Shortest refutations of the Zermelo-Fraenkel (ZF) axioms

Abstract: The Zermelo-Fraenkel (ZF) axioms evaluated are below.

1. Null Set: ∃x¬∃y(y∈x)[=∅].
2. Extensionality: ∀x∀y[∀z(z∈x↔z∈y)→x=y].
3. Pairs: ∀x∀y∃z∀w(w∈z↔w=x∨w=y).
4. Power Set: ∀x∃y∀z[z∈y↔∀w(w∈z→w∈x)] (1).
4. Power Set: ∀x∃y∀z(z∈y↔z⊆x) (2).
5. Unions: ∀x∃y∀z[z∈y↔∃w(w∈x∧z∈w)].
6. Infinity: ∃x[∅∈x∧∀y(y∈x→(y∪(y))∈x)].
7. Separation Schema: ∀uk[∀w∃v∀r(r∈v↔r∈w∧ψx,u[r,u])].
8. Replacement Schema: ∀uk[∀x∃!yϕ(x,y,u)→∀w∃v∀r(r∈v↔∃s(s∈w∧ϕx,y,u[s,r,u]))].
9. Regularity: ∀x[x≠∅→∃y(y∈x∧∀z(z∈x→¬(z∈y)))].

None are tautologous.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated *proof* value, `F` as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). Results are a 16-valued truth table in row-major and horizontal, or repeating fragments of 128-tables for more variables.

LET p, q, r, s, t, u, v, w, x, y, z: ϕ, ψ, r, s, k, u, v, w, x, y, z;
~ Not, !; + Or, ∪; & And; > Imply; < Not Imply, less than;
= Equivalent; @ Not Equivalent;
necessity, ∀, for all or every; % possibility, ∃, for one or some;
~%p=#p; (q>p) ~(q∈p), not lt.eq.; ~(q>p) (q∈p) lt.eq., ⊆.

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1. **Null Set**:This axiom asserts the existence of the empty set. Since it is provable from this axiom and the next axiom that there is a unique such set, we may introduce the notation '∅' to denote it

∃x¬∃y(y∈x) [=∅] (1.1)

(%x&~%y)&~(y>x) ; `FFFF FFFF FFFF FFFF, FFFF FFFF FFFF FFFF` (1.2)

2. **Extensionality**: This axiom asserts that when sets x and y have the same members, they are the same set.

∀x∀y[∀z(z∈x↔z∈y)→x=y] (2.1)

(#x&#y)&((#z&(~(z>x)=~(z>y)))>(x=y)) ;
`FFFF FFFF FFFF FFFF, NNNN NNNN NNNN NNNN` (2.2)

3. **Pairs**: This axiom asserts that if given any set x and y, there exists a pair set of x and y, i.e., a set which has only x and y as members. Since it is provable that there is a unique pair set for each given x and y, we introduce the notation '{x,y}' to denote it.

$$\forall x\forall y\exists z\forall w(w\in z\leftrightarrow w=x\vee w=y) \quad (3.1)$$

```
((#x&#y)&(%z&#w))&(~(w>z)=((w=x)+(w=y))) ;
                    FFFF FFFF FFFF FFFF, FFFF FFFF FFFF FFFF
```
(3.2)

4. **Power Set**: This axiom asserts that for any set x, there is a set y which contains as members all those sets whose members are also elements of x, i.e., y contains all of the subsets of x.

$$\forall x\exists y\forall z[z\in y\leftrightarrow \forall w(w\in z\rightarrow w\in x)] \quad (4.1.1)$$

```
((#x&%y)&#z)&(~(z>y)=(#w&(~(w>z)>~(w>x)))) ;
                    NNNN NNNN NNNN NNNN, FFFF FFFF FFFF FFFF
```
(4.1.2)

Since every set provably has a unique 'power set', we introduce the notation 'P(x)' to denote it. Note also that we may define the notion x is a subset of y ('$x\subseteq y$') as: $\forall z(z\in x\rightarrow z\in y)$. Then we simplify the statement of the Power Set Axiom as follows:

$$\forall x\exists y\forall z(z\in y\leftrightarrow z\subseteq x) \quad (4.2.1)$$

```
((#x&%y)&#z)&(~(z>y)=~(z>x)) ;
                    FFFF FFFF FFFF FFFF, NNNN NNNN NNNN NNNN
```
(4.2.2)

5. **Unions**: This axiom asserts that for any given set x, there is a set y which has as members all of the members of all of the members of x. Since it is provable that there is a unique 'union' of any set x, we introduce the notation '$\cup x$' to denote it.

$$\forall x\exists y\forall z[z\in y\leftrightarrow \exists w(w\in x\wedge z\in w)] \quad (5.1)$$

```
((#x&%y)&#z)&(~(z>y)=(%w&(~(w>x) &~(z>w)))) ;
                    FFFF FFFF FFFF FFFF, NNNN NNNN NNNN NNNN
```
(5.2)

6. **Infinity**: This axiom asserts the existence of an infinite set, i.e., a set with an infinite number of members. We may think of this as follows. Let us define the union of x and y ('$x\cup y$') as the union of the pair set of x and y, i.e., as $\cup\{x,y\}$. Then the Axiom of Infinity asserts that there is a set x which contains $\varnothing$ as a member and which is such that whenever a set y is a member of x, then $y\cup\{y\}$ is a member of x. Consequently, this axiom guarantees the existence of a set of the following form: $\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\},\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\},\ldots\}$ Notice that the second element, $\{\varnothing\}$, is in this set because (1) the fact that $\varnothing$ is in the set implies that $\varnothing\cup\{\varnothing\}$ is in the set and (2) $\varnothing\cup\{\varnothing\}$ just is $\{\varnothing\}$. Similarly, the third element, $\{\varnothing,\{\varnothing\}\}$, is in this set because (1) the act that $\{\varnothing\}$ is in the set implies that $\{\varnothing\}\cup\{\{\varnothing\}\}$ is in the set and (2) $\{\varnothing\}\cup\{\{\varnothing\}\}$ just is $\{\varnothing,\{\varnothing\}\}$. And so forth.

$$\exists x[\varnothing\in x\wedge\forall y(y\in x\rightarrow \cup\{y,\{y\}\}\in x)] \quad (6.0)$$

[We rewrite Eq. 6.0 by replacing the set notation of curly braces with parentheses.]

$$\exists x[\varnothing \in x \wedge \forall y(y \in x \rightarrow (y \cup (y)) \in x)] \quad (6.1)$$

```
%x&((~((%x&~%y)&~(y>x))>x)&(#y&(~(y>x)>~((y&y)>x)))) ;
                    FFFF FFFF FFFF FFFF, NNNN NNNN NNNN NNNN
```
(6.2)

7. **Separation Schema**: This axiom asserts the existence of a set that contains the elements of a given set w that satisfy a certain condition ψ. That is, suppose that ψ(x,u^) has x free and may or may not have u1,…,uk free. And let ψx,u^[r,u^] be the result of substituting r for x in ψ(x,u^). In other words, if given a formula ψ and a set w, there exists a set v which has as members precisely the members of w which satisfy the formula ψ.

$$\forall u1\ldots\forall uk[\forall w\exists v\forall r(r \in v \leftrightarrow r \in w \wedge \psi x,u\hat{}[r,u\hat{}])] \quad (7.0)$$

[We rewrite the series as the last named element only without recursion of substitutions.]

$$\forall uk[\forall w\exists v\forall r(r \in v \leftrightarrow r \in w \wedge \psi x,u[r,u])] \quad (7.1)$$

```
#(u&t)&(((#w&%v)&#r)&((~(r>v)=~(r>w))&((q&x)&((u&r)&u)))) ;
                    FFFF FFFF FFFF FFFF, FFFF FFFF FFFF FFFF
```
(7.2)

8. **Replacement Schema**: Every instance of the following schema is an axiom. Suppose that ϕ(x,y,u^)is a formula with x and y free, and let u^ represent the variables u1,…uk, which may or may not be free in ϕ. Furthermore, let ϕx,y,u^[s,r,u^] be the result of substituting s and r for x and y, respectively, in ϕ(x,y,u^). In other words, if we know that ϕ is a functional formula (which relates each set x to a unique set y), then if we are given a set w, we can form a new set v as follows: collect all of the sets to which the members of w are uniquely related by ϕ. Note that the Replacement Schema can take you 'out of' the set w when forming the set v. The elements of v need not be elements of w. By contrast, the Separation Schema of Zermelo only yields subsets of the given set w.

$$\forall u1\ldots\forall uk[\forall x\exists!y\phi(x,y,u\hat{})\rightarrow\forall w\exists v\forall r(r \in v \leftrightarrow \exists s(s \in w \wedge \phi x,y,u\hat{}[s,r,u\hat{}]))] \quad (8.0)$$

[We rewrite the series as the last named element only without recursion of substitutions.]

$$\forall uk[\forall x\exists!y\phi(x,y,u)\rightarrow\forall w\exists v\forall r(r \in v \leftrightarrow \exists s(s \in w \wedge \phi x,y,u[s,r,u]))] \quad (8.1)$$

```
#(u&t)&(((#x&%~y)&((p&x)&(y&u)))>(((#w&%v)&#r)&(~(r>v)=
(%s&(~(s>w)&(((p&x)&(y&u))&((s&r)&u)))))));
                    FFFF FFFF FFFF FFFF, NNNN NNNN NNNN NNNN
```
(8.2)

9. **Regularity**: This axiom asserts that every set is 'well-founded'. A member y of a set x with this property is called a 'minimal' element. This axiom rules out the existence of circular chains of sets (e.g., such as $x \in y \wedge y \in z \wedge z \in x$) as well as infinitely descending chains of sets (such as … $x3 \in x2 \in x1 \in x0$).

$$\forall x[x\neq\varnothing\rightarrow\exists y(y\in x\wedge\forall z(z\in x\rightarrow\neg(z\in y)))] \tag{9.1}$$

```
#x&((x@((%x&~%y)&~(y>x)))>(%y&(~(y>x)&(#z&(~(z>x)>~~(z>y)))))) ;
                  FFFF FFFF FFFF FFFF, FFFF FFFF FFFF FFFF        (9.2)
```

Eqs. 1.2-9.2 as rendered are *not* tautologous. This refutes those nine ZF axioms.