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Chaos, Solitons and Fractals 19 (2004) 1023–1030

CHAOS
SOLITONS & FRACTALS

www.elsevier.com/locate/chaos

Renormalization group and the emergence of random fractal topology in quantum field theory

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Accepted 17 June 2003

Abstract

This work reveals the close connection between the random fractal topology of space–time in microphysics and the renormalization group program (RG) of quantum field theory. As known, the primary goal of RG is to consistently remove divergences from quantum computations by factoring in the energy scale (μ) at which physical processes are probed. RG postulates that the action functional is independent of any particular choice of μ, that is, physical processes are invariant to arbitrary changes of the observation scale. In this context, we conjecture that μ represents a continuous random variable having a uniform density function. Novel results emerge in the basin of attraction of all fixed points, namely: (i) the field exponent becomes a continuous random variable and (ii) space–time coordinates become fractals with random dimensions. It is concluded that the random topology of space–time is not an exclusive attribute of the Planck scale but an inherent manifestation *of stochastic dynamics* near any fixed point of the underlying field theory.

1. Introduction

In recent years significant effort has been devoted to applications of fractal geometry, deterministic chaos and stochastic dynamics in classical and quantum physics. Due to the wide extent of this research field, a complete listing of main contributions is impractical. We mention here few examples that are representative for the topic of this work: anomalous diffusion and Levy statistics in Hamiltonian phase space [12,13], non-differentiability of Feynman paths [14,15], application of fractional Brownian motion in quantum field theory [4], vacuum fluctuations, chaotic maps and stochastic quantization in particle physics [16], mass generation in the lepton sector due to period doubling transition to chaos [17], quantum Brownian motion [18], fractional dynamics and origin of the fine-structure constant [19], Cantorian space–time and the topological foundation of coupling and mass spectra in the Standard Model [20–27].

The prevailing interpretation of El Naschie's E^{∞} model is that the Cantorian space–time topology emerges at mass scales comparable to the Planck length. Drawing on recent results regarding renormalization group in the presence of quantum fluctuations [4], our work suggests that the random topology of space–time is not an exclusive attribute of the Planck scale but an inherent manifestation *of stochastic dynamics* near any fixed point of the underlying field theory.

The paper is organized as follows: Section 2 introduces the concept of random observation scale from arguments related to equilibrium statistical mechanics. Taking the φ^4 theory as a benchmark model, Section 3 examines the behavior of the RG solution near the unique fixed point $g \to 0$. The stochastic character of the field exponent is analyzed in Section 4. Section 5 investigates the temporal evolution of the φ field from a statistical mechanics perspective. Connection of space–time coordinates to fractal objects having random dimension is discussed in Section 6. Results are summarized in Section 7.

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doi:10.1016/S0960-0779(03)00304-7

2. The random observation scale: a statistical mechanics argument

Classical statistical mechanics of systems at thermal equilibrium asserts that energy or energy and number of particles in thermodynamic ensembles are subject to incessant fluctuations. Consider, for example, a system enclosed in a heatbath at constant finite temperature T (a ''canonical ensemble''). The energy fluctuation is linearly dependent on temperature according to [1]:

$$\frac{\Delta E}{\langle E\rangle} \sim \frac{T}{N\varepsilon} \tag{1}$$

where $\langle E\rangle$ is the thermal average of the energy, N is the number of particles in the system and ε is the energy per particle. Fluctuations vanish for macroscopic systems in the limit $N \to \infty$ where the distribution of energy is sharply peaked around $\langle E\rangle$. In contrast, fluctuations survive in canonical ensembles comprising low-dimensional classical or quantum systems such as dilute Bose gases.

In the same context, we recall that quantum fields are objects with manifest statistical properties owing to the cascade of virtual processes that occur during propagation and interaction. We also recall that a basic requirement of any realistic quantum field theory is renormalizability [2]. To render all computations finite, a well-established regularization procedure needs to be implemented. The renormalization program imposes an arbitrary energy scale (μ) upon which no physical consequences must depend [2,3]. This sets the ''coarse-graining'' scale and the resolution at which the underlying physics is probed.

From these arguments it follows that, if the average energy of the quantum field system $\langle E\rangle$ sets its temperature, then μ is expected to undergo continuous fluctuations about $\langle E\rangle$. There are no preferential values in this random occurrence. Hence, invoking the uncertainty principle, we assert below that μ represents a continuous random variable with a uniform probability density function $p(\mu)$. Let μ be defined inside a range bounded by μ_M and μ_m. We have

$$\begin{aligned} &p(\mu) = \frac{1}{\mu_M - \mu_m} \leqslant \frac{\tau}{h} \quad \text{if } \mu \in \{\mu_m, \mu_M\} \\ &p(\mu) = 0 \quad \text{if otherwise} \end{aligned} \tag{2}$$

in which h is Planck's constant and τ sets the observation time window. During τ we assume that the system randomly samples all available energy scales contained in the range.

A comment on (2) is now in order. Dimensional consistency requires the right hand side of (2) to be expressed in scalar form. The simplest way to fulfill this requirement is to work with a relative energy scale μ^0 defined via

$$\mu^0 = \frac{\mu}{\mu_M} \tag{3}$$

which converts (2) into

$$\begin{aligned} &p(\mu^0) = \frac{1}{1 - \mu_m^0} \leqslant \tau^0 \quad \text{if } \mu \in \{\mu_m, \mu_M\} \\ &p(\mu^0) = 0 \quad \text{if otherwise} \end{aligned} \tag{4}$$

where $h = 1$ and $\tau^0 = \mu_M \tau$ is the non-dimensional time observation window. According to this interpretation, the time window τ^0 and energy scale μ^0 are statistically conjugate variables. This implies that there is random sampling of all available time instants contained in τ^0 in a similar fashion with energy sampling in the range $[\mu_m^0, 1]$. The natural outcome of this conjecture is that time behaves as a stochastic variable. To derive similar conclusions on the space coordinate we note that, in non-relativistic field theories, time and space scale independently as

$$\begin{aligned} &\vec{x} = s\overrightarrow{x'} \\ &t = s^z t' \end{aligned} \tag{5}$$

where $z \neq 1$ is the so-called dynamic exponent [4]. In scalar form (5) reads

$$\begin{aligned} &\vec{x}^0 = s\overrightarrow{x'}^0 \\ &t^0 = s^z t'^0 \end{aligned} \tag{6}$$

in which

$$x^0 = x\mu_M \qquad t^0 = t\mu_M \tag{7}$$

It follows that space and time are non-trivially related through

$$t^0 \sim (x^0)^z \tag{8}$$

and the analogue of (4) is given by

$$p(\mu^0) = \frac{1}{1-\mu_m^0} \leqslant (\lambda^0)^z \quad \text{if } \mu \in \{\mu_m, \mu_M\} \qquad p(\mu^0) = 0 \quad \text{if otherwise} \tag{9}$$

where $\lambda^0 = \lambda\mu_M$ is the spatial observation window. The space coordinate randomly samples all available locations contained in λ^0 as the energy scale fluctuates within $[\mu_m^0, 1]$.

3. Behavior of the RG solution near fixed points

Consider a simple field theoretic framework comprising the scalar field operator $\varphi(\vec{x}, t)$, the mass parameter m and coupling constant g. A typical prototype is the well-known φ^4 model whose free-form action is [2,3]

$$S[\varphi] = \int \left\{ \frac{1}{2}[\partial_\nu\varphi\partial^\nu\varphi - m^2\varphi^2] - \frac{\mu^{4-d}g}{4!}\varphi^4 \right\} \mathrm{d}^{d-1}x\,\mathrm{d}t \tag{10}$$

in which d stands for the dimension of the space–time domain and ν is the space–time index. Fluctuations in the observation scale are expected to create subsequent fluctuations of the operator $\varphi(\vec{x}, t)$. The field probability density function $p_\mu(\varphi, g, m)$ has dimension

$$[\varphi]^{-1} = \mu_M^{1-\frac{d}{2}} \tag{11}$$

and satisfies the RG equation [4]:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \delta(g)m\frac{\partial}{\partial m} - \frac{\gamma(g)}{2}\varphi\frac{\partial}{\partial\varphi}\right] p_\mu(\varphi, g, m) = 0 \tag{12}$$

where the coefficient functions

$$\mu\frac{\partial g}{\partial\mu} = \beta(g) \qquad \mu\frac{\partial m}{\partial\mu} = \delta(g)m \qquad \mu\frac{\partial\varphi}{\partial\mu} = -\frac{\gamma(g)}{2}\varphi \tag{13}$$

outline the scale dependence of coupling, mass and field, respectively. As known, coefficient functions are specific for each field theory. In particular, the sign of $\beta(g)$ determines the type of asymptotic behavior at large momenta, that is, whether or not the theory displays asymptotic freedom [4,5]. In general, the zeros of $\beta(g)$ define the set of fixed points $\{g_i^*\},\ i = 1, 2, \ldots, N$. In the neighborhood of these points (i.e., in their basin of attraction) Eqs. (13) become

$$\mu\frac{\partial m}{\partial\mu} = \delta(g_i^*)m \qquad \mu\frac{\partial\varphi}{\partial\mu} = -\frac{\gamma(g_i^*)}{2}\varphi \tag{14}$$

To further carry out computations involving exclusively scalar variables, it is convenient to cast (14) in a non-dimensional form. Dividing both sides by their respective units for mass and field yields

$$\mu^0 \frac{\partial m^0}{\partial \mu^0} = \delta(g_i^*) m^0$$
$$\mu^0 \frac{\partial \varphi^0}{\partial \mu^0} = -\frac{\gamma(g_i^*)}{2} \varphi^0 \tag{15}$$

where

$$\varphi^0 = \frac{\varphi}{[\varphi]}$$
$$m^0 = \frac{m}{\mu_M} \tag{16}$$

(15) is solved by the following set of closed-form solutions:

$$m_\mathrm{i}^0 = K_m (\mu^0)^{\delta(g_i^*)}$$
$$\varphi_\mathrm{i}^0 = K_\varphi (\mu^0)^{-\frac{\gamma(g_i^*)}{2}} \tag{17}$$

in which K_m, K_φ are integration constants. It can be seen that, in general, both field and mass solutions display a N-fold multiplicity and acquire properties of continuous random variables. There is only one fixed point in the φ^4 model corresponding to the IR limit ($g \to 0$) as $\mu \to 0$. In its basin of attraction we have

$$m^0 = f_m(\mu^0) = K_m (\mu^0)^{\delta_0}$$
$$\varphi^0 = f_\varphi(\mu^0) = K_\varphi (\mu^0)^{-\frac{\gamma_0}{2}} \tag{18}$$

where $\gamma_0 = \gamma(0)$ and $\delta_0 = \delta(0)$.

Since the power-law relations contained in (18) are strictly monotonic, their respective probability density functions are given by [6]

$$p(m^0) = p_\mu[f_m^{-1}(m^0)] \left| \frac{\mathrm{d}}{\mathrm{d}m^0} [f_m^{-1}(m^0)] \right|$$
$$p(\varphi^0) = p_\mu[f_\varphi^{-1}(\varphi^0)] \left| \frac{\mathrm{d}}{\mathrm{d}\varphi^0} [f_\varphi^{-1}(\varphi^0)] \right| \tag{19}$$

leading to

$$p(m^0) = \frac{K_m^{-\frac{1}{\delta_0}}}{(1-\mu_m^0)\delta_0} (m^0)^{\frac{1}{\delta_0}-1}$$
$$p(\varphi^0) = \frac{2K_\varphi^{\frac{2}{\gamma_0}}}{(1-\mu_m^0)\gamma_0} (\varphi^0)^{-\left(\frac{2}{\gamma_0}+1\right)} \tag{20}$$

One may verify from (20) that the completeness relation

$$\int_{\mu_m^0}^1 p(m^0)\,\mathrm{d}m^0 = \int_{\mu_m^0}^1 p(\varphi^0)\,\mathrm{d}\varphi^0 = 1 \tag{21}$$

is automatically satisfied. For simplicity we choose $K_m = K_\varphi = 1$.

Let $m_\mathrm{i}^0, \varphi_\mathrm{i}^0$ and $m_\mathrm{f}^0, \varphi_\mathrm{f}^0$ designate two random sets of initial and final mass and field states. Using (20) yields the following mass and field moments corresponding to these random intervals

$$\langle (m^0)^2 \rangle_{\mathrm{fi}} = \frac{1}{(1-\mu_m^0)(2\delta_0+1)} \left[(\mu_\mathrm{f}^0)^{2\delta_0+1} - (\mu_\mathrm{i}^0)^{2\delta_0+1} \right]$$
$$\langle (\varphi^0)^2 \rangle_{\mathrm{fi}} = \frac{1}{(1-\mu_m^0)(\gamma_0-1)} \left[(\mu_\mathrm{f}^0)^{1-\gamma_0} - (\mu_\mathrm{i}^0)^{1-\gamma_0} \right] \tag{22}$$

Let the initial state be fixed and the final state randomly vary. Up to an additive constant (22) may be written as

$$\begin{aligned}\langle (m^0)^2\rangle &= \frac{1}{(1-\mu_m^0)(2\delta_0+1)}(\mu^0)^{2\delta_0+1}\\ \langle (\varphi^0)^2\rangle &= \frac{1}{(1-\mu_m^0)(\gamma_0-1)}(\mu^0)^{1-\gamma_0}\end{aligned} \tag{23}$$

We shall make use of these relations in Section 6.

4. Stochastic nature of field scaling

A well-known property of critical phenomena is that, near transition points, all relevant variables scale in a similar fashion with the control parameter. The scaling behavior is defined by a set of fixed exponents that are deterministic in nature and dependent on the dimensionality of the system [5,7,8]. In contrast, we are going to show in this section that the field exponent γ_0 acquires a stochastic character due to the postulated randomness of the observation scale μ^0.

Let

$$G(\vec{x}^0, t^0) = \langle \varphi^0(\vec{x}^0, t^0)\varphi^0(0,0)\rangle \tag{24}$$

denote the field propagator from the initial space–time point (0,0) to the arbitrary point $(\vec{x}, t)$ [5]. Upon independent scaling of the non-dimensional space–time coordinates according to (6), it can be shown that the field propagator changes as [4]

$$G(\vec{x}^0, t^0) = (r^0)^{2\chi} f\left[\frac{t^0}{(r^0)^z}\right] \tag{25}$$

if the field scales as

$$\varphi^0(\vec{x}^0, t^0) = s^{\chi}\varphi^0\left(\vec{x'}^0 t'^0\right) \tag{26}$$

and the scale factor is chosen to be proportional to the modulus of the position vector

$$s \sim |\vec{x}^0| = r^0 \tag{27}$$

The asymptotic behavior of $f\left[\frac{t^0}{(r^0)^z}\right]$ is supplied by

$$\begin{aligned}&\lim f\left[\frac{t^0}{(r^0)^z}\right] \to \text{const.} \quad \text{if } t^0 \ll (r^0)^z\\ &\lim f\left[\frac{t^0}{(r^0)^z}\right] \to \left[\frac{t^0}{(r^0)^z}\right]^{2\chi/z} \quad \text{if } t^0 \gg (r^0)^z\end{aligned} \tag{28}$$

In d-dimensional space–time and at the fixed point, χ is related to the field exponent γ_0 via

$$\chi = -\left(\frac{d}{2} - 1 + \frac{\gamma_0}{2}\right) \tag{29}$$

Using the propagator interpretation as probability density amplitude for transitions involving the initial (0,0) and final space–time location $(\vec{x}, t)$ [3], we demand

$$\int_{x_i^0}^{x_f^0} [G(\vec{x}^0, t^0)]^2 \, d^{d-1}x^0 = 1 \tag{30}$$

in which (x_i^0, x_f^0) are random limits of the spatial domain. In order to explicitly carry out the integral, these limits need to be expressed in terms of μ^0. Let us rewrite (9) in the equivalent minimal form

$$[x^0 - \langle x^0\rangle]^z[\mu^0 - \langle \mu^0\rangle] = 1 \tag{31}$$

and take for simplicity

$$\langle x^0\rangle = \langle \mu^0\rangle = 0 \tag{32}$$

Thus

$$x_i^0 = (\mu_i^0)^{-\frac{1}{z}}$$
$$x_f^0 = (\mu_f^0)^{-\frac{1}{z}} \tag{33}$$

We note that (30) is a result of the general closure condition

$$\int G(\vec{x},t)G^*\left(\vec{x}',t\right)\mathrm{d}^{d-1}x = \delta^{(d-1)}\left(\vec{x}-\vec{x}'\right) \tag{34}$$

upon a suitable normalization of the δ-function to unity [9]. Since, apart from a multiplicative constant,

$$\mathrm{d}^{d-1}x^0 = (r^0)^{d-1}\mathrm{d}r^0 \tag{35}$$

we find from (30)

$$(\mu_f^0)^{\frac{d-2(2-\gamma_0)}{z}} - (\mu_i^0)^{\frac{d-2(2-\gamma_0)}{z}} = 2(2-\gamma_0) - d \tag{36}$$

Keeping the initial state fixed and letting the final state fluctuate, gives the generic relation

$$(\mu^0)^{\frac{d-2(2-\gamma_0)}{z}} = 2(2-\gamma_0) - d \tag{37}$$

highlighting the random nature of γ_0.

5. Statistical mechanics of the φ^4 model

In the previous section it was shown that the field exponent γ_0 is no longer a fixed parameter of the RG but a random variable dependent on μ^0. To gain further insight into statistical properties of the scalar field, it is desirable to analyze the effect of the stochastic observation scale on the field dynamics. To this end, a convenient starting point is the effective action formalism of the φ^4 theory [3]. Consider the vacuum expectation value of the operator $\varphi(\vec{x},t)$ in the presence of an external source $J(\vec{x},t)$

$$\varphi_c(\vec{x},t) = \frac{\langle 0|\varphi(\vec{x},t)|0\rangle_J}{\langle 0|0\rangle_J} \tag{38}$$

(38) represents the classical counterpart of the scalar field operator. The equation of motion for the free φ^4 theory is obtained by replacing $\varphi_c(\vec{x},t)$ in (10)

$$\frac{\delta S[\varphi_c]}{\delta\varphi_c} = [\partial_\nu\partial^\nu + m^2]\varphi_c(\vec{x},t) + \frac{\mu^{4-d}g}{3!}\varphi_c(\vec{x},t)^3 = 0 \tag{39}$$

For simplicity, we proceed with the assumption that the field $\varphi_c(\vec{x},t)$ is spatially uniform, $\varphi_c = \varphi_c(t)$. Furthermore, to ensure consistency with the formal treatment developed so far, we pass to a non-dimensional representation of (39) by using (7), (11) and (16). The resulting equation of motion reads

$$\left[\frac{\partial^2}{\partial(t^0)^2} + (m^0)^2\right]\varphi_c^0(t^0) + \frac{g}{3!}\varphi_c^0(t^0)^3 = 0 \tag{40}$$

In the basin of attraction of the IR fixed point, the non-linear coupling term in (40) becomes a small perturbation to the free harmonic oscillator whose equation is

$$\left[\frac{\partial^2}{\partial(t^0)^2} + (m_{\mathrm{eff}}^0)^2\right]\varphi_c^0(t^0) = 0 \tag{41}$$

The net result of this approximation is that the original mass may be replaced by an effective mass parameter [10]

$$m_{\mathrm{eff}}^0 = m^0 + \frac{g(\Phi_c^0)^2}{16m^0} \tag{42}$$

where Φ_c^0 is the amplitude for the closed-form solution of (40), i.e.

$$\varphi_c^0 = \Phi_c^0 sn(m_{\mathrm{eff}}^0 t^0) \tag{43}$$

and m^0 is a random variable depending on δ_0 (Section 3).

6. Emergence of space–time as a random fractal set

The stochastic nature of φ_c^0 can be fully accounted for by adopting a statistical interpretation of its dynamics. In this context, the field second moment may be computed as [11]

$$\langle(\varphi_c^0)^2\rangle \sim \coth\left(\frac{1}{2}m_{\text{eff}}^0 t^0\right) \tag{44}$$

In the IR limit $g \ll m^0 < 1$ and for time intervals consistent with (28) (that is, for $t^0 \ll (r^0)^z$), a reasonable approximation of (44) is

$$\langle(\varphi_c^0)^2\rangle \sim \frac{1}{m_{\text{eff}}^0 t^0} \tag{45}$$

Direct comparison of (23) with (45) gives

$$t^0 \sim (1-\mu_m^0)(\gamma_0 - 1)t_{\text{eff}}^0(\mu^0)^{\gamma_0 - 1} \tag{46}$$

in which $t_{\text{eff}}^0 = (m_{\text{eff}}^0)^{-1}$ is the period of the harmonic oscillator described by (41). Using (8) we find

$$x^0 \sim (1-\mu_m^0)(\gamma_0 - 1)t_{\text{eff}}^0(\mu^0)^{\frac{\gamma_0 - 1}{z}} \tag{47}$$

It follows from (46) and (47) that time and space coordinates scale as random fractals whose dimensions depend on both δ_0 and γ_0.

7. Summary

Starting from the viewpoint that the renormalization group scale is a continuous random variable spanning a specific range, we have shown that, near fixed points of the underlying field theory, the space–time manifold acquires properties of random fractal sets. We have found that the manifold dimension depends on the values taken by the mass and field exponents at the fixed point. The φ^4 theory has been chosen as an illustrative framework, however, results are not restricted to this model.

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