$\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravity as a Grand Unified Field Theory

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August 2018

Abstract

We argue how $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravity naturally can describe a grand unified field theory of Einstein's gravity with an $U(8)$ Yang-Mills theory. In particular, it allows for an extension of the Standard Model by including a 3-family $SU(3)_F$ symmetry group and an extra $U(1)$ symmetry. A unification of left-right $SU(3)_L \times SU(3)_R$, color $SU(3)_C$ and family $SU(3)_F$ symmetries in a maximal rank-8 subgroup of E_8 has been proposed by [33] as a landmark for future explorations beyond the Standard Model. It is warranted to explore further if this latter model also admits a similar gravitational interpretation based on the above composition of normed division algebras. Furthermore, our construction leads also to a *bimetric* theory of gravity which may have a role in dark energy. The crux of this approach is that we have *replaced* the Kaluza-Klein prescription to generate gauge symmetries in lower dimensions from isometries of the internal manifold, by $U(8)$ isometry transformations of the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued metric.

Keywords: Nonassociative Geometry, Clifford algebras, Quaternions, Octonionic Gravity, Unification, Strings.

1 Introduction

Exceptional, Jordan, Division, Clifford and Noncommutative algebras are deeply related and essential tools in many aspects in Physics, see for instance [1], [2], [3], [4], [5], [7], [6], [5], [11], [14], [13], [15], [24], [28].

Exceptional Jordan Matrix Models based on the compact E_6 involve a *double* number of the required physical degrees of freedom inherent in a complex-valued action [11]. This led Ohwashi to construct an interacting pair of mirror universes within the compact E_6 matrix model and equipped with a $Sp(4, \mathbf{H})/Z_2$ symmetry based on the quaternionic valued symplectic group. The interacting picture resembles that of the bi-Chern-Simons gravity models. A construction of nonassociative Chern-Simons membranes and 3-branes based on the large N limit of Exceptional Jordan algebras was put forward by [12].

The E_8 group was proposed long ago [30] as a candidate for a grand unification model building in $D = 4$. The supersymmetric E_8 model has more recently been studied as a fermion family and grand unification model [30] under the assumption that there is a vacuum gluino condensate but this condensate is *not* accompanied by a dynamical generation of a mass gap in the pure E_8 gauge sector. Clifford algebras and E_8 are key ingredients in Smith's $D_4 - D_5 - E_6 - E_7 - E_8$ grand unified model in $D = 8$ [16].

A complexification of ordinary gravity (not to be confused with Hermitian-Kahler geometry) has been known for a long time. Complex gravity requires that $g_{\mu\nu} = g_{(\mu\nu)} + i g_{[\mu\nu]}$ so that now one has $g_{\nu\mu} = (g_{\mu\nu})^*$, which implies that the diagonal components of the metric $g_{z_1 z_1} = g_{z_2 z_2} = g_{\bar{z}_1 \bar{z}_1} = g_{\bar{z}_2 \bar{z}_2}$ must be real. A treatment of a non-Riemannan geometry based on a complex tangent space and involving a symmetric $g_{(\mu\nu)}$ plus antisymmetric $g_{[\mu\nu]}$ metric component was first proposed by Einstein-Strauss [10] (and later on by [18]) in their unified theory of Electromagentism with gravity by identifying the EM field strength $F_{\mu\nu}$ with the antisymmetric metric $g_{[\mu\nu]}$ component.

Borchsenius [17] formulated the quaternionic extension of Einstein-Strauss unified theory of gravitation with EM by incorporating appropriately the $SU(2)$ Yang-Mills field strength into the degrees of a freedom of a quaternionc-valued metric. Oliveira and Marques [19] later on provided the Octonionic Gravitational extension of Borchsenius theory involving two interacting $SU(2)$ Yang-Mills fields and where the exceptional group G_2 was realized naturally as the automorphism group of the octonions. The non-Desarguesian geometry of the Moufang projective plane to describe Octonionic QM was discussed by [14].

It was shown in [21] how one could generalize Octonionic Gravitation into an Extended Relativity theory in Clifford spaces, involving poly-vector valued (Clifford-algebra valued) coordinates and fields, where in addition to the speed of light there is also an invariant length scale (set equal to the Planck scale) in the definition of a generalized metric distance in Clifford spaces encoding, lengths, areas, volumes and hyper-volumes metrics. An overview of the basic features of the Extended Relativity in Clifford spaces can be found in [21].

The purpose of this work is to advance further the Octonionic Geometry

(Gravity) of [19], [20] and show how $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravity naturally can describe a grand unified field theory of Einstein's gravity with an $U(8)$ Yang-Mills theory. The introduction of matter fields will be the subject of future investigation.

2 Octonions, Clifford and Lie Algebras

This introductory section is very important in order to understand some of the arguments in the next section. For this reason we deem it necessary.

2.1 Octonionic Realizations of $SO(8), SO(7), G_2, SU(3)$

Given an octonion $\mathbf{X}$ it can be expanded in a basis (e_o, e_a) as

$$\mathbf{X} = x^o\ e_o\ +\ x^a\ e_a,\ a = 1, 2, \cdots, 7. \tag{2.1}$$

where e_o is the identity element. The Noncommutative and Nonassociative algebra of octonions is determined from the relations

$$e_o^2 = e_o,\ e_o e_a = e_a e_o = e_a,\ e_a e_b = -\delta_{ab} e_o\ + C_{abc} e_c,\ a, b, c = 1, 2, 3, ...7. \tag{2.2}$$

The non-vanishing values of the fully antisymmetric structure constants C_{abc} is chosen to be $\mathbf{1}$ for the following 7 sets of index triplets (cycles) [7]

$$(124),\ (235),\ (346),\ (457),\ (561),\ (672),\ (713) \tag{2.3}$$

Each cycle represents a quaternionic subalgebra. The values of C_{abc} for the other combinations are zero. The latter 7 sets of index triplets (cycles) correspond to the 7 lines of the Fano plane.

The octonion conjugate is defined

$$\bar{\mathbf{X}} = x^o\ e_o\ -\ x^m\ e_m. \tag{2.4}$$

and the norm

$$N(\mathbf{X})\ =\ <\ \mathbf{X}\ \mathbf{X}\ >\ =\ Real\ (\bar{\mathbf{X}}\ \mathbf{X})\ = (x_o\ x_o\ + x_k\ x_k). \tag{2.5}$$

The inverse

$$\mathbf{X}^{-1}\ =\ \frac{\bar{\mathbf{X}}}{N(\mathbf{X})},\quad \mathbf{X}^{-1}\mathbf{X} = \mathbf{X}\mathbf{X}^{-1} = 1. \tag{2.6}$$

The non-vanishing associator is defined by

$$\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\} = (\mathbf{X}\mathbf{Y})\mathbf{Z} - \mathbf{X}(\mathbf{Y}\mathbf{Z}) \tag{2.7}$$

In particular, the associator

$$\{e_i, e_j, e_k\} \;=\; d_{ijkl}\; e_l, \quad d_{ijkl} \;=\; \epsilon_{ijklmnp}\; c^{mnp}, \;\; i,j,k.... = 1,2,3,.....7 \quad (2.8)$$

There are **no** matrix *representations* of the Octonions due to the non-associativity, however Dixon has shown how many Lie algebras can be obtained from the left/right action of the octonion algebra on itself [7]. $\mathbf{O}_L$ and $\mathbf{O}_R$ are identical, isomorphic to the matrix algebra $R(8)$ of 8×8 real matrices. The 64-dimensional bases are of the form $\mathbf{1}, e_{La}, e_{Lab}, e_{Labc}$, or $\mathbf{1}, e_{Ra}, e_{Rab}, e_{Rabc}$, where, for example, if $\mathbf{x} \in \mathbf{O}$, then $e_{Lab}[\mathbf{x}] = e_a(e_b\mathbf{x})$, and $e_{Rab}[\mathbf{x}] = (\mathbf{x}e_a)e_b$.

Focusing on the $left$ actions, Dixon found [7]

- $so(8) \; : \; \{e_{La};\; e_{Lab} \mid a,b = 1,\cdots,7\}$ giving a total of $7{+}21 = 28$ generators.

- $so(7) \; : \; \{e_{Lab} \mid a,b = 1,\cdots,7\}$ giving a total of 21 generators.
- $so(6) \; : \; \{e_{Lpq} \mid p,q = 1,\cdots,6\}$ giving a total of 15 generators.
- The Lie algebra g_2

$$g_2 \; : \; \{e_{Lab} - e_{Lcd} \mid e_a\; e_b \;-\; e_c\; e_d \;=\; 0, \;\; a,b,c,d = 1,\cdots,7\} \qquad (2.9)$$

g_2 is the 14-dim Lie algebra of G_2, the automorphism group of $\mathbf{O}$. The 14 generators are

$$\begin{array}{c} e_{L24} - e_{L56};\;\; e_{L56} - e_{L37};\;\; e_{L35} - e_{L67};\;\; e_{L67} - e_{L41} \\ e_{L46} - e_{L71};\;\; e_{L71} - e_{L52};\;\; e_{L57} - e_{L12};\;\; e_{L12} - e_{L63} \\ e_{L61} - e_{L23};\;\; e_{L23} - e_{L74};\;\; e_{L72} - e_{L34};\;\; e_{L34} - e_{L15} \\ e_{L13} - e_{L45};\; e_{L45} - e_{L26} \end{array} \qquad (2.10)$$

The $su(3)$ Lie algebra is a subalgebra of g_2 which leaves invariant one of the imaginary units of the octonions. In particular if one chooses e_7, $su(3)$ is the Lie algebra of $SU(3)$ which is the stability group of e_7 (a subgroup of G_2). The 8 generators of $su(3)$ are determined from the conditions

- $$su(3) \; : \; \{e_{Lpq} - e_{Lrs} \mid e_p e_q - e_r e_s = 0, \;\; p,q,r,s = 1,\cdots,6\}$$

from which one obtains the following 8 generators

$$\begin{array}{c} e_{L24} - e_{L56};\;\; e_{L35} - e_{L41};\;\; e_{L46} - e_{L52} \\ e_{L12} - e_{L63};\;\; e_{L61} - e_{L23};\;\; e_{L34} - e_{L15} \\ e_{L13} - e_{L45},\;\; e_{L45} - e_{L26} \end{array} \qquad (2.11)$$

- The generator of the $U(1)$ Lie algebra is [7]

$$e_{L45} \;+\; e_{L13} \;+\; e_{L26} \qquad (2.12)$$

and commutes with all the 8 generators of $SU(3)$. The 7-dim round sphere can be identified as the coset $S^7 \sim SO(8)/SO(7)$. The 7-dim squashed sphere can be identified as the coset $SO(7)/G_2$. Compactifications of 11-dim M-theory on 7-dim manifolds of exceptional holonomy G_2 have been extensively studied over the years

• 8×8 matrix realizations of the left/right actions. From the structure constants of the Octonion algebra one can associate to the left action of e_a on e_o and e_b

$$e_{La}\ [e_o] \ = \ e_a\ e_o \ = \ e_a, \ \ e_{La}\ [e_b] \ = \ e_a\ e_b \ = \ C_{abc}\ e_c \tag{2.13}$$

the following 8×8 *antihermitian* matrix $\mathbf{M}_{La} : e_{La} \leftrightarrow \mathbf{M}_{La}$, and whose entries are given by

$$(M^L_a)_{bc} = C_{abc},\ a,b,c = 1,2,\cdots,7;\ (M^L_a)_{00} = 0,\ (M^L_a)_{0c} = \delta_{ac},\ (M^L_a)_{c0} \ = \ -\delta_{ac} \tag{2.14}$$

Due to the non-associativity of the Octonions one has $e_1 e_2 = e_4$, but $\mathbf{M}_{L1}\mathbf{M}_{L2} \neq \mathbf{M}_{L4}$!, otherwise the generators in the above equations would have been trivially zero. As said previously, there are **no** matrix representations of the non-associative Octonion algebra, and as a result one has that

$$\mathbf{M}_{La}\ \mathbf{M}_{Lb} \ \neq \ C_{abc}\ \mathbf{M}_{Lc} \tag{2.15}$$

Given the antihermian 8×8 matrices in eq-(2.14) the $g_2, su(3), \cdots$ algebras are realized in terms of the commutators of the generators given by eqs-(2.10, 2.11). For example, in the $su(3)$ algebra case, the commutator of the first two $su(3)$ generators (2.11) is

$$[e_{L24} - e_{L56},\ e_{L35} - e_{L41}] \ \leftrightarrow \ [M_{L2}M_{L4} - M_{L5}M_{L6},\ M_{L3}M_{L5} - M_{L4}M_{L1}] \ =$$

$$M_{L2}[M_{L4}, M_{L3}]M_{L5} \ - \ M_{L5}[M_{L6}, M_{L3}]M_{L5} \ + \ \cdots \tag{2.16}$$

The commutators of the 8 $su(3)$ generators $\mathbf{L}_\alpha$ are given by

$$[\mathbf{L}_\alpha,\ \mathbf{L}_\beta] \ = \ f_{\alpha\beta\sigma}\ \mathbf{L}_\sigma, \ \ \alpha,\beta,\sigma = 1,2,\cdots,7,8 \tag{2.17}$$

where $f_{\alpha\beta\sigma}$ are the antisymmetric structure constants of the $su(3)$ algebra. The 8-dim adjoint representation of $su(3)$ can be implemented in terms of 8 *antihermitian* 8×8 matrices $\mathbf{T}_\alpha = (T_\alpha)_{\beta\sigma} = f_{\alpha\beta\sigma}$. Since the commutators of two *antihermitain* matrices is *antihermitian*, the (antisymmetric) structure constants $f_{\alpha\beta\sigma}$ are real-valued, and there are *no* i factors in the right hand side of eq-(2.17). It is not difficult to verify that the commutators in eq-(2.16) are indeed the same as those in eq-(2.17). Similarly one could have written the Lie algebra generators in terms of the *right* action of the Octonion algebra on itself.

2.2 Octonionic realization of $GL(8, R)$

The *combined* left and right action of the algebra acting on itself [8] is defined as

$$e_{La}\ e_{Rb}\ [\mathbf{x}] \ = \ e_{La}\ (\mathbf{x}\ e_{Rb});\ \ e_{Rb}\ e_{La}\ [\mathbf{x}] \ = \ (e_{La}\ \mathbf{x})\ e_{Rb}) \qquad (2.18)$$

Based on this left/right action, the authors [8] were able to find an octonionic realization (*not* a representation) of the Lie algebra $gl(8, R)$ based on the generators (8×8 matrices)

$$\mathbf{1},\ L_a,\ R_b,\ L_aR_a,\ [L_a, R_b],\quad a, b = 1, 2, \cdots, 7 \qquad (2.19)$$

obeying the relations

$$L_a\ L_b \ = \ -\delta_{ab} + C_{abc}\ L_c - [R_a,\ L_b],\ \ R_a\ R_b \ = \ -\delta_{ab} + C_{abc}\ R_c - [L_a,\ R_b],$$

$$[L_a,\ L_b] \ = \ f_{abc}L_c \ - 2\ [R_a,\ L_b],\ \ [R_a,\ R_b] \ = \ f_{abc}R_c \ - 2\ [L_a,\ R_b],$$

$$[R_a,\ L_b] \ = \ [L_a,\ R_b] \ = \ - \ [R_b,\ L_a] \ = \ - \ [L_b,\ R_a]$$

$$[R_a,\ L_a] = 0,\quad a = 1, 2, \cdots, 7 \qquad (2.20)$$

there is *no* sum over a in the eq-(2.20), and the structure constants are $f_{abc} = 2C_{abc}$.

There are $7 + 7 = 14$ generators : L_a, R_b. There are 7 generators L_aR_a (no sum over a). There are $7 \times 6 = 42$ generators $[L_a, R_b](a \neq b)$. Combined with the unit 8×8 matrix $\mathbf{1}$, it gives a total of $1 + 7 + 7 + 7 + 42 = 64$ generators, and which matches the dimension of the Lie algebra $gl(8, R)$.

The modified composition $\odot$ defined as

$$L_a \odot L_b \ = \ L_a\ L_b \ + \ [R_a, L_b] \ \Rightarrow \ L_a \odot L_b \ - \ L_b \odot L_a \ = \ f_{abc}\ L_c \qquad (2.21)$$

$$R_a \odot R_b \ = \ R_a\ R_b \ + \ [L_a, R_b] \ \Rightarrow \ R_a \odot R_b \ - \ R_b \odot R_a \ = \ f_{abc}\ R_c \qquad (2.22)$$

allows closure $[L_a, L_b]_\odot, [R_a, R_b]_\odot$ where $f_{abc} = 2C_{abc}$.

2.3 Clifford Algebraic Realization of $SU(N)$

• The dim $Cl(0, 6) = 64$, is same as the dim of $gl(8, R)$. $\mathbf{O}_L \simeq \mathbf{O}_R \simeq Cl(0, 6)$.

The $u(4)$ algebra can also be realized in terms of $so(8)$ generators, and in general, $u(N)$ algebras admit realizations in terms of $so(2N)$ generators Given the Weyl-Heisenberg "superalgebra" involving the N fermionic creation and annihilation (oscillators) operators

$$\{a_i, a_j^\dagger\} \ = \ \delta_{ij},\quad \{a_i, a_j\} \ = \ 0,\ \{a_i^\dagger, a_j^\dagger\} \ = \ 0;\quad i, j \ = \ 1, 2, 3, ...\ N. \qquad (2.23)$$

one can find a realization of the $u(N)$ algebra bilinear in the oscillators as $E_i{}^j = a_i^\dagger \, a_j$ and such that the commutators

$$[E_i{}^j, \; E_k{}^l] \; = \; a_i^\dagger \, a_j \, a_k^\dagger \, a_l \; - \; a_k^\dagger \, a_l \, a_i^\dagger \, a_j \; =$$

$$a_i^\dagger \, (\delta_{jk} - a_k^\dagger \, a_j) \, a_l \; - \; a_k^\dagger \, (\delta_{li} - a_i^\dagger \, a_l) \, a_j \; = \; a_i^\dagger \, (\delta_{jk}) \, a_l \; - \; a_k^\dagger \, (\delta_{li}) \, a_j \; =$$

$$\delta_k^j \, E_i{}^l - \; \delta_i^l \, E_k{}^j. \tag{2.24}$$

reproduce the commutators of the Lie algebra $u(N)$ since

$$-a_i^\dagger \, a_k^\dagger \, a_j \, a_l \; + \; a_k^\dagger \, a_i^\dagger \, a_l \, a_j \; = \; - \, a_k^\dagger \, a_i^\dagger \, a_l \, a_j \; + \; a_k^\dagger \, a_i^\dagger \, a_l \, a_j \; = \; 0. \tag{2.25}$$

due to the anti-commutation relations (2.23) yielding a double negative sign $(-)(-) = +$ in (2.25). Furthermore, one also has an explicit realization of the Clifford algebra $Cl(2N)$ Hermitian generators by defining the even-number and odd-number generators as

$$\Gamma_{2j} \; = \; \frac{1}{2} \, (a_j \; + \; a_j^\dagger); \quad \Gamma_{2j-1} \; = \; \frac{1}{2i} \, (a_j \; - \; a_j^\dagger). \tag{2.26}$$

The Hermitian generators of the $so(2N)$ algebra are defined as usual $\Sigma_{mn} = \frac{i}{4}[\Gamma_m, \Gamma_n]$ where $m, n = 1, 2, ...2N$. Therefore, the $u(4), so(8), Cl(8)$ algebras admit an explicit realization in terms of the fermionic Weyl-Heisenberg oscillators $a_i, a_j^\dagger$ for $i, j = 1, 2, 3, 4$.

3 $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravity and Grand Unification

Dixon [7] many years ago published a monograph pointing out the key role that the composition algebra $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$ had in the architecture of the Standard Model. More recently, it has been shown how this algebra acting on itself allows to find the Standard Model particle representations [9]. For this reason we shall construct a gravitational theory based on a $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued metric defined as

$$\mathbf{g}_{\mu\nu}(x^\mu) \; = \; g_{(\mu\nu)}(x^\mu) + g_{\mu\nu}^{IA}(x^\mu) \, (q_I \otimes e_A), \;\; q_I = q_o, q_1, q_2, q_3; \;\; e_A = e_o, e_1, e_2, \cdots, e_7 \tag{3.1}$$

where the ordinary $4D$ spacetime coordinates are $x^\mu, \mu = 0, 1, 2, 3$, and $g_{(\mu\nu)}$ is the standard Riemannian metric. The extra "internal" $C \otimes H \otimes O$-valued metric components are explicitly given by

$$(g_{(\mu\nu)} + i g_{[\mu\nu]})^{oo}, \;\; (g_{[\mu\nu]} + i g_{(\mu\nu)})^{ko}, \;\; (g_{[\mu\nu]} + i g_{(\mu\nu)})^{oa}, \;\; (g_{(\mu\nu)} + i g_{[\mu\nu]})^{ka} \tag{3.2}$$

$k = 1, 2, 3; a = 1, 2, \cdots, 7$. The index o is associated with the real units q_o, e_o. The bar conjugation amounts to $i \rightarrow -i$; $q_k \rightarrow -q_k$; $e_a \rightarrow -e_a$, so that $\bar{\mathbf{g}}_{\mu\nu} = \mathbf{g}_{\nu\mu}$.

The generalization of the line interval considered in [19], [20] based on the metric (3.1) is then given by

$$ds^2 \ = \ < \ \mathbf{g}_{\mu\nu} \ dx^\mu \ dx^\nu \ > \ = \ (\ g_{(\mu\nu)} \ + \ g^{oo}_{(\mu\nu)} \ ) \ dx^\mu \ dx^\nu \tag{3.3}$$

where the operation $< \cdots >$ denotes taking the *real* components. From eq-(3.3) one learns that the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued metric leads to a *bimetric* theory of gravity where the two metrics are, respectively, $g_{(\mu\nu)}, g^{oo}_{(\mu\nu)} = h_{(\mu\nu)}$.

The $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued affinity is given by

$$\mathbf{\Upsilon}^\rho_{\mu\nu} \ = \Gamma^\rho_{\mu\nu}(g_{\mu\nu}) \ + \ \mathbf{\Theta}^\rho_{\mu\nu} \ = \ \Gamma^\rho_{\mu\nu}(g_{\mu\nu}) \ + \ \delta^\rho_\mu \ \mathbf{A}_\nu \ =$$

$$\Gamma^\rho_{\mu\nu}(g_{\mu\nu}) + \delta^\rho_\mu \ \left(\ A^{oo}_\nu \ (q_o \otimes e_o) + A^{ia}_\nu \ (q_i \otimes e_a) + A^{io}_\nu \ (q_i \otimes e_o) + A^{oa}_\nu \ (q_o \otimes e_a) \ \right) \tag{3.4}$$

Thus we have decomposed the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued affinity $\mathbf{\Upsilon}^\rho_{\mu\nu}$ into a real-valued "external" part Γ plus an "internal" part $\mathbf{\Theta}^\rho_{\mu\nu}$. The base spacetime connection is chosen to be the torsionless Christoffel connection

$$\Gamma^\rho_{\mu\nu} \ = \ \Gamma^\rho_{\nu\mu} \ = \ \frac{1}{2} \ g^{\rho\sigma} \ (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{3.5}$$

The $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued curvature tensor $\mathbf{R}^\sigma_{\rho\mu\nu} = \mathcal{R}^\sigma_{\rho\mu\nu} + \mathbf{\Omega}^\rho_{\sigma\mu\nu}$, involving the base spacetime and internal space curvature is defined by

$$\mathbf{R}^\sigma_{\rho\mu\nu} \ = \ \mathbf{\Upsilon}^\sigma_{\rho\mu,\nu} \ - \ \mathbf{\Upsilon}^\sigma_{\rho\nu,\mu} \ + \ \mathbf{\Upsilon}^\sigma_{\tau\nu} \ \mathbf{\Upsilon}^\tau_{\rho\mu} \ - \ \mathbf{\Upsilon}^\sigma_{\tau\mu} \ \mathbf{\Upsilon}^\tau_{\rho\nu}. \tag{3.6}$$

$$\mathbf{R}^\sigma_{\rho\mu\nu} \ = \ \mathcal{R}^\sigma_{\rho\mu\nu}(\Gamma^\rho_{\mu\nu}) \ + \ \delta^\sigma_\rho \ \mathbf{F}_{\mu\nu}. \tag{3.7}$$

where $\mathcal{R}^\sigma_{\rho\mu\nu}(\Gamma^\rho_{\mu\nu})$ is the base spacetime Riemannian curvature associated to the symmetric Christoffel connection $\Gamma^\rho_{\mu\nu}$.

The "internal" space $\mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued curvature is

$$\mathbf{\Omega}^\rho_{\sigma\mu\nu} \ = \ \delta^\rho_\sigma \ \mathbf{F}_{\mu\nu} \tag{3.8}$$

with

$$\mathbf{F}_{\mu\nu} \ = \ \mathbf{A}_{\mu,\nu} - \mathbf{A}_{\nu,\mu} \ - \ [\ \mathbf{A}_\mu \ , \ \mathbf{A}_\nu \ ]. \tag{3.9}$$

and where the field $\mathbf{A}_\mu$ can be read directly in terms of the internal space affinity from the relation

$$\mathbf{\Theta}^\rho_{\mu\nu} \ = \ \delta^\rho_\mu \ \mathbf{A}_\nu \tag{3.10}$$

There are 32 complex-valued fields (64-real valued fields)

$$\mathbf{A}_\mu \ = \ \{A^{oo}_\mu, A^{io}_\mu, A^{oa}_\mu, A^{ia}_\mu\} \tag{3.11}$$

and the commutators in eq-(3.9) are defined by

$$[q_I \otimes e_A, \ q_J \otimes e_B] \ = \ \frac{1}{2} \ \{q_I, q_J\} \otimes [e_A, e_B] \ + \ \frac{1}{2} \ [q_I, q_J] \otimes \{e_A, e_B\} \tag{3.12}$$

which lead to the following explicit components for $\mathbf{F}_{\mu\nu}$

$$F^{oo}_{\mu\nu} \ = \ \partial_\mu A^{oo}_\nu \ - \ \partial_\nu A^{oo}_\mu \tag{3.13}$$

$$F^{oc}_{\mu\nu} \ = \ \partial_\mu A^{oc}_\nu \ - \ \partial_\nu A^{oc}_\mu \ + \ (A^{oa}_\mu \ A^{ob}_\nu \ - \delta_{ij} \ A^{ia}_\mu \ A^{jb}_\nu) \ C^c_{ab} \tag{3.14}$$

$$F^{ko}_{\mu\nu} \ = \ \partial_\mu A^{ko}_\nu \ - \ \partial_\nu A^{ko}_\mu \ + \ (A^{io}_\mu \ A^{jo}_\nu \ - \delta_{ab} \ A^{ia}_\mu \ A^{jb}_\nu) \ f^k_{ij} \tag{3.15}$$

$$F^{kc}_{\mu\nu} \ = \ \partial_\mu A^{kc}_\nu \ - \ \partial_\nu A^{ko}_\mu \ + \ A^{oa}_\mu \ A^{kb}_\nu \ C^c_{ab} \ + \ A^{io}_\mu \ A^{jc}_\nu \ f^k_{ij} \tag{3.16}$$

Embedding the Standard Model Gauge Fields into the Internal Connection $\Theta^\rho_{\mu\nu}$

The next step is to establish the Gravity/Gauge correspondence (not unlike the AdS/CFT correspondence) which in essence amounts to embed the 12 Gauge Fields of the Standard Model $SU(3) \times SU(2) \times U(1)$ into the fields appearing inside the internal connection $\mathbf{\Theta}^\rho_{\mu\nu} = \delta^\rho_\mu \mathbf{A}_\nu$.

Eqs-(3.13-3.16) yield the following 32 complex-valued non-vanishing field strengths

$$F^{oo}_{\mu\nu}, \quad F^{ko}_{\mu\nu}, \quad F^{oc}_{\mu\nu}, \quad F^{kc}_{\mu\nu}, \quad k = 1,2,3; \quad c = 1,2,\cdots,7 \tag{3.17}$$

Given the $U(1)$ Maxwell field

$$\mathcal{F}_{\mu\nu} \ = \ \partial_\mu \mathcal{A}_\nu \ - \ \partial_\nu \mathcal{A}_\mu \tag{3.18}$$

the Maxwell kinetic term in the Standard Model action is embedded as follows

$$\mathcal{F}_{\mu\nu} \ \mathcal{F}^{\mu\nu} \ \subset \ F^{oo}_{\mu\nu} \ (F^{\mu\nu}_{oo})^* \tag{3.19}$$

Given the $SU(2)$ field strength

$$\mathcal{F}^k_{\mu\nu} \ = \ \partial_\mu \mathcal{A}^k_\nu \ - \ \partial_\nu \mathcal{A}^k_\mu \ + \ \mathcal{A}_\mu{}^i \ \mathcal{A}^j_\nu \ \epsilon^k_{ij} \tag{3.20}$$

the $SU(2)$ Yang-Mills term is embedded as

$$\mathcal{F}^i_{\mu\nu} \ \mathcal{F}^{\mu\nu}_i \ (i = 1,2,3) \ \subset \ (F^{ko}_{\mu\nu}) \ (F^{\mu\nu}_{ko})^* \ (k = 1,2,3) \tag{3.21}$$

Since the $SU(2)$ algebra is isomorphic to the algebra of quaternions, the embedding (3.21) is very natural. The chain of subgroups

$$SO(8) \ \supset \ SO(7) \ \supset \ G_2 \ \supset \ SU(3) \tag{3.22}$$

related to the round and squashed seven-spheres : $S^7 \simeq SO(8)/SO(7), S^7_* \simeq SO(7)/G_2$, reflect how the $SU(3)$ group is embedded. The number of generators of $SO(8)$, $SO(7)$ are 28 and 21 respectively. There are $7 + 21 = 28$ complex-valued (42 real-valued) field strengths, respectively

$$F^{oc}_{\mu\nu}, \quad F^{kc}_{\mu\nu}, \quad k = 1,2,3; \quad c = 1,2,\cdots,7 \tag{3.23}$$

such that the $SU(3)$ Yang-Mills terms can be embedded into the contribution of the above $7 + 21 = 28$ complex-valued fields as follows

$$\mathcal{F}^{\alpha}_{\mu\nu}\ \mathcal{F}^{\mu\nu}_{\alpha}\ (\alpha = 1,2,\ldots,7,8)\ \subset\ (F^{oc}_{\mu\nu})\ (F^{\mu\nu}_{oc})^{*}\ +\ (F^{kc}_{\mu\nu})\ (F^{\mu\nu}_{kc})^{*}\ (c = 1,2,\ldots,7) \tag{3.24}$$

and where the $SU(3)$ field strength is given by

$$\mathcal{F}^{\gamma}_{\mu\nu}\ =\ \partial_{\mu}\mathcal{A}^{\gamma}_{\nu}\ -\ \partial_{\nu}\mathcal{A}^{\gamma}_{\mu}\ +\ \mathcal{A}_{\mu}{}^{\alpha}\ \mathcal{A}^{\beta}_{\nu}\ f^{\gamma}_{\alpha\beta} \tag{3.25}$$

The Gravitational Action

To begin with one can realize that there are problems with quadratic curvature actions like

$$\int\ <\ \mathbf{g}^{\mu\nu}\ \mathbf{g}^{\rho\sigma}\ \bar{\mathbf{F}}_{\mu\rho}\ \mathbf{F}_{\nu\sigma}\ >,\quad \int\ <\ \bar{\mathbf{R}}_{\mu\nu\rho\sigma}\ \mathbf{R}^{\mu\nu\rho\sigma}\ >,\quad \cdots ... \tag{3.26}$$

(as usual $< \cdots >$ denotes taking the real part) because the composition algebra $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$ is non-commutative, non-associative, *and* non-alternative [7]. To raise the four indices in $< \bar{\mathbf{R}}\mathbf{R} >$ requires the product of 4 factors of the metric $\mathbf{g}$ making matters more problematic because the Moufang identities, like $(AB)(CA) = A(BC)A$ are *no* longer obeyed due to the loss of alternativity.

For the time being we shall discard the other metric component $g^{oo}_{(\mu\nu)}$, and raise/lower spacetime indices with the base spacetime metric $g_{\mu\nu}$ to simplify things. Actions based on terms linear in the curvature $\int < \mathbf{R} >$ furnish the standard Einstein-Hilbert action $\int \mathcal{R}$ if one chooses for the integral measure $\sqrt{det\ |g_{\mu\nu}|}$. In doing so, we also may build quadratic curvature actions like

$$\int\ < g^{\mu\nu}\ g^{\rho\sigma}\ \bar{\mathbf{F}}_{\mu\rho}\ \mathbf{F}_{\nu\sigma} >\ \ =\ \int\ g^{\mu\nu}\ g^{\rho\sigma}\ (F^{IA}_{\mu\rho})^{*}\ F^{JB}_{\nu\sigma}\ \delta_{AB}\ \delta_{IJ} \tag{3.27}$$

$(I = 0,1,2,3; A = 0,1,2,3,\cdots,7)$, and

$$\int\ c_1\ \mathcal{R}\ + c_2\ (\mathcal{R}_{\mu\nu})^2\ + c_3\ (\mathcal{R}_{\mu\nu\rho\sigma})^2 \tag{3.28}$$

To sum up, given the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued curvature tensor $\mathbf{R}^{\sigma}_{\rho\mu\nu} = \mathcal{R}^{\sigma}_{\rho\mu\nu} + \mathbf{\Omega}^{\rho}_{\sigma\mu\nu}$, we shall raise/lower indices with the base spacetime metric $g_{\mu\nu}$ to construct the following action linear in $\mathcal{R}$, and quadratic in $\mathbf{F}$:

$$S\ =\ \frac{1}{16\pi G}\ \int\ d^4x\ \sqrt{|det\ g_{\mu\nu}|}\ \left(\ \mathcal{R}\ -\kappa^2\ (F^{IA}_{\mu\nu})\ (F^{\mu\nu}_{IA})^{*}\ \right) \tag{3.29}$$

κ is a length parameter, and the metric signature is chosen to be Lorentzian $(-,+,+,+)$.

The 32 complex-valued fields A_μ^{IA}, and field strengths $F_{\mu\nu}^{IA}$, have a one-to-one *correspondence* with the 64 real-valued fields $\mathcal{A}_\mu^\alpha(\alpha = 1,2,\cdots,64)$ associated with the $u(8)$ Lie algebra of the compact group $U(8) = SU(8) \times U(1)$. Hence, the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravity / Gauge correspondence is

$$\frac{1}{16\pi G} \int d^4x \sqrt{|det\ g_{\mu\nu}|} \ \left(\mathcal{R} \ - \kappa^2 \ (F_{\mu\nu}^{IA}) \ (F_{IA}^{\mu\nu})^* \ \right) \ \Leftrightarrow$$

$$\int d^4x \sqrt{|det\ g_{\mu\nu}|} \ \left(\frac{\mathcal{R}}{16\pi G} \ - \frac{1}{4g^2} \ (\mathcal{F}_{\mu\nu}^\alpha) \ (\mathcal{F}_\alpha^{\mu\nu}) \right) \tag{3.30}$$

α runs over $1,2,3,\cdots,64$ which is the number of generators of the $u(8)$ Lie algebra. The $U(8)$ gauge coupling g is $\frac{1}{4g^2} = \frac{\kappa^2}{16\pi G} \Rightarrow g^2\kappa^2 = 4\pi G = 4\pi L_P^2$, where L_P is the Planck scale.

The results of section **2** permit to *associate* the *internal* $\mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$ part of the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued metric $\mathbf{g}_{\mu\nu}$ to a 8×8 matrix-valued metric $\mathbf{G}_{\mu\nu} = G_{\mu\nu}^{MN}$ comprised of 8×8 complex entries. Namely, the 64 matrix entries in $G_{\mu\nu}^{MN}$ are comprised of *tensorial* quantities. The $\mathbf{R}$-component of the metric $\mathbf{g}_{\mu\nu}$ is associated to the diagonal 8×8 matrix $g_{\mu\nu}\delta^{MN}$. In this way one can rewrite the line element (3.3) in terms of the **trace** of the 8×8 complex-valued matrices with tensorial-valued entries as follows

$$ds^2 \ = \ \frac{1}{16} \ \left(\ Trace \ \{ \ G_{\mu\nu}^{MN} \ dx^\mu \ dx^\nu \ \} \ \right) \ + \ complex \ conjugate \tag{3.31}$$

The *isometry* group that leaves *invariant* the line element in eq-(3.31) is precisely the unitary $U(8)$ group. Under $U(8)$ transformations acting on the matrix (and not on the coordinates) one has

$$Trace \ \{ \ \mathbf{G'}_{\mu\nu} \ dx^\mu \ dx^\nu \ \} \ = \ Trace \ \{ \ \mathbf{U} \ \mathbf{G}_{\mu\nu} \ \mathbf{U}^\dagger \ dx^\mu \ dx^\nu \ \} \ =$$

$$Trace \ \{ \ \mathbf{U}^\dagger \ \mathbf{U} \ \mathbf{G}_{\mu\nu} \ dx^\mu \ dx^\nu \ \} \ = \ Trace \ \{ \ \mathbf{G}_{\mu\nu} \ dx^\mu \ dx^\nu \ \} \tag{3.32}$$

due to the unitary matrix $\mathbf{U}^\dagger \ \mathbf{U} = \mathbf{1}$, and the cyclic property of the trace. Consequently, we have replaced the Kaluza-Klein prescription to generate gauge symmetries in lower dimensions from isometries of the internal manifold, by $U(8)$ isometry transformations of the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued metric, described by eq-(3.2). A related approach based on Clifford spaces can be found in [22]. The Lorentz transformations act on the spacetime coordinates and spacetime indices of $\mathbf{G}_{\mu\nu}$ only. Thus the interval (3.32) is also Lorentz invariant.

This $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued gravitational model is not complete until *matter* is introduced and solutions to the corresponding Einstein's equations are found. There is a long history of $SU(8)$ unification models in the literature; see [31] and the encyclopedic work by [32]. An interesting $SU(8)$ family unification with boson-fermion balance was constructed by [30] where the 56 of scalars breaks $SU(8)$ to $SU(3)_{family} \times SU(5) \times U(1)/Z_5$. The embedding conditions (3.19-3.24) correspond to the following branching/decomposition of $U(8)$

$$U(8) = SU(8) \times U(1) \ \rightarrow \ SU(3)_F \times SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1) \times U(1) \tag{3.33}$$

The subgroups in the right hand side of (3.34) appear in *pairs* due to the doubling of degrees of freedom resulting from the complex-valued fields which appear in the right-hand side of eqs-(3.19, 3.24). The rank of $U(8)$ is 8 and matches the total rank of the groups in the right-hand side (3.33) : $2+2+1+1+1+1 = 8$. $SU(3)_F$ is the 3-family symmetry group; $SU(3)_C$ is the color group. $SU(2)_L \times SU(2)_R$ is the left/right chiral isospin group, and one of the $U(1)$'s can be identified with the $U(1)_Y$.

A unification of left-right $SU(3)_L \times SU(3)_R$, color $SU(3)_C$ and family $SU(3)_F$ symmetries in a maximal rank-8 subgroup of E_8 was proposed by [33] as a landmark for future explorations beyond the Standard Model (SM). This model is called the $SU(3)$-family extended SUSY trinification model [33]. Among the key properties of this model are the unification of SM Higgs and lepton sectors, a common Yukawa coupling for chiral fermions, the absence of the μ-problem, gauge couplings unification and proton stability to all orders in perturbation theory.

One may notice that after a symmetry breaking $SU(3)_L \rightarrow SU(2)_L \times U(1)$, and $SU(3)_R \rightarrow SU(2)_R \times U(1)$ of the $SU(3)$-family extended SUSY trinification model $[SU(3)]^4$ of [33], one recovers precisely the branching of $U(8)$ described by the right hand side of eq-(3.33). Therefore it is warranted to explore further the model of [33] within the context of the results described in this work. Arguments for a Grand Unified Model, including gravity, based on the complex Clifford algebra $Cl(5,C) \sim [Cl(4,R)]^4$, were advanced by the author [34]. The dimension of $Cl(5,C) = 64$, is also the dimension of the real Clifford algebra $Cl(0,6;R) \simeq \mathbf{O}_L \simeq \mathbf{O}_R$ [7].

Concluding, $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravity naturally can describe a Grand Unified Field Theory of Einstein **gravity** with a $U(8)$ Yang-Mills theory. In particular, the embedding conditions (3.19-3.24) suggest that an extension of the Standard Model group should include a 3-family $SU(3)_F$ symmetry group, and an extra $U(1)$ symmetry. The fact that so far only 3 families have been observed is very encouraging that this Grand Unification approach based on $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravity is on the right track. The role of the extra metric element $h_{\mu\nu} = g^{oo}_{(\mu\nu)}$ within the context of *bimetric* theories of gravity (and dark energy) deserves further scrutiny.

Acknowledgements

We are indebted to M. Bowers for invaluable assistance in preparing the manuscript. Special thanks to T. Smith for numerous discussions of his work.

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