

Sine Function at Rational Argument, Finite Product of Gamma Functions and Infinite Product Representation

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"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life." - John 6:63.

ABSTRACT. I corrected the Theorem 21 of previous paper, obtaining an identity for sine function at rational argument involving finite sum of the gamma functions; hence, the representation of infinite product arose.

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1. INTRODUCTION

In present paper, I corrected the Theorem 21 in [1, p. 9], obtaining the following identity

$$\frac{q}{p\pi} \sin\left(\frac{p\pi}{q}\right) = \prod_{s=1}^q \frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right)\Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)},$$

which enabled me to prove the following infinite product

$$\frac{q}{p\pi} \sin\left(\frac{p\pi}{q}\right) = \prod_{\ell=0}^{\infty} \prod_{s=1}^q \left(1 - \frac{p^2}{(q^2\ell + qs)^2}\right);$$

more specifically, I get

$$\begin{aligned} \frac{2}{\pi} &= \prod_{\ell=0}^{\infty} \left(1 - \frac{1}{(4\ell + 2)^2}\right) \left(1 - \frac{1}{(4\ell + 4)^2}\right) \\ &= \frac{3}{4} \cdot \frac{15}{16} \cdot \frac{35}{36} \cdot \frac{63}{64} \cdot \frac{99}{100} \cdot \frac{143}{144} \cdot \dots, \end{aligned}$$

$$\begin{aligned} \frac{3\sqrt{3}}{2\pi} &= \prod_{\ell=0}^{\infty} \left(1 - \frac{1}{(9\ell + 3)^2}\right) \left(1 - \frac{1}{(9\ell + 6)^2}\right) \left(1 - \frac{1}{(9\ell + 9)^2}\right) \\ &= \frac{8}{9} \cdot \frac{35}{36} \cdot \frac{80}{81} \cdot \frac{143}{144} \cdot \frac{224}{225} \cdot \frac{323}{324} \cdot \dots, \end{aligned}$$

$$\begin{aligned} \frac{2\sqrt{2}}{\pi} &= \prod_{\ell=0}^{\infty} \left(1 - \frac{1}{(16\ell+4)^2}\right) \left(1 - \frac{1}{(16\ell+8)^2}\right) \left(1 - \frac{1}{(16\ell+12)^2}\right) \left(1 - \frac{1}{(16\ell+16)^2}\right) \\ &= \frac{15}{16} \cdot \frac{63}{64} \cdot \frac{143}{144} \cdot \frac{255}{256} \cdot \frac{399}{400} \cdot \frac{575}{576} \cdot \frac{783}{784} \cdot \frac{1023}{1024} \cdots \end{aligned}$$

and so on.

2. PRELIMINARY

I use the following classical formula, [2, Section 12.13; 3], which is a Corollary of the Weierstrass infinite product representation for the gamma function:

Corollary 2.1. *If k is a positive integer and $a_1 + a_2 + \dots + a_k = b_1 + b_2 + \dots + b_k$, where the a_j and b_j are complex numbers and no b_j is zero or a negative integer, then*

$$\prod_{\ell=0}^{\infty} \frac{(\ell + a_1) \cdots (\ell + a_k)}{(\ell + b_1) \cdots (\ell + b_k)} = \frac{\Gamma(b_1) \cdots \Gamma(b_k)}{\Gamma(a_1) \cdots \Gamma(a_k)}. \quad (2.1)$$

Proof. See [2, Section 12.13]. □

3. THE MAIN THEOREM

3.1. The sine function at rational argument and the finite product of gamma functions.

Theorem 3.1. *If p and q are positive integers and $p \leq q$, then*

$$\frac{q}{p\pi} \sin\left(\frac{p\pi}{q}\right) = \prod_{s=1}^q \frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right) \Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)}, \quad (3.1)$$

where $\Gamma(z)$ denotes the gamma function and $\sin(z)$ denotes the sine function.

Proof. . Consider the Euler's infinite product representation for sine function [4, p. 321]

$$\frac{\sin(\pi z)}{\pi z} = \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right). \quad (3.2)$$

Let $z = p/q$ in (3.2), with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$, and encounter

$$\begin{aligned} \frac{q \sin\left(\frac{p\pi}{q}\right)}{p\pi} &= \prod_{j=1}^{\infty} \left(1 - \frac{p^2}{j^2 q^2}\right) \\ &= \prod_{k=0}^{\infty} \left(1 - \frac{p^2}{(k+1)^2 q^2}\right). \end{aligned} \quad (3.3)$$

Now, notice that for any $a \in \mathbb{Z}$ and $b \in \mathbb{N}$; there exists unique $c, d \in \mathbb{Z}$, such that $a = bc + d$ and $0 \leq d < b$ (division law in \mathbb{Z} , see [7, Lemma 7, p. 4]). Hither, this means that any $(k \in \mathbb{N}_0, q \in \mathbb{N})$ uniquely determine the integer r and s , such that $k = qr + s$, where $r = 0, 1, 2, \dots$ and $s = 1, 2, 3, \dots, q - 1$. Thereupon, it follows (by uniform convergence) that

$$\begin{aligned}
\frac{q \sin\left(\frac{p\pi}{q}\right)}{p\pi} &= \prod_{r=0}^{\infty} \prod_{s=0}^{q-1} \left(1 - \frac{p^2}{(qr+s+1)^2 q^2}\right) \\
&= \prod_{s=0}^{q-1} \prod_{r=0}^{\infty} \left(1 - \frac{p^2}{(qr+s+1)^2 q^2}\right) \\
&= \prod_{s=0}^{q-1} \frac{q^2 \Gamma^2\left(1 + \frac{s+1}{q}\right)}{(s+1)^2 \Gamma\left(\frac{q(s+1)-p}{q^2}\right) \Gamma\left(\frac{q(s+1)+p}{q^2}\right)} \\
&= \prod_{s=0}^{q-1} \frac{q^2}{(s+1)^2} \cdot \prod_{s=0}^{q-1} \frac{\Gamma^2\left(1 + \frac{s+1}{q}\right)}{\Gamma\left(\frac{q(s+1)-p}{q^2}\right) \Gamma\left(\frac{q(s+1)+p}{q^2}\right)} \\
\Rightarrow \frac{q \sin\left(\frac{p\pi}{q}\right)}{p\pi} &= \frac{q^{2q}}{\Gamma^2(q+1)} \cdot \prod_{s=0}^{q-1} \frac{\Gamma^2\left(1 + \frac{s+1}{q}\right)}{\Gamma\left(\frac{q(s+1)-p}{q^2}\right) \Gamma\left(\frac{q(s+1)+p}{q^2}\right)} \\
&= \frac{q^{2q}}{\Gamma^2(q+1)} \cdot \prod_{s=1}^q \frac{\Gamma^2\left(1 + \frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right) \Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)},
\end{aligned}$$

using the identity $\Gamma(1+z) = z\Gamma(z)$ in previous equation, I get

$$\begin{aligned}
\frac{q \sin\left(\frac{p\pi}{q}\right)}{p\pi} &= \frac{q^{2q}}{\Gamma^2(q+1)} \cdot \prod_{s=1}^q \left(\frac{s}{q}\right)^2 \frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right) \Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)} \\
&= \frac{q^{2q}}{\Gamma^2(q+1)} \cdot \prod_{s=1}^q \left(\frac{s}{q}\right)^2 \cdot \prod_{s=1}^q \frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right) \Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)} \\
&= \frac{q^{2q}}{\Gamma^2(q+1)} \cdot \frac{\Gamma^2(q+1)}{q^{2q}} \cdot \prod_{s=1}^q \frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right) \Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)} \\
&= \prod_{s=1}^q \frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right) \Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)} \\
\Rightarrow \frac{q}{p\pi} \sin\left(\frac{p\pi}{q}\right) &= \prod_{s=1}^q \frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right) \Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)},
\end{aligned}$$

which is the desired result. □

Example 3.2. Set $p = 1$ and $q = 2$ in Theorem 3.1

$$\frac{2}{\pi} = 4 \frac{\Gamma^2\left(\frac{1}{2}\right) \Gamma^2\left(\frac{2}{2}\right)}{\Gamma^2\left(\frac{1}{4}\right) \Gamma^2\left(\frac{3}{4}\right)},$$

using simplification, I obtain

$$\frac{1}{2\pi} = \frac{\Gamma^2(\frac{1}{2})\Gamma^2(\frac{2}{2})}{\Gamma^2(\frac{1}{4})\Gamma^2(\frac{3}{4})}.$$

Example 3.3. Set $p = 1$ and $q = 3$ in Theorem 3.1

$$\frac{3\sqrt{3}}{2\pi} = 9 \frac{\Gamma^2(\frac{1}{3})\Gamma^2(\frac{2}{3})\Gamma^2(\frac{3}{3})}{\Gamma(\frac{1}{9})\Gamma(\frac{2}{9})\Gamma(\frac{4}{9})\Gamma(\frac{5}{9})\Gamma(\frac{7}{9})\Gamma(\frac{8}{9})},$$

using simplification, I obtain

$$\frac{\sqrt{3}}{6\pi} = \frac{\Gamma^2(\frac{1}{3})\Gamma^2(\frac{2}{3})\Gamma^2(\frac{3}{3})}{\Gamma(\frac{1}{9})\Gamma(\frac{2}{9})\Gamma(\frac{4}{9})\Gamma(\frac{5}{9})\Gamma(\frac{7}{9})\Gamma(\frac{8}{9})}.$$

Example 3.4. Set $p = 1$ and $q = 4$ in Theorem 3.1

$$\frac{4}{\pi\sqrt{2}} = 16 \frac{\Gamma^2(\frac{1}{4})\Gamma^2(\frac{2}{4})\Gamma^2(\frac{3}{4})\Gamma^2(\frac{4}{4})}{\Gamma(\frac{1}{16})\Gamma(\frac{3}{16})\Gamma(\frac{5}{16})\Gamma(\frac{7}{16})\Gamma(\frac{9}{16})\Gamma(\frac{11}{16})\Gamma(\frac{13}{16})\Gamma(\frac{15}{16})},$$

using simplification, I obtain

$$\frac{\sqrt{2}}{8\pi} = \frac{\Gamma^2(\frac{1}{4})\Gamma^2(\frac{2}{4})\Gamma^2(\frac{3}{4})\Gamma^2(\frac{4}{4})}{\Gamma(\frac{1}{16})\Gamma(\frac{3}{16})\Gamma(\frac{5}{16})\Gamma(\frac{7}{16})\Gamma(\frac{9}{16})\Gamma(\frac{11}{16})\Gamma(\frac{13}{16})\Gamma(\frac{15}{16})}.$$

Example 3.5. Set $p = 1$ and $q = 5$ in Theorem 3.1

$$\begin{aligned} & \frac{5}{\pi} \sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}} \\ &= 25 \frac{\Gamma^2(\frac{1}{5})\Gamma^2(\frac{2}{5})\Gamma^2(\frac{3}{5})\Gamma^2(\frac{4}{5})\Gamma^2(\frac{5}{5})}{\Gamma(\frac{1}{25})\Gamma(\frac{4}{25})\Gamma(\frac{6}{25})\Gamma(\frac{9}{25})\Gamma(\frac{11}{25})\Gamma(\frac{14}{25})\Gamma(\frac{16}{25})\Gamma(\frac{19}{25})\Gamma(\frac{21}{25})\Gamma(\frac{24}{25})} \end{aligned}$$

using simplification, I obtain

$$\frac{1}{5\pi} \sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}} = \frac{\Gamma^2(\frac{1}{5})\Gamma^2(\frac{2}{5})\Gamma^2(\frac{3}{5})\Gamma^2(\frac{4}{5})\Gamma^2(\frac{5}{5})}{\Gamma(\frac{1}{25})\Gamma(\frac{4}{25})\Gamma(\frac{6}{25})\Gamma(\frac{9}{25})\Gamma(\frac{11}{25})\Gamma(\frac{14}{25})\Gamma(\frac{16}{25})\Gamma(\frac{19}{25})\Gamma(\frac{21}{25})\Gamma(\frac{24}{25})}.$$

3.2. New infinite product representation for the sine function at rational argument.

Corollary 3.6. If p and q are positive integers and $p \leq q$, then

$$\frac{q}{p\pi} \sin\left(\frac{p\pi}{q}\right) = \prod_{\ell=0}^{\infty} \prod_{s=1}^q \left(1 - \frac{p^2}{(q^2\ell + qs)^2}\right), \quad (3.4)$$

where $\sin(z)$ denotes the sine function.

Proof. Note that

$$\frac{s}{q} + \frac{s}{q} = \frac{s}{q} - \frac{p}{q^2} + \frac{s}{q} + \frac{p}{q^2},$$

satisfy the condition $a_1 + a_2 = b_1 + b_2$; $k = 2$ is a positive integer; the a_j and b_j are complex numbers and no b_j is zero or a negative integer. From Corollary 2.1 and the right hand side of the (3.1), it follows that

$$\begin{aligned} \frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right)\Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)} &= \prod_{\ell=0}^{\infty} \frac{\left(\ell + \frac{s}{q} - \frac{p}{q^2}\right)\left(\ell + \frac{s}{q} + \frac{p}{q^2}\right)}{\left(\ell + \frac{s}{q}\right)\left(\ell + \frac{s}{q}\right)} \\ &= \prod_{\ell=0}^{\infty} \left(1 - \frac{p^2}{(q^2\ell + qs)^2}\right). \end{aligned} \quad (3.5)$$

From Theorem 3.1 and (3.5), I conclude that

$$\frac{q}{p\pi} \sin\left(\frac{p\pi}{q}\right) = \prod_{\ell=0}^{\infty} \prod_{s=1}^q \left(1 - \frac{p^2}{(q^2\ell + qs)^2}\right),$$

which is the desired result. \square

Example 3.7. Set $p = 1$ and $q = 2$ in Corollary 3.6

$$\begin{aligned} \frac{2}{\pi} &= \prod_{\ell=0}^{\infty} \left(1 - \frac{1}{(4\ell + 2)^2}\right) \left(1 - \frac{1}{(4\ell + 4)^2}\right) \\ &= \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{4^2}\right) \left(1 - \frac{1}{6^2}\right) \left(1 - \frac{1}{8^2}\right) \left(1 - \frac{1}{10^2}\right) \left(1 - \frac{1}{12^2}\right) \cdots \\ &= \frac{3}{4} \cdot \frac{15}{16} \cdot \frac{35}{36} \cdot \frac{63}{64} \cdot \frac{99}{100} \cdot \frac{143}{144} \cdots \end{aligned}$$

Example 3.8. Set $p = 1$ and $q = 3$ in Corollary 3.6

$$\begin{aligned} \frac{3\sqrt{3}}{2\pi} &= \prod_{\ell=0}^{\infty} \left(1 - \frac{1}{(9\ell + 3)^2}\right) \left(1 - \frac{1}{(9\ell + 6)^2}\right) \left(1 - \frac{1}{(9\ell + 9)^2}\right) \\ &= \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{6^2}\right) \left(1 - \frac{1}{9^2}\right) \left(1 - \frac{1}{12^2}\right) \left(1 - \frac{1}{15^2}\right) \left(1 - \frac{1}{18^2}\right) \cdots \\ &= \frac{8}{9} \cdot \frac{35}{36} \cdot \frac{80}{81} \cdot \frac{143}{144} \cdot \frac{224}{225} \cdot \frac{323}{324} \cdots \end{aligned}$$

Example 3.9. Set $p = 1$ and $q = 4$ in Corollary 3.6

$$\begin{aligned} \frac{2\sqrt{2}}{\pi} &= \prod_{\ell=0}^{\infty} \left(1 - \frac{1}{(16\ell + 4)^2}\right) \left(1 - \frac{1}{(16\ell + 8)^2}\right) \left(1 - \frac{1}{(16\ell + 12)^2}\right) \left(1 - \frac{1}{(16\ell + 16)^2}\right) \\ &= \left(1 - \frac{1}{4^2}\right) \left(1 - \frac{1}{8^2}\right) \left(1 - \frac{1}{12^2}\right) \left(1 - \frac{1}{16^2}\right) \\ &\quad \left(1 - \frac{1}{20^2}\right) \left(1 - \frac{1}{24^2}\right) \left(1 - \frac{1}{28^2}\right) \left(1 - \frac{1}{32^2}\right) \cdots \\ &= \frac{15}{16} \cdot \frac{63}{64} \cdot \frac{143}{144} \cdot \frac{255}{256} \cdot \frac{399}{400} \cdot \frac{575}{576} \cdot \frac{783}{784} \cdot \frac{1023}{1024} \cdots \end{aligned}$$

Example 3.10. Set $p = 3$ and $q = 4$ in Corollary 3.6

$$\begin{aligned} \frac{2\sqrt{2}}{3\pi} &= \prod_{\ell=0}^{\infty} \left(1 - \frac{9}{(16\ell+4)^2}\right) \left(1 - \frac{9}{(16\ell+8)^2}\right) \left(1 - \frac{9}{(16\ell+12)^2}\right) \left(1 - \frac{9}{(16\ell+16)^2}\right) \\ &= \left(1 - \frac{9}{4^2}\right) \left(1 - \frac{9}{8^2}\right) \left(1 - \frac{9}{12^2}\right) \left(1 - \frac{9}{16^2}\right) \\ &\quad \left(1 - \frac{9}{20^2}\right) \left(1 - \frac{9}{24^2}\right) \left(1 - \frac{9}{28^2}\right) \left(1 - \frac{9}{32^2}\right) \cdots \\ &= \frac{7}{16} \cdot \frac{55}{64} \cdot \frac{135}{144} \cdot \frac{247}{256} \cdot \frac{391}{400} \cdot \frac{567}{576} \cdot \frac{775}{784} \cdot \frac{1015}{1024} \cdots \end{aligned}$$

Example 3.11. Set $p = 1$ and $q = 5$ in Corollary 3.6

$$\begin{aligned} \frac{5}{\pi} \sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}} &= \prod_{\ell=0}^{\infty} \left(1 - \frac{1}{(25\ell+5)^2}\right) \left(1 - \frac{1}{(25\ell+10)^2}\right) \\ &\quad \left(1 - \frac{1}{(25\ell+15)^2}\right) \left(1 - \frac{1}{(25\ell+20)^2}\right) \left(1 - \frac{1}{(25\ell+25)^2}\right) \\ &= \left(1 - \frac{1}{5^2}\right) \left(1 - \frac{1}{10^2}\right) \left(1 - \frac{1}{15^2}\right) \left(1 - \frac{1}{20^2}\right) \left(1 - \frac{1}{25^2}\right) \\ &\quad \left(1 - \frac{1}{30^2}\right) \left(1 - \frac{1}{35^2}\right) \left(1 - \frac{1}{40^2}\right) \left(1 - \frac{1}{45^2}\right) \left(1 - \frac{1}{50^2}\right) \cdots \\ &= \frac{24}{25} \cdot \frac{99}{100} \cdot \frac{224}{225} \cdot \frac{399}{400} \cdot \frac{624}{625} \cdot \frac{899}{900} \cdot \frac{1224}{1225} \cdot \frac{1599}{1600} \cdot \frac{2024}{2025} \cdot \frac{2499}{2500} \cdots \end{aligned}$$

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