Modified general relativity and the Klein-Gordon equation in curved spacetime

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Abstract

The Klein-Gordon equation in curved spacetime can be symmetrized into symmetric and antisymmetric rank 2 tensors for bosons with spins 0,1,2 and spinor-tensors for fermions with spins $1/2, 3/2$. The tensors in a modified equation of general relativity which add to zero are shown to belong to the symmetric part of the Klein-Gordon equation. Modified general relativity is intrinsically hidden in the Klein-Gordon equation and the formalism of quantum field theory. The metric as a field variable describing gravitons vanishes from the massless spin-2 Klein-Gordon equation in the long-range to particle regimes of a spacetime described by a 4-dimensional time oriented Lorentzian manifold with a torsionless and metric compatible connection. Massless gravitons do not exist as force mediators of gravity in these regimes of spacetime.

Keywords
quantum mechanics; Klein-Gordon; general relativity; dark energy; dark matter, quantum field theory, quantum gravity

1. Introduction

It has been nearly a century since Schrödinger [1] wrote down his equation describing non-relativistic quantum mechanics. In the same year of 1926, Klein [2], Gordon [3] and Fock [4] developed the relativistic quantum mechanical wave equation; mainly referred to as the Klein-Gordon (KG) equation. Over a decade before that, Einstein [5] formulated general relativity (GR) in 1915. And yet today, there is still not a full understanding of the relationship between the two fundamental theories of physics: quantum theory and general relativity. The quantization of gravity has been the major approach to unite the two theories, with string theory and loop quantum gravity the two mainstream proposals. However, those and other theories of quantum gravity have well documented successes and failures [6, 7]. Rather than trying to force quantum theory on general relativity, or vice versa, this article investigates if a connection between quantum theory and general relativity exists naturally.

The Klein-Gordon equation for a free field with a particular spin in Minkowski spacetime is fundamental to the formulation of quantum field theory (QFT). In curved spacetime, the covariant KG equation is assumed to be the rudimentary equation for the development of quantum theory. It is an asymmetric wave equation which can be symmetrized into symmetric and antisymmetric rank 2 tensors describing bosons with spins 0,1,2; and spinor-tensors for fermions with spins $1/2, 3/2$. This requires the existence of a smooth, real, non-vanishing vector field, A^β. It, its scalar product $\varphi = A_\beta A^\beta$, its covariant derivative $\Psi^{\alpha\beta} = \nabla^\alpha A^\beta$ and related spinor-tensors, represent the fields of the KG equation in curved spacetime. A^β has the role of a fundamental quantum vector field. It directly represents the spin-1 Proca field and is the vector field of the electromagnetic field for massless spin-1 bosons. As GR involves symmetric tensors, the only possibility to associate GR directly with quantum theory in curved spacetime is through the symmetric part of the KG equation. In that regard, it is noteworthy that the Lie derivative

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of the metric with respect to A^β *is* the symmetric part of the KG equation, $\tilde{\Psi}_{\alpha\beta}$. This immediately exhibits a geometrical property of quantum theory which has been neglected.

Recently, a modified Einstein equation of general relativity was developed by Nash [8] with the introduction of a symmetric tensor, $\Phi_{\alpha\beta}$, that describes the energy-momentun of the gravitational field. It is constructed from the Lie derivative of both the metric and the unit vectors collinear with one of the pair of regular vectors in the line element field (X,-X). Thus, the Lie derivative of the metric is involved with both $\tilde{\Psi}_{\alpha\beta}$ and $\Phi_{\alpha\beta}$. This links general relativity to quantum theory as dicussed in detail in section 2. The relationship between $\tilde{\Psi}_{\alpha\beta}$ and the KG equation for each spin is investigated in section 3 to prove that GR resides intrinsically in the symmetric part of the KG equation for spins $0, 1, 2, 1/2$ and $3/2$.

In section 4, some interesting results appear from the study of spin-2 massless "particles". Gravity, described by the metric, is a long-range *effective* force which is assumed to extend down to the particle regime of fermi lengths. If gravitons are the exchange particles of gravity, they must be massless in those regimes. In a spacetime described by a four-dimensional time oriented Lorentzian manifold with a torsionless and metric compatible connection, it is shown that massless spin-2 gravitons cannot be described with the metric in the long-range to particle regimes of spacetime; and massless spin-2 "particles" do not couple to a non-zero energy-momentum tensor. Massless gravitons therefore do not act as force mediators of gravity in those regimes. This result should not be viewed controversially. It is well known that GR, as a classical field theory, does not require particle exchange to describe the effective force of gravity; that is nicely done by the curvature of spacetime. Furthermore, there is nothing in the formalism of QFT that requires GR to be quantized. That was noted by Feynman who said: [9] "It is still possible that quantum theory does not absolutely guarantee that gravity *has* to be quantized". Gravity, unlike the other three known forces in nature, does not require the exchange of particles to describe its long-range force behavior. This explains why gravity is so much weaker than the long-range electromagnetic force which involves the photon as the exchange particle.

As discussed in section 5, quantum field theory in curved spacetime is formulated from free field solutions to the KG equation. GR resides beautifully within QFT.

The wave-particle duality is discussed in section 6. The symmetric part of the KG equation contains a topological tensor that satisfies a wave equation for all spins. Since the particle nature of spin-1 bosons and spin-1/2 fermions is described by the antisymmetric part of the KG equation, those particles have a wave nature (and conversely) at all times because both parts of the symmetrized KG equation have the same space time coordinates. This explains the experimental results that photons and electrons have simultaneous wave and particle characteristics.

In this article, the Lorentzian metric $g_{\alpha\beta}$, which describes both the gravitational field and the geometry of spacetime, is preserved in the sense that it is not treated in any way as a perturbation to the flat spacetime Minkowski metric; or to any other metric acting as a background field. The results obtained from the KG equation which depend on the metric do not involve linear or higher order approximations to any non-linear quantity.

2. Decomposition of the Klein-Gordon Equation in curved spacetime

Curved spacetime is described by the 4-dimensional time oriented Lorentzian manifold with a +2 signature metric, $(M, g_{\alpha\beta})$. The connection on the manifold is torsionless and metric compatible. The Lorentzian manifold is assumed to be compact with a vanishing Euler-Poincaré characteristic. It admits a smooth regular vector field A^β and a smooth regular line element field $(X^\beta, -X^\beta)$. More specifically, a compact orientable manifold with a vanishing Euler-Poincaré characteristic admits a smooth regular vector field [10, 13]; a compact orientable manifold with a Lorentzian metric and a vanishing Euler-Poincaré characteristic admits a smooth non-vanishing line element field [11, 12, 13].

The KG equation of special relativity is $\partial_a \partial^a \Psi = k^2 \Psi$ where $k = \frac{m_0 c}{\hbar}$. In the transition to curved spacetime, it becomes

$$\nabla_\mu \nabla^\mu \Psi = k^2 \Psi. \tag{1}$$

This is an asymmetric wave equation with Ψ representing spins 0,1,2,1/2 and 3/2. The KG equation can be constructed from the regular vector field A^β with real components and the (0,2) tensor $\Psi_{\alpha\beta} = \nabla_\alpha A_\beta$, for all spins, as discussed in section 3. In that sense, A^β is a fundamental quantum vector field. $\Psi_{\alpha\beta}$ can be symmetrized according to

$$\begin{aligned}\Psi_{\alpha\beta} &= \frac{1}{2}(\nabla_\alpha A_\beta + \nabla_\beta A_\alpha) + \frac{1}{2}(\nabla_\alpha A_\beta - \nabla_\beta A_\alpha) \\ &:= \frac{1}{2}\tilde{\Psi}_{\alpha\beta} + \frac{1}{2}K_{\alpha\beta}.\end{aligned} \tag{2}$$

The Lie derivative of the metric with respect to A^β is the symmetric part of the KG equation, $\tilde{\Psi}_{\alpha\beta}$.

The modified equation of general relativity developed in [8]

$$-\frac{8\pi G}{c^4}\tilde{T}_{\alpha\beta} + G_{\alpha\beta} + \Phi_{\alpha\beta} = 0 \tag{3}$$

contains a symmetric tensor $\Phi_{\alpha\beta}$ that represents the energy-momentum of the gravitational field:

$$\begin{aligned}\Phi_{\alpha\beta} &= \frac{1}{2}\pounds_X g_{\alpha\beta} + \pounds_X u_\alpha u_\beta \\ &= \frac{1}{2}(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha) + u^\lambda(u_\alpha \nabla_\beta X_\lambda + u_\beta \nabla_\alpha X_\lambda)\end{aligned} \tag{4}$$

where X is one of the pair of the line element field that exists, and u is a timelike unit vector collinear with X. $\tilde{T}_{\alpha\beta}$ is the total matter energy-momentum tensor and $G_{\alpha\beta}$ is the Einstein tensor. The energy-momentum tensor $T^{\alpha\beta} = \tilde{T}^{\alpha\beta} - \frac{c^4}{8\pi G}\Phi^{\alpha\beta}$ is locally conserved, $\nabla_\alpha T^{\alpha\beta} = 0$. The modified Einstein equation can then be written in the same form as he wrote it:

$$\frac{8\pi G}{c^4}T_{\alpha\beta} = G_{\alpha\beta}. \tag{5}$$

The trace of $\Phi_{\alpha\beta}$ with respect to the metric dynamically replaces the cosmological constant, which does not appear in equation (3). From (3) and (4)

$$\begin{aligned}\pounds_X g_{\alpha\beta} &= 2(\Phi_{\alpha\beta} - \pounds_X u_\alpha u_\beta) \\ &= 2(\frac{8\pi G}{c^4}\tilde{T}_{\alpha\beta} - G_{\alpha\beta} - \Phi_{\alpha\beta} + \frac{1}{2}(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha)) \\ &= \tilde{\Psi}_{\alpha\beta}\end{aligned} \tag{6}$$

if the the quantum vector is one of the pair in the line element field, and conversely. On an orientable Lorentzian manifold with a vanishing Euler-Poincaré characteristic, that is always possible. Thus, a geometrical link between modified general relativity and quantum theory is established. Modified general relativty is intrinsically hidden in the symmetric part of the KG equation.

$\tilde{\Psi}_{\alpha\beta}$ must be divergenceless

$$\nabla_\alpha \tilde{\Psi}^{\alpha\beta} = 0 \tag{7}$$

to eliminate four of the ten degrees of freedom inherent in it. $\tilde{\Psi}_{\alpha\beta}$ can then be expressed as the sum of a symmetric traceless pure spin-2 field $C_{\alpha\beta}$ and a spin-0 field accompanying the metric

$$\tilde{\Psi}_{\alpha\beta} = C_{\alpha\beta} + \frac{1}{4}\tilde{\Psi}g_{\alpha\beta} \tag{8}$$

where $\tilde{\Psi} = 2\nabla_\alpha A^\alpha$ has the property $\partial_\alpha \tilde{\Psi} = 0$. $\tilde{\Psi}_{\alpha\beta}$ is traceless only when the Lorentz constraint

$$\nabla_\alpha A^\alpha = 0 \tag{9}$$

is involved, which is inherently the case with the spin-1 KG equation as discussed in subsection 3.1. The traceless criteria was used in [8] as a condition to derive the modified Einstein equation. This introduced a traceless expression that can represent $C_{\alpha\beta}$ which contains a collection of divergenceless tensor fields $h_{\alpha\beta}$ that are independent of the Lovelock tensors $g_{\alpha\beta}$ and $G_{\alpha\beta}$; and has the property $\partial_\alpha h = 0$ where h is the trace of $h_{\alpha\beta}$:

$$C_{\alpha\beta} = \frac{c^3}{16\pi G}(h_{\alpha\beta} - \frac{1}{4}hg_{\alpha\beta}). \tag{10}$$

This expression intrinsically contains the tensors of the modified Einstein equation which add to zero. The tensor fields $h_{\alpha\beta}$ have dimension L^{-2} which eliminates the superenergy divergenceless tensors, such as the traces of the Chevreton and Bach tensors which depend on L^{-4}.

Equation (10) is a more general expression for the decomposition of a symmetric tensor than it appears to be. Anderson [14] investigated a general symmetric collection of tensors of the form $D_{\alpha\beta} = D_{\alpha\beta}(g_{\mu\nu}; g_{\mu\nu,\lambda}; g_{\mu\nu,\lambda\sigma}; \varrho_Q; \varrho_{Q,\lambda}; \varrho_{Q,\lambda\sigma})$ which satisfies

$$\nabla_\alpha D^{\alpha\beta} = 0 \tag{11}$$

where ϱ_Q is a collection of tensor fields of arbitrary rank and weight which are independent of the metric; and proved that $D_{\alpha\beta}$ is independent of the fields ϱ_Q and their derivatives. In particular,

$$\frac{\partial D_{\alpha\beta}}{\partial \varrho_Q} = 0 \tag{12}$$

for all ϱ_Q. By choosing $h_{\alpha\beta}$ as one of the ϱ_Q fields, $C_{\alpha\beta}$ can be constructed from it and the metric. Because equation (11) is unique up to a divergenceless symmetric tensor, $C_{\alpha\beta}$ can be added to $D_{\alpha\beta}$ and not affect (12) because $\frac{\partial C_{\alpha\beta}}{\partial \varrho_Q} = 0$. Just as modified GR depends on the metric to describe the gravitational field, quantum theory requires the metric to describe a graviton. The spin-2 KG field $\tilde{\Psi}_{\alpha\beta}$ must have the general structure of $D_{\alpha\beta}$ with the traceless divergenceless residual tensor $C_{\alpha\beta}$. Therefore,

$$C_{\alpha\beta} = \frac{c^3}{16\pi G}(h_{\alpha\beta} - \frac{1}{4}hg_{\alpha\beta}) \tag{13}$$

is a general expression for the traceless spin-2 field which involves the metric as a field variable. It intrinsically contains the tensors of the modified Einstein equation which add to zero, and satisfies (7).

As will become evident, it is important to preserve the form of (1) in curved spacetime.

3. Constructing the Klein-Gordon equation for each spin from the quantum vector A^β in curved spacetime

The KG equation can be constructed from the vector field A^β with real components and the (2,0) tensor $\Psi^{\alpha\beta} = \nabla^\alpha A^\beta$, for all spins. While this seems trivial, there are some subtle points to discuss.

3.1. Spin-1 Klein-Gordon equation

If Ψ is the real field vector A^β, (1) yields the covariant equation

$$\nabla_\alpha \nabla^\alpha A^\beta = k^2 A^\beta. \tag{14}$$

In curved spacetime, covariant derivatives generally do not commute, so the ansatz

$$\nabla_\alpha K^{\alpha\beta} = k^2 A^\beta \tag{15}$$

is not equivalent to $\nabla_\alpha \nabla^\alpha A^\beta = k^2 A^\beta$ and $\nabla_\alpha A^\alpha = 0$, contrary to the situation in Minkowski spacetime. Rather,

$$\nabla_\alpha K^{\alpha\beta} = k^2 A^\beta - \nabla_\alpha \nabla^\beta A^\alpha. \tag{16}$$

Since the Proca equation in curved spacetime [15, 22] is traditionally described by (15), the KG equation must be expressed with a new term as

$$\nabla_\alpha \nabla^\alpha A^\beta = k^2 A^\beta + \nabla_\alpha \nabla^\beta A^\alpha. \tag{17}$$

This method restricts the left side of (14) to that of an antisymmetric tensor and modifies the form of the KG equation.

Contrary to this approach, we choose to retain the form of the KG equation given by (14) and abandon the *sole* reliance on the antisymmetric tensor $K^{\alpha\beta}$. The KG equation can be preserved and still involve the useful tensor $K^{\alpha\beta}$ by using (2) and expressing the spin-1 KG equation as

$$\nabla_\alpha(\tilde{\Psi}^{\alpha\beta} + K^{\alpha\beta}) = 2k^2 A^\beta. \tag{18}$$

which simplifies to

$$\nabla_\alpha K^{\alpha\beta} = 2k^2 A^\beta \tag{19}$$

using (7). In general, this equation has both spin-0 and spin-1 characteristics; but the Lorentz constraint (9) ensures it describes a neutral vector boson. The Lorentz constraint is intrinsic to the spin-1 KG equation which follows directly from (19):

$$-[\nabla_\alpha, \nabla_\beta]K^{\alpha\beta} = -2R_{\alpha\beta}K^{\alpha\beta} = 4k^2\nabla_\beta A^\beta = 0 \tag{20}$$

because $R_{\alpha\beta}$ and $K^{\alpha\beta}$ have opposite symmetries.

From the commutation relation with the Lorentz constraint

$$\begin{aligned}[\nabla_\alpha, \nabla_\lambda]A^\alpha &= \nabla_\alpha\nabla_\lambda A^\alpha \\ &= A^\sigma R_{\sigma\lambda}\end{aligned} \tag{21}$$

it follows that

$$\nabla_\alpha K^{\alpha\beta} = k^2 A^\beta - A^\sigma R^\beta_\sigma. \tag{22}$$

By retaining the form of the KG equation in curved spacetime, (19) requires

$$k^2 A^\beta = -A^\sigma R^\beta_\sigma \tag{23}$$

which relates the geometry of spacetime to the mass of the quantum vector components.

3.2. Spin-0 Klein-Gordon equation

When Ψ is a scalar field φ, the spin-0 Klein-Gordon equation is

$$\nabla_\alpha\nabla^\alpha\varphi = k^2\varphi. \tag{24}$$

With φ defined in terms of the quantum vector field A^β as

$$\varphi := A^\beta A_\beta, \tag{25}$$

and using (2) and (7), we have

$$\begin{aligned}\nabla_\alpha\nabla^\alpha\varphi &= 2\nabla_\alpha(A_\beta\Psi^{\alpha\beta}) \\ &= A_\beta\nabla_\alpha K^{\alpha\beta} + \frac{1}{2}(\tilde{\Psi}_{\alpha\beta}\tilde{\Psi}^{\alpha\beta} + K_{\alpha\beta}K^{\alpha\beta})\end{aligned} \tag{26}$$

where $\tilde{\Psi}^{\alpha\beta}K_{\alpha\beta} = 0$ because the tensors in the product have opposite symmetries.

3.3. Spin-1/2 Klein-Gordon equation

The spin-1/2 KG equation in curved spacetime expressed in terms of its spinor indices is

$$\nabla_\alpha\nabla^\alpha\Psi^{A\dot{A}} = k^2\Psi^{A\dot{A}}. \tag{27}$$

In accordance with [16, 17], the two index spinors $\varphi^{A\dot{B}}$ and $\varphi_{A\dot{B}}$ can be expressed in terms of the associated tensors A^β and A_β as

$$\varphi^{A\dot{B}} = \sigma^{A\dot{B}}_\beta A^\beta \tag{28}$$

and

$$\varphi_{A\dot{B}} = \sigma^\beta_{A\dot{B}} A_\beta. \tag{29}$$

The Hermitian connecting quantities $\sigma_\beta^{A\dot{B}}$ transform as a spacetime vector on the index β and as spinors on the index $A = 1,2$ and conjugate index $\dot{B} = 1,2$. Covariant derivatives of spinors are introduced in the same formalism as that for tensors by adopting the spinor affinities $\Gamma^A_{\alpha B}$ and defining

$$\nabla_\alpha \Psi_A = \partial_\alpha \Psi_A - \Gamma^B_{\alpha A}\Psi_B, \;\; \nabla_\alpha \Psi^A = \partial_\alpha \Psi^A + \Gamma^A_{\alpha B}\Psi^B \tag{30}$$

for the spinors Ψ_A and Ψ^A respectively. The covariant derivative of a mixed index spinor-tensor is defined as

$$\nabla_\alpha \Psi^{\beta A} = \partial_\alpha \Psi^{\beta A} + \Gamma^\beta_{\alpha\kappa}\Psi^{\kappa A} + \Gamma^A_{\alpha B}\Psi^{\beta B} \tag{31}$$

and the covariant derivative of the connection quantities is postulated to vanish

$$\nabla_\kappa \sigma^\alpha_{A\dot{B}} = 0. \tag{32}$$

This means

$$\partial_\kappa \sigma^\alpha_{A\dot{B}} + \sigma^\beta_{A\dot{B}}\Gamma^\alpha_{\kappa\beta} - \sigma^\alpha_{C\dot{B}}\Gamma^C_{\kappa A} - \sigma^\alpha_{A\dot{C}}\Gamma^{\dot{C}}_{\kappa\dot{B}} = 0. \tag{33}$$

The spinor-tensor covariant derivative is equivalent to the tensor covariant derivative if and only if this postulate holds. We can prove that from the covariant derivatives of a vector A^α and its spinor equivalent as follows:

$$\begin{aligned}\nabla_\kappa A^\alpha &= \sigma^\alpha_{A\dot{B}}\nabla_\kappa A^{A\dot{B}} \\ &= \sigma^\alpha_{A\dot{B}}(\partial_\kappa A^{A\dot{B}} + \Gamma^A_{\kappa C}A^{C\dot{B}} + \Gamma^{\dot{B}}_{\kappa\dot{C}}A^{A\dot{C}}) \\ &= \partial_\kappa A^\alpha + \Gamma^\alpha_{\kappa\beta}A^\beta - A^{A\dot{B}}(\partial_\kappa \sigma^\alpha_{A\dot{B}} + \sigma^\beta_{A\dot{B}}\Gamma^\alpha_{\kappa\beta} - \sigma^\alpha_{C\dot{B}}\Gamma^C_{\kappa A} - \sigma^\alpha_{A\dot{C}}\Gamma^{\dot{C}}_{\kappa\dot{B}}) \\ &= A^\alpha{}_{;\kappa}\end{aligned} \tag{34}$$

Equation (27) is then equivalent to

$$\sigma_\beta^{A\dot{A}}\nabla_\alpha\nabla^\alpha A^\beta = k^2\Psi^{A\dot{A}} \tag{35}$$

using (28). This can be rewritten in terms of $\Psi^{\alpha\beta} = \nabla^\alpha A^\beta$ and symmetrized as in (2) to give

$$\sigma_\beta^{A\dot{A}}\nabla_\alpha(\tilde{\Psi}^{\alpha\beta} + K^{\alpha\beta}) = 2k^2\sigma_\beta^{A\dot{A}}A^\beta \tag{36}$$

which simplifies to

$$\sigma_\beta^{A\dot{A}}\nabla_\alpha K^{\alpha\beta} = 2k^2\sigma_\beta^{A\dot{A}}A^\beta \tag{37}$$

using (7).

Remark 1. *The Dirac equations in curved spacetime must be modified to be solutions of their parent spin-*1/2 *KG equation.*

The Dirac equations in curved spacetime

$$(\gamma^\nu\nabla_\nu - k)\Psi^A = 0, \;\; (\gamma^\nu\nabla_\nu + k)\Psi^{\dot{A}} = 0 \tag{38}$$

are taken to be factorizations of the spin-1/2 KG equation. The gamma matrices γ^μ in curved spacetime are assumed to satisfy the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{39}$$

The product of the Dirac factorizations

$$(\gamma^\mu\nabla_\mu + k)(\gamma^\nu\nabla_\nu - k)\Psi^{A\dot{A}} = 0 \tag{40}$$

must yield the spin-1/2 KG equation. Using

$$\nabla_\mu\gamma^\nu = 0 \tag{41}$$

we have

$$\gamma^\mu \gamma^\nu \nabla_\mu \nabla_\nu \Psi^{A\dot{A}} = k^2 \Psi^{A\dot{A}} \tag{42}$$

which is expressed in the literature [18, 19] (with the metric having a +2 signature) as

$$\nabla_\mu \nabla^\mu \Psi^{A\dot{A}} = (k^2 + \frac{1}{4}R)\Psi^{A\dot{A}}. \tag{43}$$

With only the algebra of (39), the spin-1/2 KG equation in curved spacetime is not precisely recoverable due to the additional $\frac{1}{4}R$ term. This term can be eliminated by inserting the scalar $\Omega = \frac{1}{2}\sqrt{R}$ into (38) to obtain the modified Dirac equations

$$(\gamma^\nu \nabla_\nu + \Omega - k)\Psi^A = 0, \;\; (\gamma^\nu \nabla_\nu + \Omega + k)\Psi^{\dot{A}} = 0. \tag{44}$$

By defining the algebras

$$\{\gamma^\mu \nabla_\mu, \Omega\} = 0 \tag{45}$$

and

$$\frac{1}{2}\{\Omega, \Omega\} = \frac{R}{4}, \tag{46}$$

the product of the modified Dirac equations in curved spacetime yields their parent spin-1/2 KG equation (27).

3.4. Spin-2 and spin-3/2 Klein-Gordon equations

The KG equation for a spin-2 field is

$$\nabla_\mu \nabla^\mu \Psi_{\alpha\beta} = k^2 \Psi_{\alpha\beta}. \tag{47}$$

The spin-2 field is unique in that it and the metric are second rank tensors. Gravitons are the particles associated with the metric. The symmetric spin-2 field must therefore involve the metric as a field variable to enable gravitons to be described by the spin-2 KG equation. It must also be traceless with respect to the metric, and divergenceless; it has 5 degrees of freedom if $k \neq 0$. The general decomposition (13)

$$C_{\alpha\beta} = \frac{c^3}{16\pi G}(h_{\alpha\beta} - \frac{1}{4}hg_{\alpha\beta}) \tag{48}$$

satisfies these properties and automatically incorporates the metric as a field variable in the symmetric part of the spin-2 field. With a metric compatible connection, $\Psi_{\alpha\beta}$ must satisfy

$$\nabla_\mu \nabla^\mu h_{\alpha\beta} = k^2(h_{\alpha\beta} - \frac{1}{4}hg_{\alpha\beta}) \tag{49}$$

with $h = h^{\alpha\beta}g_{\alpha\beta}$ a solution to $\partial_\mu h = 0$ and the wave equation

$$\nabla_\mu \nabla^\mu h = 0. \tag{50}$$

The action

$$S^{(2)} = \frac{1}{2}\int (\nabla_\mu \Psi_{\alpha\beta} \nabla^\mu \Psi^{\alpha\beta} + k^2 \Psi_{\alpha\beta}\Psi^{\alpha\beta})\sqrt{-g}d^4x \tag{51}$$

can be used to derive the spin-2 KG equation (47).

Spin-3/2. A spin-3/2 field can be described by the vector-spinor wave equation

$$\nabla_\mu \nabla^\mu \Psi_{\alpha A\dot{A}} = k^2 \Psi_{\alpha A\dot{A}}. \tag{52}$$

Using (29), this equation can be written as

$$\sigma^\beta_{A\dot{A}}(\nabla_\mu \nabla^\mu - k^2)\Psi_{\alpha\beta} = 0. \tag{53}$$

It follows that modified GR is intrinsically hidden in the symmetric part of the KG equation for all spins, and the following theorem can be stated:

Theorem 3.1. *The equation* $-\frac{8\pi G}{c^4}\tilde{T}_{\alpha\beta} + R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Phi_{\alpha\beta} = 0$ *describing general relativity with a total matter energy-momentum tensor* $\tilde{T}_{\alpha\beta}$ *which includes all types of matter (and dark matter if it exists), and the energy-momentum of the gravitational field with the tensor* $\Phi_{\alpha\beta}$*, is contained in the symmetric part of the Klein-Gordon equation* $\nabla_\mu\nabla^\mu\Psi = k^2\Psi$ *in an open subset of a 4-dimensional time oriented Lorentzian manifold, with* Ψ *having spins 0,1,2,1/2 and 3/2.*

Some consequences of these results are now discussed.

4. Massless spin-2 "particles"

Gravitons are taken to be massless particles because of the $\frac{1}{r^2}$ long-range effective force behaviour of gravity. They have spin-2 so that they can couple to the energy-momentum tensor. However, in a 4-dimensional spacetime with a metric compatible connection, when the mass vanishes, the metric as a field variable vanishes from the symmetric spin-2 KG equation leaving

$$\nabla_\mu\nabla^\mu h_{\alpha\beta} = 0 \tag{54}$$

as the equation describing a massless spin-2 "particle". Because $h_{\alpha\beta}$ is independent of the metric, this equation cannot describe a massless spin-2 graviton in the long-range to particle regimes of spacetime.

Furthermore, massless spin-2 "particles" cannot couple to a non-zero energy-momentum tensor as force mediators for gravity in these regimes. If we calculate the interaction of the total matter energy-momentum tensor with $h_{\alpha\beta}$, we obtain

$$\begin{aligned} S_h^{int} &= -\frac{1}{2c}\int \tilde{T}^{\alpha\beta}h_{\alpha\beta}\sqrt{-g}d^4x \\ &= -\frac{c^3}{16\pi G}\int G^{\alpha\beta}h_{\alpha\beta}\sqrt{-g}d^4x \end{aligned} \tag{55}$$

since $\int \Phi^{\alpha\beta}h_{\alpha\beta}\sqrt{-g}d^4x = 0$ because $h_{\alpha\beta}$ is divergenceless. The variation of the functional S_h^{int} with respect to $h_{\alpha\beta}$ must vanish. This requires $G^{\alpha\beta}$ to vanish because by definition, $h_{\alpha\beta}$ is independent of both the metric and the Einstein tensor. $\Phi^{\alpha\beta}$ is orthogonal to and independent of $h_{\alpha\beta}$. Thus, $\int \Phi^{\alpha\beta}\delta h_{\alpha\beta}\sqrt{-g}d^4x = 0$ and $\Phi^{\alpha\beta}$ vanishes. It follows that $\tilde{T}^{\alpha\beta}$ vanishes and there is no coupling to the matter energy-momentum tensor. Massless spin-2 "particles" do not couple to any types of matter but can occupy the vacuum in accordance with $R^{\alpha\beta} = 0$.

The hierarchy problem of particle physics can be stated as the question: why is the force of gravity so much weaker than the other three known forces in nature? In the case of electrodynamics, if both gravity and electrodynamics have long-range massless force mediators, why is the electromagnetic force 10^{40} times stronger than that of gravity? The electroweak force is 10^{24} times stronger than gravity. And as the name suggests, the strong nuclear force presents the largest disparity to gravity at nuclear dimensions.

At the basis of this problem is the notion that gravity *must* be quantized in the particle and long-range regimes. However, the spin-2 KG equation in a 4-dimensional Lorentzian spacetime with a metric compatible connection excludes massless gravitons as force mediators of gravity in these regimes. This starkly contrasts with the spin-1 KG equation for a massless photon which mediates the electromagnetic field; similarly for the electroweak force and the massive spin-1 W and Z bosons, and the spin-1 massless gluons mediating the strong nuclear force. Gravity has no massless particles that act as force mediators in the particle and long-range regimes. Thus, the hierarchy problem is at least substantially if not fully explained without the need of extra spatial dimensions inherent in string theory; or any other theory that involves massless gravitons in a Lorentzian spacetime with a metric compatible connection.

5. Quantum field theory in curved spacetime

QFT in curved spacetime is formulated from the free field solutions to the KG equation for a particular spin. In curved spacetime, the existence and uniqueness of solutions to the KG equation are formulated in a globally hyperbolic spacetime. This formulation follows directly from the time oriented spacetime described in section 2 when the domain of dependence [20] of an achronal closed set $\Sigma \subset M$ equals M. Then, Σ is a Cauchy surface for the spacetime $(M, g_{\alpha\beta})$. If a spacetime admits a Cauchy surface, the spacetime is globally hyperbolic. Since modified GR is hidden in the symmetric part of the KG free field solutions, it is hidden in the formulation of QFT. The two main approaches to QFT in curved spacetime are quickly reviewed to demonstrate how the free field solutions to the KG equation for spins 0, 1 and 1/2 are involved.

5.1. Canonical QFT in Curved Spacetime

As discussed in [21] and summarized nicely by Mostepanenko [22], canonical quantization in curved spacetime for spin-0, 1 and 1/2 fields involves constructing a complete orthonormal set $\{\Psi_\alpha^{(+)}; \Psi_\alpha^{(-)}\}$ from the solutions to the equations (24) (with a term $\xi R \Psi_\alpha$ added for gravitational coupling to the scalar field); (19) and (37). The upper $\pm$ indices signify the positive and negative frequency solutions as clarified in [21, 22]. The orthonormal conditions are

$$(\Psi_\alpha^{(+)}, \Psi_\beta^{(+)}) = \mp\delta_{\alpha\beta}, \ \ (\Psi_\alpha^{(-)}, \Psi_\beta^{(-)}) = \delta_{\alpha\beta}, \ \ (\Psi_\alpha^{(+)}, \Psi_\beta^{(-)}) = 0 \tag{56}$$

where the integration is performed over a globally spacelike hypersurface. The field Ψ can then be expressed as

$$\Psi = \Sigma_\alpha [\Psi_\alpha^{(-)} a_\alpha^{(-)} + \Psi_\alpha^{(+)} a_\alpha^{(+)}] \tag{57}$$

where the expressions for antiparticle creation and particle annihilation operators follow from the orthonormality conditions

$$a_\alpha^{(+)} = \mp(\Psi_\alpha^{(+)}, \Psi), \ \ a_\alpha^{(-)} = (\Psi_\alpha^{(-)}, \Psi). \tag{58}$$

The $\mp$ signs refer to the boson and fermion cases, respectively. Quantization of the field requires the commutation (anticommutation) relations

$$[a_\alpha^{*(-)}, a_\beta^{(+)}]\mp = [a_\alpha^{(-)}, a_\beta^{*(+)}]\mp = \delta_{\alpha\beta}, \ \ [a_\alpha^{(\pm)}, a_\beta^{(\pm)}]\mp = [a_\alpha^{*(\pm)}, a_\beta^{*(\pm)}]\mp = 0. \tag{59}$$

These are equivalent to the equal time canonical commutation (anticommutation) relations

$$[\Psi(t,x), \Psi(t,x')]\mp = [\pi(t,x), \pi(t,x')]\mp = 0, \ \ [\Psi(t,x), \pi(t,x')]\mp = i\delta(x - x') \tag{60}$$

where π is the canonically conjugate momentum operator defined by

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi)} = \partial_0 \Psi. \tag{61}$$

In curved spacetime, there is no unique way of defining the positive and negative frequency modes and therefore no unique "vacuum state". The concept of old-new states is introduced and if mixing of positive and negative frequency solutions occurs, then "particles" are created by the gravitational field. The new particle density in the old vacuum state is

$$< 0|\Psi_\beta|0 >= \Sigma_\mu \mid \Psi_{\beta\mu} \mid^2 \tag{62}$$

where the Bogolubov coefficients $\Psi_{\beta\mu}$ connect the old-new modes.

We have proved that the modified Einstein equation of GR is hidden in the symmetric part of the KG equation. The field operators and the commutation (anticommutation) operators of canonical QFT in curved spacetime are obtained from the free field solutions to the KG equation. Canonical quantization of QFT in curved spacetime is formulated from the infinite superposition of KG free field solutions with mixed frequencies. Modified GR is hidden in the canonical formalism of QFT.

5.2. Algebraic QFT in Curved Spacetime

As stated in Wald [20], Hollands and Wald [23], and Brunetti and Fredenhagen [24], the standard canonical formulation of QFT relies heavily on concepts like vacuum and particles. However, these concepts lose their meaning in curved spacetime. Although natural notions of "vacuum state" and "particles" can be defined for a free field in stationary spacetimes, no such notions exist in a general curved spacetime. It is not that "particles" cannot be defined at all in curved spacetime but rather that many definitions exist and none appears preferred. This difficulty is not present in the algebraic approach. Algebraic QFT can be formulated without requiring a preferred representation of the canonical commutation relations, and without requiring the definition of a preferred notion of "particles".

In [20], algebraic QFT begins with a real scalar field satisfying the KG equation in a formulation so that all the basic ideas carry over without any essential change to real, linear bosonic fields. Minor modifications are needed to treat complex, linear, bosonic fields. The modifications needed to formulate the theory of linear fermionic fields essentially consist of several key sign changes. Hollands and Wald [23] start with a real scalar free field solution of the KG equation. The KG equation with a source $j(x^\alpha)$

$$(\nabla_\mu \nabla^\mu - k^2)\phi = j \tag{63}$$

on a globally hyperbolic spacetime has a well posed initial value formulation in the sense defined in the article. The field equation of the algebra of observables generated by the fundamental field ϕ and a distribution f as a test function, is defined in the algebra as

$$\phi((\nabla_\mu \nabla^\mu - k^2)f) = 0. \tag{64}$$

The algebraic theory in [24] is based on the locally covariant approach to quantum field theory which uses the language of categories to incorporate the principle of general covariance. The functor defining the quantum theory depends on the free field KG with an additional coupling term

$$(\nabla_\mu \nabla^\mu - k^2 - \xi R)\phi = 0 \tag{65}$$

where ξ is a coupling constant to the Riemann scalar R.

An algebraic approach to QFT in curved spacetime must involve the KG equation in some fundamental role because it is the rudimentary equation of relativistic quantum mechanics in curved spacetime. Therefore, modified GR is hidden in algebraic QFT in curved spacetime.

Although each major theory of physics appears to act independently of the other, modified GR resides beautifully within QFT; hidden in the symmetric part of the free field solutions to the KG equation. Furthermore, massless spin-2 gravitons described by the metric do not exist as the force mediators for gravity. GR down to the detectable size of elementary particles, such as the quark radius [25] of $\sim 10^{-19}$m, is not quantized. In that sense, QFT in curved spacetime *is* quantum gravity in the particle and long-range regimes of a 4-dimensional spacetime with a metric compatible connection. Modified general relativity uniquely describes the effective force of gravity geometrically without the exchange of massless particles.

However, it is not known if the KG equation is capable of explaining the high energy physics near the Planck length $\sim 10^{-35}$m. A more general theory of quantum gravity may be required which yields the KG equation in the particle-long range regimes.

6. Wave-particle duality

The wave-particle duality of matter is a well known aspect of the quantum mechanical description of nature. However, it was not understood that light behaves as a particle and a wave similtaneously until that property of photons was observed by Peruzzo et al. [26]. Later, Piazza et al. [27] observed the similtaneous wave-particle behaviour for a plasmonic field.

It has been established that the antisymmetric tensor $K_{\alpha\beta}$ can describe the particle nature of quantum theory for spin-1 bosons from (19) and spin-1/2 fermions by (37). The symmetric tensor $\tilde{\Psi}_{\alpha\beta}$ applies to all spins and satifies the wave equation (54). Thus, quantum theory described the the KG equation has an intrinsic wave property in its symmetric part and a particle characteristic in its antisymmetric part. Because the symmetries belong to the same equation with the same spacetime coordinates, the wave and particle natures of the spin-1 bosons and the spin-1/2 fermions must act similtaneously at the same spatial coordinates. They have *both* wave and particle characteristics at all times; *not* one or the other as is commonly understood. This opens the door to a discussion of the foundations of quantum physics from a new perspective.

7. Conclusion

The results in this article were obtained directly from the rudimentary Klein-Gordon equation of relativistic quantum mechanics in curved spacetime and the modified Einstein equation of general relativity. The KG equation was constructed from a smooth, real, regular quantum vector field A^β for spins $0, 1, 2, 1/2$ and $3/2$. Its symmetric part is the Lie derivative of the metric along the quantum vector. This provided a geometrical link to the modified equation of general relativity from which a very general decomposition for $\tilde{\Psi}_{\alpha\beta}$ was obtained in terms of the metric and a collection of fields independent of the metric; and the first and second derivatives of both. It followed that the tensors of the modified Einstein equation which add to zero are contained in $\tilde{\Psi}_{\alpha\beta}$.

Thus, quantum theory intrinsically contains modified general relativity in the symmetric part of the Klein-Gordon equation in curved spacetime. This leads to the following conclusions:

1. Both the canonical and algebraic approaches to QFT fundamentally involve free field solutions to the KG equation. As the modified Einstein equation is hidden in the symmetric part of the KG equation for each spin, general relativity is hidden in the formalism of quantum field theory. Although each major theory of physics appears to act independently of the other, GR resides beautifully within QFT.

2. On a 4-dimensional time oriented Lorentzian manifold with a torsionless and metric compatible connection, the metric as a field variable does not appear in the spin-2 KG equation for a massless graviton in the long-range to particle regimes. Massless gravitons do not couple to a non-zero energy-momentum tensor as the force mediators for gravity. Thus, massless gravitons in the spacetime as described, do not exist.

3. That massless gravitons do not exist in such spacetimes explains the hierarchy problem of particle physics. Furthermore, unlike the other three known fundamental forces in nature, no particle exchange is required to explain the force of gravity; that is nicely done by the curvature of spacetime.

4. Massless spin-2 "particles" do not couple to any types of matter and can occupy the vacuum.

5. Spin-1 bosons and the spin-1/2 fermions act similtaneously at the same spatial coordinates. They have both wave and particle characteristics at all times.

That the fundamental equation of quantum theory contains modified general relativity is a testament to the complex mathematical structures that nature seems to blend together so beautifully.

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