The Pochhammer's Symbol at Rational Argument and the Finite Product of Gamma Functions

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"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life." - John 6:63.

Abstract. I derive some finite product representations of gamma functions for the Pochhammer's symbol at rational argument.

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1. Introduction

In present paper, I derive the identity below

$$\left(\frac{p}{q}\right)_n = q^n \cdot \prod_{k=0}^{q-1} \frac{\Gamma\left(\frac{p+kq}{q^2} + \frac{n}{q}\right)}{\Gamma\left(\frac{p+kq}{q^2}\right)},$$

which enabled me to prove the following finite products of gamma functions

$$\left(\frac{1}{2}\right)_n = 2^n \cdot \frac{\Gamma\left(\frac{1}{4}+\frac{n}{2}\right)\Gamma\left(\frac{3}{4}+\frac{n}{2}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)},$$

$$\left(\frac{1}{3}\right)_n = 3^n \cdot \frac{\Gamma\left(\frac{1}{9}+\frac{n}{3}\right)\Gamma\left(\frac{4}{9}+\frac{n}{3}\right)\Gamma\left(\frac{7}{9}+\frac{n}{3}\right)}{\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{7}{9}\right)},$$

$$\left(\frac{2}{3}\right)_n = 3^n \cdot \frac{\Gamma\left(\frac{2}{9}+\frac{n}{3}\right)\Gamma\left(\frac{5}{9}+\frac{n}{3}\right)\Gamma\left(\frac{8}{9}+\frac{n}{3}\right)}{\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{8}{9}\right)}$$

and so on.

2. Preliminary

The Gauss multiplication formula for gamma function assures me that

Theorem 2.1. *(Gauss)*

$$\forall z \notin \left\{-\frac{m}{n}: m \in \mathbb{N}\right\}: \prod_{k=0}^{n-1} \Gamma\left(z+\frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{1/2-nz} \Gamma(nz), \tag{2.1}$$

where $\Gamma(z)$ *denotes the gamma function.*

Proof. See [1]. □

3. THE MAIN THEOREM

3.1. The Pochhammer symbol at rational argument.

Theorem 3.1. *If* p *and* q *are positive integers and* $p \leqslant q$*, then*

$$\left(\frac{p}{q}\right)_n = q^n \cdot \prod_{k=0}^{q-1} \frac{\Gamma\left(\frac{p+kq}{q^2}+\frac{n}{q}\right)}{\Gamma\left(\frac{p+kq}{q^2}\right)}. \tag{3.1}$$

Proof. I define the following function, by virtue of the right hand side of (3.1)

$$\begin{aligned} E_n(p,q) &:= q^n \cdot \prod_{k=0}^{q-1} \frac{\Gamma\left(\frac{p+kq}{q^2}+\frac{n}{q}\right)}{\Gamma\left(\frac{p+kq}{q^2}\right)} \\ &= q^n \cdot \prod_{k=0}^{q-1} \frac{\Gamma\left(\frac{p}{q^2}+\frac{n}{q}+\frac{k}{q}\right)}{\Gamma\left(\frac{p}{q^2}+\frac{k}{q}\right)} \\ &= q^n \cdot \frac{\prod_{k=0}^{q-1}\Gamma\left(\frac{p}{q^2}+\frac{n}{q}+\frac{k}{q}\right)}{\prod_{k=0}^{q-1}\Gamma\left(\frac{p}{q^2}+\frac{k}{q}\right)} \\ &= q^n \cdot \frac{\Gamma\left(\frac{p}{q^2}+\frac{n}{q}\right)\prod_{k=1}^{q-1}\Gamma\left(\frac{p}{q^2}+\frac{n}{q}+\frac{k}{q}\right)}{\Gamma\left(\frac{p}{q^2}\right)\prod_{k=1}^{q-1}\Gamma\left(\frac{p}{q^2}+\frac{k}{q}\right)}. \end{aligned} \tag{3.2}$$

From Theorem 2.1, I obtain with a bit of manipulation

$$\begin{aligned} &\Gamma(z)\prod_{k=1}^{n-1}\Gamma\left(z+\frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{1/2-nz}\Gamma(nz) \\ &\Rightarrow \prod_{k=1}^{n-1}\Gamma\left(z+\frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{1/2-nz}\frac{\Gamma(nz)}{\Gamma(z)}. \end{aligned} \tag{3.3}$$

Replace n by q in (3.3)

$$\prod_{k=1}^{q-1}\Gamma\left(z+\frac{k}{q}\right) = (2\pi)^{(q-1)/2} q^{1/2-qz}\frac{\Gamma(qz)}{\Gamma(z)}. \tag{3.4}$$

Replace z by $\frac{p}{q^2}+\frac{n}{q}$ in (3.4)

$$\prod_{k=1}^{q-1}\Gamma\left(\frac{p}{q^2}+\frac{n}{q}+\frac{k}{q}\right)=(2\pi)^{(q-1)/2}q^{1/2-p/q-n}\frac{\Gamma\left(\frac{p}{q}+n\right)}{\Gamma\left(\frac{p}{q^2}+\frac{n}{q}\right)}. \tag{3.5}$$

Replace z by $\frac{p}{q^2}$ in (3.4)

$$\prod_{k=1}^{q-1}\Gamma\left(\frac{p}{q^2}+\frac{k}{q}\right)=(2\pi)^{(q-1)/2}q^{1/2-p/q}\frac{\Gamma\left(\frac{p}{q}\right)}{\Gamma\left(\frac{p}{q^2}\right)}. \tag{3.6}$$

From (3.2), (3.5) and (3.6), it follows that

$$E_n(p,q)=q^n\cdot\frac{\Gamma\left(\frac{p}{q^2}+\frac{n}{q}\right)(2\pi)^{(q-1)/2}q^{1/2-p/q-n}\frac{\Gamma\left(\frac{p}{q}+n\right)}{\Gamma\left(\frac{p}{q^2}+\frac{n}{q}\right)}}{\Gamma\left(\frac{p}{q^2}\right)(2\pi)^{(q-1)/2}q^{1/2-p/q}\frac{\Gamma\left(\frac{p}{q}\right)}{\Gamma\left(\frac{p}{q^2}\right)}}=\frac{\Gamma\left(\frac{p}{q}+n\right)}{\Gamma\left(\frac{p}{q}\right)}=\left(\frac{p}{q}\right)_n,$$

which is the desired result.The last step is justified by the definition of the Pochhammer symbol $\Gamma(z+n)/\Gamma(z):=(z)_n$ [2]. This completes the proof. □

Example 3.2. Set $p=1$ and $q=2$ in Theorem 1

$$\left(\frac{1}{2}\right)_n=2^n\cdot\frac{\Gamma\left(\frac{1}{4}+\frac{n}{2}\right)\Gamma\left(\frac{3}{4}+\frac{n}{2}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}. \tag{3.7}$$

Example 3.3. Set $p=1$ and $q=3$ in Theorem 1

$$\left(\frac{1}{3}\right)_n=3^n\cdot\frac{\Gamma\left(\frac{1}{9}+\frac{n}{3}\right)\Gamma\left(\frac{4}{9}+\frac{n}{3}\right)\Gamma\left(\frac{7}{9}+\frac{n}{3}\right)}{\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{7}{9}\right)}. \tag{3.8}$$

Example 3.4. Set $p=2$ and $q=3$ in Theorem 1

$$\left(\frac{2}{3}\right)_n=3^n\cdot\frac{\Gamma\left(\frac{2}{9}+\frac{n}{3}\right)\Gamma\left(\frac{5}{9}+\frac{n}{3}\right)\Gamma\left(\frac{8}{9}+\frac{n}{3}\right)}{\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{8}{9}\right)}. \tag{3.9}$$

Example 3.5. Set $p=1$ and $q=4$ in Theorem 1

$$\left(\frac{1}{4}\right)_n=4^n\cdot\frac{\Gamma\left(\frac{1}{16}+\frac{n}{4}\right)\Gamma\left(\frac{5}{16}+\frac{n}{4}\right)\Gamma\left(\frac{9}{16}+\frac{n}{4}\right)\Gamma\left(\frac{13}{16}+\frac{n}{4}\right)}{\Gamma\left(\frac{1}{16}\right)\Gamma\left(\frac{5}{16}\right)\Gamma\left(\frac{9}{16}\right)\Gamma\left(\frac{13}{16}\right)}. \tag{3.10}$$

Example 3.6. Set $p=2$ and $q=4$ in Theorem 1

$$\left(\frac{2}{4}\right)_n=4^n\cdot\frac{\Gamma\left(\frac{2}{16}+\frac{n}{4}\right)\Gamma\left(\frac{6}{16}+\frac{n}{4}\right)\Gamma\left(\frac{10}{16}+\frac{n}{4}\right)\Gamma\left(\frac{14}{16}+\frac{n}{4}\right)}{\Gamma\left(\frac{2}{16}\right)\Gamma\left(\frac{6}{16}\right)\Gamma\left(\frac{10}{16}\right)\Gamma\left(\frac{14}{16}\right)}. \tag{3.11}$$

Example 3.7. Set $p=3$ and $q=4$ in Theorem 1

$$\left(\frac{3}{4}\right)_n = 4^n \cdot \frac{\Gamma\left(\frac{3}{16}+\frac{n}{4}\right)\Gamma\left(\frac{7}{16}+\frac{n}{4}\right)\Gamma\left(\frac{11}{16}+\frac{n}{4}\right)\Gamma\left(\frac{15}{16}+\frac{n}{4}\right)}{\Gamma\left(\frac{3}{16}\right)\Gamma\left(\frac{7}{16}\right)\Gamma\left(\frac{11}{16}\right)\Gamma\left(\frac{15}{16}\right)}. \tag{3.12}$$

REFERENCES

[1] https://proofwiki.org/wiki/Gauss_Multiplication_Formula.

[2] Weisstein, Eric W., Pochhammer Symbol, from *MathWorld* - A Wolfram Web Resource. http://mathworld.wolfram.com/PochhammerSymbol.html.

4. APPENDIX

Theorem 4.1. *If* $n \in \mathbb{Z}_{\geqslant 3}$, *then*

$$\Gamma\left(1-\frac{1}{n}\right)\prod_{k=1}^{n-2}\frac{\Gamma\left(\frac{k}{n}\right)}{\Gamma\left(\frac{k}{n-1}\right)} = \sqrt{2\pi\left(1-\frac{1}{n}\right)}. \tag{4.1}$$

Proof. From Theorem 2.1, I obtain with a bit of manipulation

$$\begin{aligned} &\Gamma(z)\prod_{k=1}^{n-1}\Gamma\left(z+\frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{1/2-nz}\Gamma(nz) \\ &\Rightarrow \prod_{k=1}^{n-1}\Gamma\left(z+\frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{1/2-nz}\frac{\Gamma(nz)}{\Gamma(z)}. \end{aligned} \tag{4.2}$$

On the other hand, replace z by $1/\ell$, let ℓ tends to infinity in both members of (4.2)

$$\lim_{\ell\to\infty}\prod_{k=1}^{n-1}\Gamma\left(\frac{1}{\ell}+\frac{k}{n}\right) = \lim_{\ell\to\infty}(2\pi)^{(n-1)/2} n^{1/2-n/\ell}\frac{\Gamma\left(\frac{n}{\ell}\right)}{\Gamma\left(\frac{1}{\ell}\right)}. \tag{4.3}$$

Note that

$$\lim_{\ell\to\infty}\Gamma\left(\frac{1}{\ell}+\frac{k}{n}\right) = \Gamma\left(\frac{k}{n}\right), \tag{4.4}$$

$$\lim_{\ell\to\infty} n^{-n/\ell} = 1 \tag{4.5}$$

and

$$\lim_{\ell\to\infty}\frac{\Gamma\left(\frac{n}{\ell}\right)}{\Gamma\left(\frac{1}{\ell}\right)} = \frac{1}{n}. \tag{4.6}$$

From (4.3), (4.4), (4.5) and (4.6), I conclude that

$$\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) = \frac{(2\pi)^{(n-1)/2} n^{1/2}}{n}. \tag{4.7}$$

Now, replace n by $n-1$ in (3.7)

$$\prod_{k=1}^{n-2} \Gamma\left(\frac{k}{n-1}\right) = \frac{(2\pi)^{(n-2)/2} (n-1)^{1/2}}{n-1}. \tag{4.8}$$

Divide (4.7) by (4.8) and encounter the desired result. ☐

Example 4.2. Set $n=3$ in Theorem 3.1

$$\Gamma\left(\frac{2}{3}\right)\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{2}\right)} = 2\sqrt{\frac{\pi}{3}}. \tag{4.9}$$

Example 4.3. Set $n=4$ in Theorem 3.1

$$\Gamma\left(\frac{3}{4}\right)\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{3}\right)}\frac{\Gamma\left(\frac{2}{4}\right)}{\Gamma\left(\frac{2}{3}\right)} = \sqrt{\frac{3\pi}{2}}. \tag{4.10}$$

Example 4.4. Set $n=5$ in Theorem 3.1

$$\Gamma\left(\frac{4}{5}\right)\frac{\Gamma\left(\frac{1}{5}\right)}{\Gamma\left(\frac{1}{4}\right)}\frac{\Gamma\left(\frac{2}{5}\right)}{\Gamma\left(\frac{2}{4}\right)}\frac{\Gamma\left(\frac{3}{5}\right)}{\Gamma\left(\frac{3}{4}\right)} = 2\sqrt{\frac{2\pi}{5}}. \tag{4.11}$$