A thought about the Riemann hypothesis.

Abstract. In these papers we will try to face the Riemann hypothesis, basing on the study of the functional equation of the Riemann zeta function.

0 Introduction and known properties. *[1]*

The Riemann zeta function is defined by the Dirichlet series:

0.1 $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

for every complex number s with real part Re(s) greater than 1.

However, this function can be analytically continued to a holomorphic function on the whole complex s-plane, except for a simple pole at s=1, which satisfies the following functional equation:

1 $\zeta(s) = 2^s \pi^{s-1} sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$, where s is a complex number and Γ is the gamma function.

The zeta function has some zeros, called the trivial zeros, and others, known as the non-trivial zeros, which lie in the critical strip **0<Re(s)<1**, symmetrically about the real line.
Thus for every non-trivial zero z = $\sigma + it$:

0.2 $\exists! \, z_n \mid z_n = \sigma - it$

The Riemann hypothesis is about the non-trivial zeros and it asserts that:
"Every non-trivial zero s has Re(s)=1/2".

In addition, for the zeta function **(0.1)** there is the following identity, which is the Euler product formula:

0.3 $\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \frac{1}{1-p^{-s}}$,

where s is a complex number with real part Re(s) greater than 1 and p is a prime number.

Proof p(0) *[Euler]***:**

We consider the zeta function **(0.1)**

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$

We multiply it by the factor $\frac{1}{2^s}$

$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \cdots$

Subtracting the second equation from the first we get:

$(1 - \frac{1}{2^s})\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots$

Doing this, we have removed all elements that have a factor of 2.

Repeating the same procedure for the next terms:

$(1 - \frac{1}{7^s})(1 - \frac{1}{5^s})\left(1 - \frac{1}{3^s}\right)(1 - \frac{1}{2^s})\,\zeta(s) = 1$

In this way, we are progressively removing all the multiples of every term after 1, which is a prime number since it is not a multiple of any lesser number. Therefore, the numbers of the product are all primes.

Repeating infinitely for $\frac{1}{p^s}$, where p is a prime, we obtain:

$$\dots(1-\frac{1}{11^s})\,(1-\frac{1}{7^s})(1-\frac{1}{5^s})\left(1-\frac{1}{3^s}\right)(1-\frac{1}{2^s})\,\zeta(\mathrm{s})=1$$

Dividing both sides by everything but $\zeta(\mathrm{s})$,we get:

$$\zeta(\mathrm{s})=\frac{1}{(1-\frac{1}{2^s})}\,\frac{1}{\left(1-\frac{1}{3^s}\right)}\,\frac{1}{(1-\frac{1}{5^s})}\,\frac{1}{(1-\frac{1}{7^s})}\,\frac{1}{(1-\frac{1}{11^s})}\cdots,$$

which is equivalent to:

$$\zeta(\mathrm{s})=\prod_{p\ prime}\frac{1}{1-p^{-s}}\ \textbf{(0.3)}$$

From the Euler product formula **(0.3)** we deduce an important consequence:

(0.4) *"the zeta function has no zeros in the half-plane Re(s)>1".*

1 Study of the functional equation

In order to study the functional equation, we factor it.

1.1 factor: 2^s

The factor 2^s is an exponential function of the form y=a^s, with a=2 and s ∈ ℂ. Since the base of the exponent is different from 0, there is no value of s such that cancels the factor 2^s and therefore the functional equation **(1)**. Thus:

1.11 $\boldsymbol{\nexists\ \mathrm{s} \in C \mid 2^S = 0\ \rightarrow (1) = 0}$

1.2 factor: π^{s-1}

Doing a similar reasoning to that used for the factor **(1.1),** we can assert that there is no value of s such that cancels the factor π^{s-1} and therefore the functional equation **(1).** Thus:

1.21 $\boldsymbol{\nexists\ s\ \in C \mid \pi^{S-1} = 0\ \rightarrow (1) = 0}$

1.3 factor: $\sin(\frac{\pi\, s}{2})$

Since the factor $\sin\left(\frac{\pi s}{2}\right)$ is a goniometric function of the form y=sin(x), it cancels the functional equation **(1)** when the argument of the sin equals to kπ , with k$\in$ ℤ.

In this case we have that:

$$\frac{\pi s}{2}=k\pi$$

$$\pi s=2k\pi$$

1.31 $\boldsymbol{s = 2k}$**, with** $\boldsymbol{k \in \mathbb{Z}}$

Because the zeta function **(0.1)** has no zeros for **(0.4)**, the formula **(1.31)** becomes:

1.32 s = -2n, with n $\boldsymbol{\in \mathbb{N} - \{0\}}$

Thus, when s = -2n, with n ∈ $\mathbb{N}-\{0\}$, the functional equation **(1)** equals 0. Therefore:

1.33 $\boldsymbol{\zeta(-2n) = 0}$**, with n** $\boldsymbol{\in \mathbb{N} - \{0\}}$

The trivial zeros of the zeta function derive from the formula **(1.33)**.

1.4 factor: $\mathbf{\Gamma(1-s)}$ *[2]*

Since the gamma function has the property to be non- zero everywhere, there is no value of s such that cancels the factor $\Gamma(1-s)$ and therefore the functional equation **(1)**. Thus:

1.41 $\boldsymbol{\nexists\ s \in C \mid \Gamma(1-s) = 0 \rightarrow (1) = 0}$

1.5 factor: $\boldsymbol{\zeta(1-s)}$

[In the next paragraph we will concentrate upon the factor ***(1.5)****].*

2 Study of the factor $\boldsymbol{\zeta(1-s)}$

In order to study this factor, we will think about the following question:
"What happen when $\zeta(1-s)$=0, canceling the functional equation ***(1)****?"*
We formulate two hypothesis:

Hypothesis A: ***"If*** $\boldsymbol{\zeta(1-s)}$***=0, then*** $\boldsymbol{\zeta(1-s) \neq \zeta(s)}$**".**

Hypothesis B: ***"If*** $\boldsymbol{\zeta(1-s)}$***=0, then*** $\boldsymbol{\zeta(1-s) = \zeta(s)}$**".**

From the functional equation we demonstrate that the hypothesis **A** is false, while the hypothesis **B** is true.

Proof p(1):

Considered the functional equation **(1)**, if $\zeta(1-s)$*=0*, for the zero-product property, the functional equation equals 0. From this, we work out the following relation:

2.1 $\boldsymbol{\zeta(1-s)=0 \rightarrow \zeta(s)=0}$

In fact:
If $\zeta(1-s)$*=0*

$$\zeta(s) = 2^{s}\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

$$\zeta(s) = 2^{s}\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\ 0$$

$\zeta(s)$ =0

Therefore, $\zeta(1-s)$*=0* $\rightarrow \zeta(s)$ *=0* **(2.1)**

From the relation **(2.1)**, the hypothesis **(A)** is false, whilst the hypothesis (**B**) is true:

(A) $\zeta(1-s)=0 \rightarrow \zeta(s)=0 \rightarrow \zeta(1-s) \neq \zeta(s) \rightarrow (false\ for\ proof\ \boldsymbol{p}(\mathbf{1}))$.

(B) $\zeta(1-s)=0 \rightarrow \zeta(s)=0 \rightarrow \zeta(1-s) = \zeta(s) \rightarrow (true\ for\ proof\ \boldsymbol{p}(\mathbf{1}))$.

From the veracity of the hypothesis **(B)** we deduce a key condition satisfied when $\zeta(1-s)$*=0* that is:

2.2 $\boldsymbol{\zeta(s) = \zeta(1-s)}$

From the condition **(2.2)** we deduce an important consequence:
"The trivial zeros of the zeta function don't derive from the factor $\zeta(1-s)$".

Proof p(2):

Let s = -2n, with n ∈ $\mathbb{N} - \{0\}$, we substitute it in the functional equation **(1)**:

$$\zeta(-2n) = 2^{-2n}\pi^{-2n-1}\sin\left(\frac{\pi(-2n)}{2}\right)\Gamma\big(1-(-2n)\big)\zeta\big(1-(-2n)\big)$$

$\zeta(-2n) = 2^{-2n}\pi^{-2n-1}\sin(-n\pi)\,\Gamma(1+2n)\zeta(1+2n)$

Done the substitution and the related calculations, $\zeta(s)$ and $\zeta(1-s)$ have become respectively $\zeta(-2n)$ and $\zeta(1+2n)$.

For the formula **(1.33)** $\zeta(-2n) = 0$, while $\zeta(1+2n) \neq 0$, because the zeta function of the form **(0.1)** has no zeros.

Therefore $\zeta(-2n) \neq \zeta(1+2n)$ and thus $\zeta(s) \neq \zeta(1-s)$.

Consequently the condition **(2.2)** is not satisfied and so the trivial zeros don't derive from the factor **(1.5)**, but only from **(1.3).**

3 Study of the condition $\zeta(s) = \zeta(1-s)$ in the critical strip

We consider the functional equation **(1).** From what we have said in the previous paragraphs, we deduce that the non-trivial zeros of the zeta function don't derive from the factors **(1.1), (1.2), (1.4)**, because they don't cancel the functional equation, and even from the factor **(1.3)**, from which the sole trivial zeros derive. Thus they only derive from the factor **(1.5)** and satisfy the condition **(2.2)**.

Now, we suppose that z_0 is a non-trivial zero with real part $\sigma \neq \frac{1}{2}$ and imaginary part it:

$z_0 = \sigma + it$

We substitute z_0 in the condition **(2.2)**:

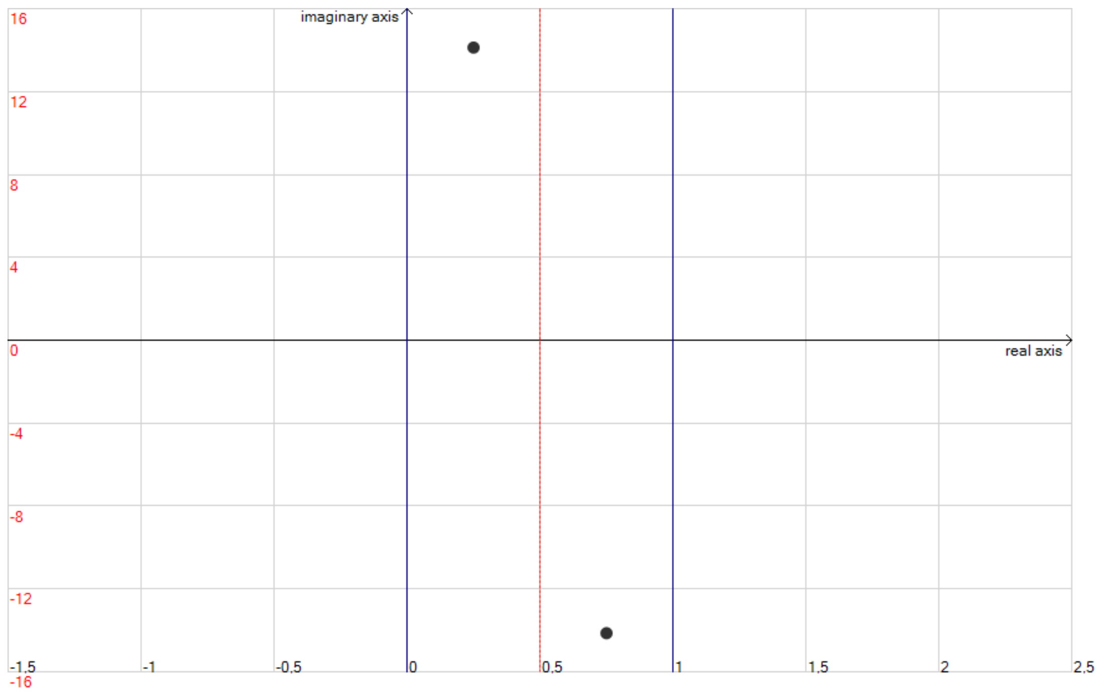
$\zeta(z_0) = \zeta(1-z_0)$

From this, we obtain a second non-trivial zero z_1, which is equal to 1-z_0:

$z_1 = 1 - z_0 = (1-\sigma) - it$

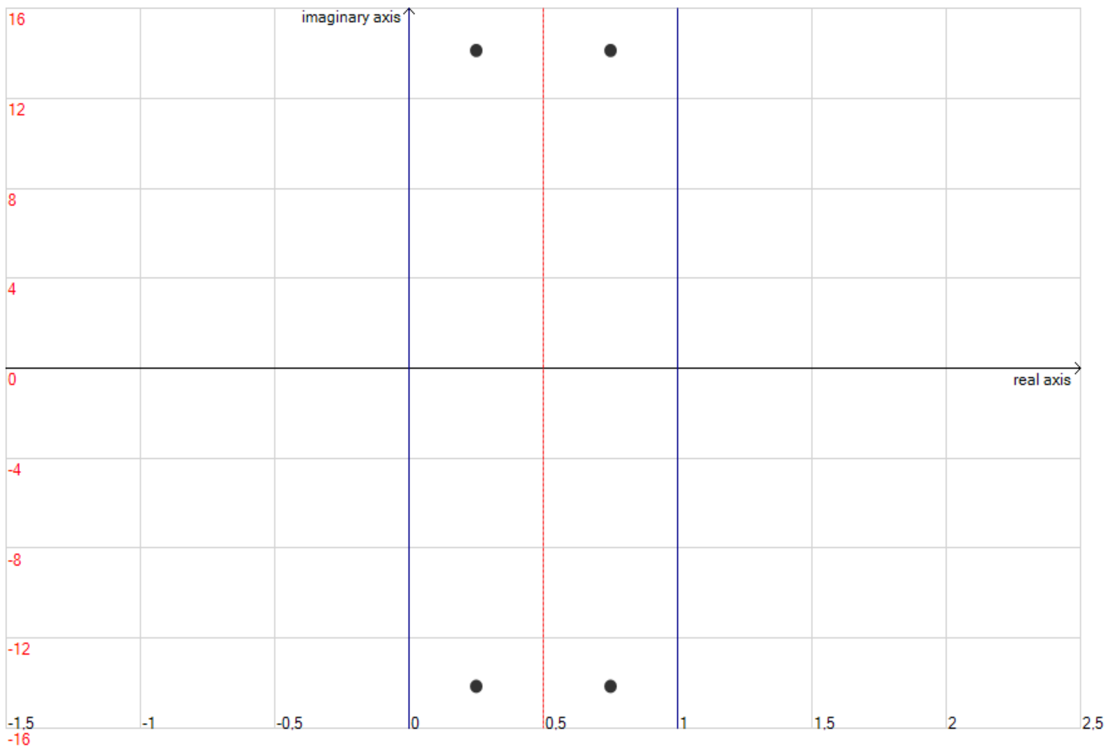
We represent z_0 and z_1 in the complex plane:

Picture **1**



From the property **(0.2)** of the non-trivial zero we obtain z_2 and z_3:
$z_2 = \sigma - it$
$z_3 = (1-\sigma) + it$
We represent all the non-trivial zeros in the complex plane:

Picture **2**



Since z_0, z_1, z_2 and z_3 are non-trivial zeros, we deduce that $\zeta(z_0) = \zeta(z_1) = \zeta(z_2) = \zeta(z_3)$.
Therefore every couple of non-trivial zeros satisfies the following relation:

3.1 $\boldsymbol{\zeta(z_j) = \zeta(z_k)}$

So, for the non-trivial zeros both relations **(2.2)** and **(3.1)** are satisfied and combining them together we obtain this final relation, which must be satisfied by all the couples of non-trivial zeros:

3.2 $\boldsymbol{\begin{cases} \zeta(z_k) = \zeta(s) \\ \zeta(z_j) = \zeta(1-s) \end{cases}}$

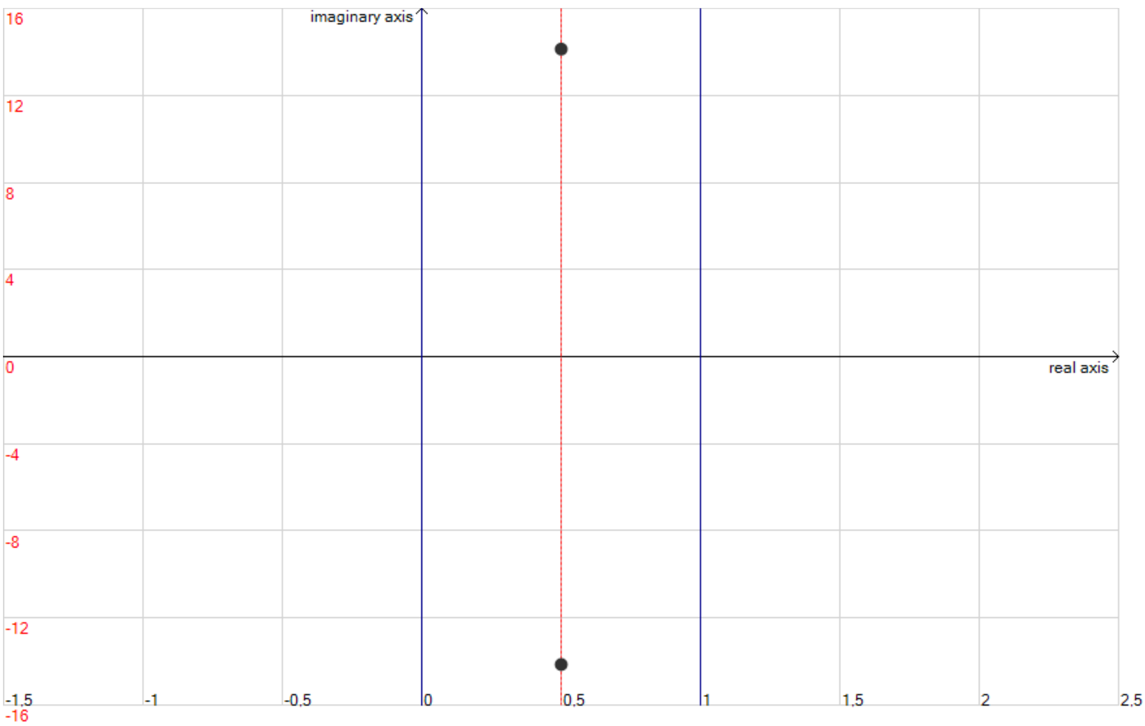
We have shown that if z_0 is a non-trivial zero with real part $\sigma \neq \frac{1}{2}$, there will be another three non-trivial zeros (z_1, z_2, z_3) such that we will have the situation represented in the picture **2.**
Now, we consider all the possible couples of non-trivial zeros:
$\{(z_0; z_1), (z_0; z_2), (z_0; z_3), (z_1; z_2), (z_1; z_3), (z_2; z_3)\}$
From the relation **(3.2)** we deduce that only the couples $(z_0; z_1)$ and $(z_2; z_3)$ satisfy the relation. In fact, since in the relation **(3.2)** one zero is equal to $\zeta(s)$ and the other is equal to $\zeta(1-s)$, these zeros must be symmetric about the point P$(\frac{1}{2}; 0)$.

Therefore the disposition of the non-trivial zeros, shown in the picture **2**, is an absurd because the couples $(z_0; z_2)$, $(z_0; z_3)$, $(z_1; z_2)$ and $(z_1; z_3)$ don't satisfy the relation **(3.2)**; in fact they aren't symmetric about the point P($\frac{1}{2}$; 0), but about the real line, for $(z_0; z_2)$ and $(z_1; z_3)$, and about the critical line x=$\frac{1}{2}$ for $(z_0; z_3)$ and $(z_1; z_2)$.
The only case, for which the relation **(3.2)** and the property **(0.2)** of the non-trivial zeros are both true, is when z_0 and z_1 coincide respectively with z_3 and z_2.
Thus we have:

$$z_0 = z_3$$
$$\sigma + it = (1 - \sigma) + it$$
$$\sigma = 1 - \sigma$$
$$2\,\sigma = 1$$
$$\sigma = \frac{1}{2}$$

We represent the situation in the complex plane:

Picture **3**



From the result, represented in the picture **3**, we substitute a complex number of the form $\frac{1}{2} + \sigma$ in the condition **(2.2)** and we get:

$$\mathbf{3.3}\quad \boldsymbol{\zeta\left(\frac{1}{2} + it\right) = \zeta\left(\frac{1}{2} - it\right)}$$

This relation is true only if the complex number $\frac{1}{2} + \sigma$ is a zero that is when $\zeta\left(\frac{1}{2} + it\right) = 0$.
Therefore the formula (3.3) is true if:

$$\mathbf{3.4}\quad \boldsymbol{\zeta\left(\frac{1}{2} \pm it\right) = 0}$$

The formula **(3.4)** confirms the relation **(3.2)** and the property **(0.2)** of the non-trivial zeros, stated in the introduction.
All the zeros, known as the non-trivial zeros of the zeta function, derive from this formula.

In conclusion, in the paragraph **3** we have shown that all the non-trivial zeros of the zeta function must be of the form $\frac{1}{2}+\sigma$ so that the relation **(3.2)** and the property **(0.2)**,both satisfied by all the non-trivial zeros, are true and this confirms the veracity of the Riemann hypothesis.

References

[1] *Wikipedia, Riemann zeta function: https://en.wikipedia.org/wiki/Riemann_zeta_function*
[2] *Wikipedia, Gamma function: https://en.wikipedia.org/wiki/Gamma_function*