

# On the Decomposition of the Pochhammer's symbol

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*And Jesus said unto them, I am the bread of life: he that cometh to me shall never hunger; and he that believe on me shall never thirst. John 6:35.*

ABSTRACT. I derive an identity for the decomposition of the Pochhammer's symbol.

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## 1. INTRODUCTION

In present paper, I derive the identity below

$$(a)_n = \frac{a}{a+n} \cdot \frac{(a+1)_{k-1}}{(a+n+1)_{2k-1}} \cdot (a+k)_{n+k}.$$

## 2. PRELIMINARY

**Lemma 2.1.** *Let  $u_r$  and  $v_r$  two sequences, then*

$$\prod_{r=1}^n \frac{u_r}{v_r} = 1 + \sum_{k=1}^n \left(1 - \frac{v_k}{u_k}\right) \prod_{r=1}^k \frac{u_r}{v_r}. \quad (2.1)$$

*provided none of the denominators in (2.1) are zero.*

**Proof.** See [1, p. 4, (2.2)]. □

## 3. THE MAIN THEOREM

### 3.1. The Finite sum for $(a)_n/(b)_n$ .

**Lemma 3.1.** *If  $a, b \in \mathbb{R}$  and  $n \in \mathbb{Z}^+$ , then*

$$\frac{(a)_n}{(b)_n} = \frac{a(n+b)}{b(n+a)} \left[ 1 + \sum_{k=1}^n \left( \frac{a-b}{a+k} \right) \frac{(a+1)_k}{(b+1)_k} \right], \quad (3.1)$$

*provided none of the denominators in (3.1) are zero.*

**Proof.** Consider the function defined by

$$E_n(a, b) := \frac{(a)_n}{(b)_n} \quad (3.2)$$

I know [2, p. 1, (4)] that

$$(\ell)_n = \prod_{r=0}^{n-1} (\ell + r). \quad (3.3)$$

From (3.2) and (3.3), I conclude that

$$E_n(a, b) = \prod_{r=0}^{n-1} \frac{a+r}{b+r} = \frac{a(n+b)}{b(n+a)} \prod_{r=1}^n \frac{a+r}{b+r}. \quad (3.4)$$

On the other hand, replace  $u_r$  by  $a+r$  and  $v_r$  by  $b+r$  in Lemma 2.1

$$\begin{aligned} \prod_{r=1}^n \frac{a+r}{b+r} &= 1 + \sum_{k=1}^n \left( 1 - \frac{b+k}{a+k} \right) \prod_{r=1}^k \frac{a+r}{b+r} \\ &= 1 + \sum_{k=1}^n \frac{a-b}{a+k} \prod_{r=1}^k \frac{a+r}{b+r} \\ &= 1 + \sum_{k=1}^n \left( \frac{a-b}{a+k} \right) \frac{(a+1)_k}{(b+1)_k}. \end{aligned} \quad (3.5)$$

From (3.4) and (3.5), it follows that

$$E_n(a, b) = \frac{a(n+b)}{b(n+a)} \left[ 1 + \sum_{k=1}^n \left( \frac{a-b}{a+k} \right) \frac{(a+1)_k}{(b+1)_k} \right]. \quad (3.6)$$

Now, from (3.2) and (3.6), I conclude that

$$\frac{(a)_n}{(b)_n} = \frac{a(n+b)}{b(n+a)} \left[ 1 + \sum_{k=1}^n \left( \frac{a-b}{a+k} \right) \frac{(a+1)_k}{(b+1)_k} \right],$$

which is the desired result. □

### 3.2. The Decomposition of $(a)_n / (b)_n$ .

**Theorem 3.2.** *If  $a, b \in \mathbb{R}$ ,  $n \in \mathbb{Z}^+$  and  $k \in \mathbb{Z}^+$ , then*

$$\frac{(a)_n}{(b)_n} = \frac{a}{b} \cdot \frac{b+n}{a+n} \cdot \frac{(a+1)_{k-1}}{(b+1)_{k-1}} \cdot \frac{(b+n+1)_{2k-1}}{(a+n+1)_{2k-1}} \cdot \frac{(a+k)_{n+k}}{(b+k)_{n+k}}, \quad (3.7)$$

provided none of the denominators in (3.7) are zero.

**Proof.** Suppose the definition below

$$S_n(a, b) := \frac{b(n+a)}{a(n+b)} \cdot \frac{(a)_n}{(b)_n} = 1 + \sum_{k=1}^n \left( \frac{a-b}{a+k} \right) \frac{(a+1)_k}{(b+1)_k}, \quad (3.8)$$

by virtue of the Lemma 3.1.

Replace  $n$  by  $n+1$  in the right hand side of (3.8)

$$\begin{aligned} S_{n+1}(a, b) &= 1 + \sum_{k=1}^{n+1} \left( \frac{a-b}{a+k} \right) \frac{(a+1)_k}{(b+1)_k} \\ &= 1 + \sum_{k=1}^n \left( \frac{a-b}{a+k} \right) \frac{(a+1)_k}{(b+1)_k} + \left( \frac{a-b}{a+n+1} \right) \frac{(a+1)_{n+1}}{(b+1)_{n+1}} \\ &= S_n(a, b) + \left( \frac{a-b}{a+n+1} \right) \frac{(a+1)_{n+1}}{(b+1)_{n+1}} \\ &\Rightarrow S_{n+1}(a, b) - S_n(a, b) = \left( \frac{a-b}{a+n+1} \right) \frac{(a+1)_{n+1}}{(b+1)_{n+1}}. \end{aligned} \quad (3.9)$$

On the other hand, by definition above, I have

$$S_{n+1}(a, b) - S_n(a, b) = \frac{b(n+a+1)}{a(n+b+1)} \cdot \frac{(a)_{n+1}}{(b)_{n+1}} - \frac{b(n+a)}{a(n+b)} \cdot \frac{(a)_n}{(b)_n}. \quad (3.10)$$

From (3.9) and (3.10), it follows that

$$\frac{b(n+a+1)}{a(n+b+1)} \cdot \frac{(a)_{n+1}}{(b)_{n+1}} - \frac{b(n+a)}{a(n+b)} \cdot \frac{(a)_n}{(b)_n} = \left( \frac{a-b}{a+n+1} \right) \frac{(a+1)_{n+1}}{(b+1)_{n+1}}. \quad (3.11)$$

With a bit of manipulation (3.11) becomes

$$\frac{(a)_n}{(b)_n} = \frac{(n+a+1)(n+b)}{(n+a)(n+b+1)} \cdot \frac{(a)_{n+1}}{(b)_{n+1}} - \frac{a(a-b)(n+b)}{b(n+a)(a+n+1)} \cdot \frac{(a+1)_{n+1}}{(b+1)_{n+1}}. \quad (3.12)$$

Note that

$$\frac{(a+1)_{n+1}}{(a)_{n+1}} = \frac{n+a+1}{a} \Rightarrow (a)_{n+1} = \frac{a(a+1)_{n+1}}{n+a+1} \quad (3.13)$$

and

$$\frac{(b+1)_{n+1}}{(b)_{n+1}} = \frac{n+b+1}{b} \Rightarrow (b)_{n+1} = \frac{b(b+1)_{n+1}}{n+b+1}. \quad (3.14)$$

Divide (3.13) by (3.14)

$$\frac{(a)_{n+1}}{(b)_{n+1}} = \frac{a(n+b+1)}{b(n+a+1)} \cdot \frac{(a+1)_{n+1}}{(b+1)_{n+1}}. \quad (3.15)$$

From (3.12) and (3.15), I conclude that

$$\frac{(a)_n}{(b)_n} = \frac{a(n+b)(n+b+1)}{b(n+a)(n+a+1)} \cdot \frac{(a+1)_{n+1}}{(b+1)_{n+1}}. \quad (3.16)$$

Replace  $a$  by  $a + 1$ ,  $b$  by  $b + 1$  and  $n$  by  $n + 1$  in (3.16)

$$\frac{(a+1)_{n+1}}{(b+1)_{n+1}} = \frac{(a+1)(n+b+2)(n+b+3)}{(b+1)(n+a+2)(n+a+3)} \cdot \frac{(a+2)_{n+2}}{(b+2)_{n+2}}. \quad (3.17)$$

Substitute the right hand side of (3.17) in the right hand side of (3.16)

$$\frac{(a)_n}{(b)_n} = \frac{a(a+1)(n+b)(n+b+1)(n+b+2)(n+b+3)}{b(b+1)(n+a)(n+a+1)(n+a+2)(n+a+3)} \cdot \frac{(a+2)_{n+2}}{(b+2)_{n+2}}. \quad (3.18)$$

Replace  $a$  by  $a + 2$ ,  $b$  by  $b + 2$  and  $n$  by  $n + 2$  in (3.16)

$$\frac{(a+2)_{n+2}}{(b+2)_{n+2}} = \frac{(a+2)(n+b+4)(n+b+5)}{(b+2)(n+a+4)(n+a+5)} \cdot \frac{(a+3)_{n+3}}{(b+3)_{n+3}}. \quad (3.19)$$

Substitute the right hand side of (3.19) in the right hand side of (3.18)

$$\frac{(a)_n}{(b)_n} = \frac{a(a+1)(a+2)}{b(b+1)(b+2)} \cdot \frac{(n+b)(n+b+1)(n+b+2)(n+b+3)(n+b+4)(n+b+5)}{(n+a)(n+a+1)(n+a+2)(n+a+3)(n+a+4)(n+a+5)} \cdot \frac{(a+3)_{n+3}}{(b+3)_{n+3}}. \quad (3.20)$$

I note easily that for the  $k$ -th iteration, I get

$$\frac{(a)_n}{(b)_n} = \left( \prod_{r=0}^{k-1} \frac{a+r}{b+r} \right) \left( \prod_{r=0}^{2k-1} \frac{n+b+r}{n+a+r} \right) \cdot \frac{(a+k)_{n+k}}{(b+k)_{n+k}}. \quad (3.21)$$

On the other hand, I know [2, p. 1, (4)] that

$$\prod_{r=1}^k (a+r-1) = (a)_k. \quad (3.22)$$

Applying (3.22) in the right hand side of (3.21), I encounter

$$\frac{(a)_n}{(b)_n} = \frac{a}{b} \cdot \frac{b+n}{a+n} \cdot \frac{(a+1)_{k-1}}{(b+1)_{k-1}} \cdot \frac{(b+n+1)_{2k-1}}{(a+n+1)_{2k-1}} \cdot \frac{(a+k)_{n+k}}{(b+k)_{n+k}},$$

which is the desired result.  $\square$

**Corollary 3.3.** *If  $a \in \mathbb{R}$ ,  $n \in \mathbb{Z}^+$  and  $k \in \mathbb{Z}^+$ , then*

$$(a)_n = \frac{a}{a+n} \cdot \frac{(a+1)_{k-1}}{(a+n+1)_{2k-1}} \cdot (a+k)_{n+k}, \quad (3.23)$$

*provided none of the denominators in (3.23) are zero.*

**Proof.** Separate the variables  $a$  and  $b$  from the Theorem 3.2 and compare the both members.  $\square$

- [1] Bhatnagar, Gaurav, *In Praise of an Elementary Identity of Euler*, [arXiv:1102.0659v3](#) [math.CO] 12 Jun 2011.
- [2] Guedes, Edigles, *Infinite Product Representations for Binomial Coefficient, Pochhammer's Symbol, Newton's Binomial and Exponential Function*, June 27, 2016, [viXra:1611.0049](#).
- [3] Guedes, Edigles, *News Limit Formulas for Exponential of the Digamma Function, k-Power and Exponential Function*, July 10, 2018, [viXra:1807.0228](#).

#### 4. APPENDIX

I present below a new proof for an old identity of finite sum:

**Corollary 4.1.** *If  $|x| < 1$ ,  $x \neq 0$  and  $n \in \mathbb{Z}^+$ , then*

$$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}. \quad (4.1)$$

**Proof.** In [3, p. 4, Corollary 3.2], I have the following limit formula for the  $n$ -power

$$x^n = \lim_{\ell \rightarrow \infty} \frac{(x\ell)_n}{(\ell)_n} \quad (4.2)$$

Replace  $a$  by  $x\ell$  and  $b$  by  $\ell$  in Lemma 3.1

$$\begin{aligned} \frac{(x\ell)_n}{(\ell)_n} &= \frac{x\ell(n+\ell)}{\ell(n+x\ell)} \left[ 1 + \sum_{k=1}^n \left( \frac{x\ell - \ell}{x\ell + k} \right) \frac{(x\ell + 1)_k}{(\ell + 1)_k} \right] \\ &= x \frac{n+\ell}{n+x\ell} \left[ 1 + \sum_{k=1}^n \left( \frac{x\ell - \ell}{x\ell + k} \right) \frac{(x\ell + 1)_k}{(\ell + 1)_k} \right]. \end{aligned} \quad (4.3)$$

From (4.2) and (4.3), I conclude that

$$x^n = x \lim_{\ell \rightarrow \infty} \frac{n+\ell}{n+x\ell} \left[ 1 + \sum_{k=1}^n \left( \frac{x\ell - \ell}{x\ell + k} \right) \frac{(x\ell + 1)_k}{(\ell + 1)_k} \right]. \quad (4.4)$$

On the other hand, I know that

$$\lim_{\ell \rightarrow \infty} \frac{n+\ell}{n+x\ell} = \frac{1}{x}, \quad (4.5)$$

$$\lim_{\ell \rightarrow \infty} \frac{x\ell - \ell}{x\ell + k} = \frac{x-1}{x} = 1 - \frac{1}{x} \quad (4.6)$$

and

$$\lim_{\ell \rightarrow \infty} \frac{(x\ell + 1)_k}{(\ell + 1)_k} = x^k, \quad (4.7)$$

From (4.4),(4.5),(4.6) and (4.7), it follows that

$$\begin{aligned}x^n &= 1 + \left(1 - \frac{1}{x}\right) \sum_{k=1}^n x^k \\ \Rightarrow x^n - 1 &= \left(1 - \frac{1}{x}\right) \sum_{k=1}^n x^k \\ \Rightarrow \frac{x^n - 1}{x - 1} &= \sum_{k=1}^n x^{k-1} \\ \Rightarrow \sum_{k=0}^{n-1} x^k &= \frac{x^n - 1}{x - 1},\end{aligned}\tag{4.8}$$

which is the desired result.

□