2) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ and} \\ \cot \pi z_k < 0 \text{ and } \cot \pi t_k < 0 \text{ and } (\cot \pi z_k)(\cot \pi t_k) > 1 \\ \textit{or} \\ 2\left(2k - p_{\lambda_k} - p_{\mu_k}\right) - 2t_k - 2z_k + 1 = 0, \text{ if } \cot \pi z_k > 0 \text{ and } \cot \pi t_k = \frac{1}{\cot \pi z_k} \\ \textit{or} \\ 2\left(2k - p_{\lambda_k} - p_{\mu_k}\right) - 2t_k - 2z_k + 3 = 0, \text{ if } \cot \pi z_k < 0 \text{ and } \cot \pi t_k = \frac{1}{\cot \pi z_k} \end{cases} \tag{16}$$

We have $\cot(\pi z_k)\cot(\pi t_k) = 1$ if $\cos\pi(z_k + t_k) = 0$, that is, $z_k + t_k = \frac{1}{2}$ or $z_k + t_k = \frac{3}{2}$ because $0 < z_k < 1$ and $0 < t_k < 1$. Hence, (16) is equivalent to

$$\begin{cases} -\frac{\pi}{2} < \pi(2k - x_k - y_k + 1) < \frac{\pi}{2} \text{ and } (\cot \pi z_k)(\cot \pi t_k) < 1 \\ \textit{or} \\ -\frac{\pi}{2} < \pi(2k - x_k - y_k) < \frac{\pi}{2} \text{ and } \cot \pi z_k > 0 \text{ and} \\ \cot \pi t_k > 0 \text{ and } (\cot \pi z_k)(\cot \pi t_k) > 1 \\ \textit{or} \\ -\frac{\pi}{2} < \pi(2k - x_k - y_k + 2) < \frac{\pi}{2} \text{ and } \cot \pi z_k < 0 \text{ and} \\ \cot \pi t_k < 0 \text{ and } (\cot \pi z_k)(\cot \pi t_k) > 1 \\ \textit{or} \\ z_k + t_k = \frac{1}{2}, \text{ if } \cot \pi z_k > 0 \text{ and } z_k + t_k = \frac{1}{2} \\ \textit{or} \\ z_k + t_k = \frac{3}{2}, \text{ if } \cot \pi z_k < 0 \text{ and } z_k + t_k = \frac{3}{2} \end{cases} \tag{17}$$

We have

$$\cot(\pi z_k)\cot(\pi t_k) > 1 \Leftrightarrow \frac{\cos(\pi z_k)}{\sin(\pi z_k)}\frac{\cos(\pi t_k)}{\sin(\pi t_k)} > 1$$

The inequality $\sin(\pi t) > 0$ is verified for all $0 < t < 1$. Then

$$\cot(\pi z_k)\cot(\pi t_k) > 1 \Leftrightarrow \cos(\pi z_k)\cos(\pi t_k) - \sin(\pi z_k)\sin(\pi t_k) = \cos\pi(z_k + t_k) > 0$$

that is,

$$0 < z_k + t_k < \frac{1}{2} \text{ or } \frac{3}{2} < z_k + t_k < 2$$

Also,

$$\begin{aligned}\cot(\pi z_k)\cot(\pi t_k) &< 1 \Leftrightarrow \frac{\cos(\pi z_k)}{\sin(\pi z_k)}\frac{\cos(\pi t_k)}{\sin(\pi t_k)} < 1 \\ &\Leftrightarrow \cos\pi(z_k + t_k) < 0 \Leftrightarrow \frac{1}{2} < z_k + t_k < \frac{3}{2}\end{aligned}$$

Also, $\cot(\pi t) > 0$, if $0 < t < \frac{1}{2}$ and $\cot(\pi t) < 0$, if $\frac{1}{2} < t < 1$. Thus, (17) becomes

$$\begin{cases} -\frac{1}{2} < 2k - p_{\lambda_k} - p_{\mu_k} - t_k - z_k + 1 < \frac{1}{2} \text{ and } \frac{1}{2} < z_k + t_k < \frac{3}{2} \\ \textit{or} \\ -\frac{1}{2} < 2k - p_{\lambda_k} - p_{\mu_k} - t_k - z_k < \frac{1}{2} \text{ and } \cot\pi z_k > 0 \text{ and } \cot\pi t_k > 0 \text{ and} \\ 0 < z_k + t_k < \frac{1}{2} \text{ or } \frac{3}{2} < z_k + t_k < 2 \\ \textit{or} \\ -\frac{1}{2} < 2k - p_{\lambda_k} - p_{\mu_k} - t_k - z_k + 2 < \frac{1}{2} \text{ and } \cot\pi z_k < 0 \text{ and } \cot\pi t_k < 0 \text{ and} \\ 0 < z_k + t_k < \frac{1}{2} \text{ or } \frac{3}{2} < z_k + t_k < 2 \\ \textit{or} \\ z_k + t_k = \frac{1}{2} \text{ and } \cot\pi z_k > 0 \text{ and } z_k + t_k = \frac{1}{2} \\ \textit{or} \\ z_k + t_k = \frac{3}{2} \text{ and } \cot\pi z_k < 0 \text{ and } z_k + t_k = \frac{3}{2} \end{cases}$$

By using (9) we get

$$\begin{cases} \frac{1}{2} < z_k + t_k < \frac{3}{2} \text{ and } \frac{1}{2} < z_k + t_k < \frac{3}{2} \\ \textit{or} \\ 0 < t_k + z_k < \frac{1}{2} \text{ and } 0 < z_k < \frac{1}{2} \text{ and } 0 < t_k < \frac{1}{2} \text{ and} \\ 0 < z_k + t_k < \frac{1}{2} \text{ or } \frac{3}{2} < z_k + t_k < 2 \\ \textit{or} \\ \frac{3}{2} < t_k + z_k < \frac{5}{2} \text{ and } \frac{1}{2} < z_k < 1 \text{ and } \frac{1}{2} < t_k < 1 \text{ and} \\ 0 < z_k + t_k < \frac{1}{2} \text{ or } \frac{3}{2} < z_k + t_k < 2 \\ \textit{or} \\ z_k + t_k = \frac{1}{2} \text{ and } 0 < z_k < \frac{1}{2} \text{ and } z_k + t_k = \frac{1}{2} \\ \textit{or} \\ z_k + t_k = \frac{3}{2} \text{ and } \frac{1}{2} < z_k < 1 \text{ and } z_k + t_k = \frac{3}{2} \end{cases}$$

i.e.,

$$\left\{\begin{array}{c} 0 < t_k + z_k < \frac{1}{2} \text{ and } 0 < z_k < \frac{1}{2} \text{ and } 0 < t_k < \frac{1}{2} \\ \textit{or} \\ z_k + t_k = \frac{1}{2} \text{ and } 0 < z_k < \frac{1}{2} \\ \textit{or} \\ \frac{1}{2} < z_k + t_k < \frac{3}{2} \\ \textit{or} \\ z_k + t_k = \frac{3}{2} \text{ and } \frac{1}{2} < z_k < 1 \\ \textit{or} \\ \frac{3}{2} < t_k + z_k < 2 \text{ and } \frac{1}{2} < z_k < 1 \text{ and } \frac{1}{2} < t_k < 1 \end{array}\right. \tag{18}$$

The assumption $0 < t_k + z_k < \frac{1}{2}$ implies that $0 < z_k < \frac{1}{2}$ and $0 < t_k < \frac{1}{2}$. The assumption $z_k + t_k = \frac{1}{2}$ implies that $0 < z_k < \frac{1}{2}$ and $0 < t_k < \frac{1}{2}$. The assumption $z_k + t_k = \frac{3}{2}$ implies that $\frac{1}{2} < z_k < 1$ and $\frac{1}{2} < t_k < 1$. The assumption $\frac{3}{2} < t_k + z_k < 2$ implies that $\frac{1}{2} < z_k < 1$ and $\frac{1}{2} < t_k < 1$ since we have $0 < t_k < 1$ and $0 < z_k < 1$. Otherwise all these implications are not true. Thus, (18) can be reduced to:

$$\left\{\begin{array}{c} 0 < t_k + z_k < \frac{1}{2} \text{ or } z_k + t_k = \frac{1}{2} \text{ or } \frac{1}{2} < z_k + t_k < \frac{3}{2} \\ \text{or} \\ z_k + t_k = \frac{3}{2} \text{ or } \frac{3}{2} < t_k + z_k < 2 \end{array}\right. \tag{19}$$

The last statement (19) is true for all $k \geq 3$ since $0 < t_k < 1$ and $0 < z_k < 1$ and it is a trivial case as it presents all the locations of the real number $z_k + t_k$. ■

References

[1] G. H. Hardy and E. M. Wright, An introduction to the theory of numbers, Oxford University Press (1960), § 22.3, 4th edition, pp, 343-345.

[2] To be completed.