Relativistic Forces Between Rotating Bodies

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Abstract
The force experienced by a rotating body that lies in the **G** field of another rotating body depends both on the **G** field and on its own angular velocity of rotation that affects the magnitude and direction of the exerted force. The force is in general not central and not symmetric. The cases of the non rotating observer and the far away observer are examined for rotation with and without slippage. It is shown that the force may be repulsive or attractive or alternating between attractive and repulsive depending on the angular velocities of the rotations and distance.

1 Introduction

This paper is a continuation of [2] and [3]. In [2] the path of signals emanating from the origin of rotating frames was studied. In [3] based on the findings of [2] the field (called $\mathbf{G}$) created by a rotating body that emanates signals was determined for different observers. In this paper we go one step further to examine the force felt by a rotating body in a $\mathbf{G}$ field. The body will in general feel a non central force and will be obliged to move accordingly. The magnitude of the force felt depends on the magnitude of the field but also on the ability of the body to receive signals, which is proportional to its mass. However, if the body itself rotates, the force due to the field $\mathbf{G}$ that is experienced by the rotating body is affected both in magnitude and direction by the mass and rotation of the receiving body because signals approaching the receiving rotating body are affected by its rotation. This leads us among other things to attractive and repulsive forces. The interaction between two rotating bodies is studied and the strength and direction of forces determined for the cases of the non-rotating and the far away observer and for the sub-cases of rotation with and without slippage. (By slippage we mean exponentially decreasing angular velocity of rotation).
This paper is organized as follows: In section 2 we review previous results. In section 3 we find the force between two rotating bodies for rotation with and without slippage and for different observers. In section 4 we visualize the signals' path to understand how repulsive and attractive forces are formed. In section 5 we show how we can generalize to bodies with non parallel axes of rotation. In section 6 we conclude.

2 Review of previous theory

Below is a summary of the results found in [2] and [3], on which this paper stands. We present the transformation of cylindrical coordinates for rotating frames and for different kind of observers and the corresponding $\mathbf{G}$ field that a rotating body at the origin of the rotating frame (which rotates about its z axis), emitting signals, creates.

The formulas for the **G** field differ slightly from those in [3] in that the constant 4π is absorbed by the constant k_G which now becomes equal to the usual gravitational constant ($6.67*10^{-11}$ m^3 kg^{-1} s^{-2})

A. Rotation without slippage (the angular velocity w of rotation of signals is constant with respect to the distance from the rotating body). Precession of the rotating body is assumed having a very small amplitude and is thus neglected.

A.I Observer O' at the origin but not rotating with the body. (The transformation holds for $|z| \le \frac{c}{w}$).

$$\rho' = c \sin \xi I(\xi, t) \tag{1}$$

$$\Theta' = \Theta + vt \tag{2}$$

$$z' = z \tag{3}$$

$$t' = t \tag{4}$$

$$\pi' = \pi \frac{\rho}{\rho'} \frac{c}{\sqrt{c^2 + w^2 \rho^2}} = \pi \frac{t}{I(\xi, t)} \frac{c}{\sqrt{c^2 + w^2 \rho^2}} \tag{5}$$

$$v' = v \tag{6}$$

where $\rho, \Theta, z, t, \pi, v$ are the radial distance in cylindrical coordinates, the angle of rotation as fraction of a circle (for example degrees), the z direction that coincides with the axis of rotation, time, the number pi, and the frequency of rotation respectively for observer O , who is located at the origin and rotates with the body. And where $\rho', \Theta', z', t', \pi', v'$ are the same quantities for observer O', who is located at the origin but not rotating with the body. The speed of light is *c*. The angle of inclination of the signal is the same, ξ, for both observers *O* and O'.

Further, where,

$$I(\xi, t) = \int_0^t \cos \varphi dt \tag{7}$$

$$\cos \varphi = \sqrt{\frac{1 - w^2 t^2 \cos^2 \xi}{1 + w^2 t^2 \sin^2 \xi}} = \sqrt{\frac{c^2 - w^2 z^2}{c^2 + w^2 \rho^2}} \tag{8}$$

with $\cos \xi = \frac{z}{\sqrt{\rho^2 + z^2}}$, $\sin \xi = \frac{\rho}{\sqrt{\rho^2 + z^2}}$, $\rho = ct \sin \xi$, $z = ct \cos \xi$, φ is the deflection angle of the field signal from the radial direction.

From the above we can find the transformation of the angular velocity w using the formula ($w = 2\pi v$ and $w' = 2\pi' v$) and the angle of rotation θ measured in radians (using $\theta = 2\pi\Theta$ and $\theta' = 2\pi'\Theta'$) as,

$$\frac{w'}{w} = \frac{\pi'}{\pi} \tag{9}$$

$$\theta' = (\theta + wt) \frac{\pi'}{\pi} \tag{10}$$

The **G** field which in this case we denote as $\mathbf{G}'$ is given by

$$\mathbf{G}' = -\frac{k_G m'}{\left(z^2 + \frac{\rho^2 z^2 w^4 U(\xi, t)}{I(\xi, t)(\rho^2 + z^2)} + \frac{\rho^2 \cos \varphi \sqrt{\rho^2 + z^2}}{c I(\xi, t)} \right) \frac{c}{\sqrt{c^2 + w^2 \rho^2}}} \hat{\mathbf{v}}' \tag{11}$$

where

$$U(\xi,t)=\int_0^t \frac{t^4}{\sqrt{1-w^2t^2\cos^2\xi}(1+w^2t^2\sin^2\xi)^{\frac{3}{2}}}dt \tag{12}$$

And $\hat{\mathbf{v}}'$ is the unit vector in the direction of the velocity of the signals of the field.

$$\hat{\mathbf{v}}'=(v'_\rho,v'_\theta,v'_z)=(\sin\xi\cos\varphi,\sin\xi\sin\varphi,\cos\xi) \tag{13}$$

When $z=0$ the above become,

$$I(\xi,t)\big]_{z=0}=I(\frac{\pi}{2},t)=\int_0^t \frac{dt}{\sqrt{1+w^2t^2}}=\frac{1}{w}\operatorname{arcsinh} wt=\frac{1}{w}\operatorname{arcsinh}\frac{w\rho}{c} \tag{14}$$

$$\rho'\big]_{z=0}=\frac{c}{w}\operatorname{arcsinh}\frac{w\rho}{c} \tag{15}$$

And

$$\mathbf{G}'\big]_{z=0}=-\frac{k_Gm'(c^2+w^2\rho^2)\operatorname{arcsinh}\frac{w\rho}{c}}{\rho^3cw}\hat{\mathbf{v}}'=-\frac{k_Gm'(c^2+w^2\rho^2)\rho'}{c^2\rho^3}\hat{\mathbf{v}}' \tag{16}$$

A.II Observer O'' is the far away observer outside the cylindrical volume defined by $\rho''\le\frac{c}{w}$) for which the transformation below holds.

$$\rho''=\rho\frac{c}{\sqrt{c^2+w^2\rho^2}}=\rho\cos\varphi'' \tag{17}$$

$$\Theta''=\Theta+vt \tag{18}$$

$$z''=z \tag{19}$$

$$t''=t \tag{20}$$

$$\pi''=\pi \tag{21}$$

$$v''=v \tag{22}$$

$$\theta''=\theta+wt \tag{23}$$

The angle φ'' is the angle of deflection of the field signal from the radial direction.

$$\tan\varphi''=wt(1+w^2t^2\sin^2\xi) \tag{24}$$

And the angle of inclination of the signal from the z axis is given by

$$\tan\xi''=\tan\xi\frac{\sqrt{1+w^2t^2(1+w^2t^2\sin^2\xi)^2}}{(1+w^2t^2\sin^2\xi)^2}=\tan\xi\frac{\sqrt{c^2+w^2(\rho^2+z^2)(c^2+w^2\rho^2)^2}}{(c^2+w^2\rho^2)^2} \tag{25}$$

Where $\tan\xi=\frac{\rho}{z}$

The $\mathbf{G}$ field, which in this case we denote as $\mathbf{G}''$, is given by

$$\mathbf{G}''=-\frac{k_Gm'(c^2+\rho^2w^2)^{\frac{3}{2}}}{c^3\left(\rho^2+z^2\right)}\hat{\mathbf{v}}''==-\frac{km'c^3}{(c^2\rho''^2+(c^2-w^2\rho''^2)z^2)\sqrt{c^2-w^2\rho''^2}}\hat{\mathbf{v}}'' \tag{26}$$

Where $\hat{\mathbf{v}}''$ is the unit vector in the direction of the velocity of the signals of the field as O'' sees them,

$$\hat{\mathbf{v}}''=(v''_\rho,v''_\theta,v''_z)=(\sin\xi''\cos\varphi'',\sin\xi''\sin\varphi'',\cos\xi'') \tag{27}$$

The field for observer O'' extends up to a cylinder defined by $\rho'' \le \frac{c}{w}$ where it forms a barrier and himself lies outside that cylinder. In contrast, O' (case A.I), who lies inside the above cylinder sees the field as extending to infinity in the radial direction but is restricted in the z direction to within $|z| \le \frac{c}{w}$.

B. Rotation with slippage. (The angular velocity of rotation of signals decreases exponentially with respect to the distance from the rotating body). This case has more meaning physically and we also avoid the unnatural boundaries that appear at $|z| = c/w$ and at $\rho'' = c/w$. The angular velocity is given by $w = w_0 e^{-(\lambda\rho+\mu z)} = w_0 e^{-ct(\lambda\sin\xi+\mu\cos\xi)}$ and the frequency of rotation is $v = v_0 e^{-(\lambda\rho+\mu z)} = v_0 e^{-ct(\lambda\sin\xi+\mu\cos\xi)}$ where λ, μ are the slippage parameters.

B.I Observer O' at the origin but not rotating.

$$\rho' = c\sin\xi I(\xi, t, \lambda, \mu) \tag{28}$$

$$\Theta' = \Theta + \int_0^t v_0 e^{-ct(\lambda\sin\xi+\mu\cos\xi)} dt = \Theta + \frac{v_0(1-e^{-ct(\lambda\sin\xi+\mu\cos\xi)})}{c(\lambda\sin\xi+\mu\cos\xi)} \tag{29}$$

$$z' = z \tag{30}$$

$$t' = t \tag{31}$$

$$\pi' = \pi\frac{\rho}{\rho'}\frac{c}{\sqrt{c^2 + w_0^2\rho^2 e^{-2(\lambda\rho+\mu z)}}} \tag{32}$$

where

$$I(\xi, t, \lambda, \mu) = \int_0^t \cos\varphi dt \tag{33}$$

where φ is the angle of deflection of the signal from the radial direction and

$$\cos\varphi = \sqrt{\frac{1 - w_0^2 t^2 e^{-2ct(\lambda\sin\xi+\mu\cos\xi)}\cos^2\xi}{1 + w_0^2 t^2 e^{-2ct(\lambda\sin\xi+\mu\cos\xi)}\sin^2\xi}} = \sqrt{\frac{c^2 - w_0^2 z^2 e^{-2(\lambda\rho+\mu z)}}{c^2 + w_0^2\rho^2 e^{-2(\lambda\rho+\mu z)}}} \tag{34}$$

$$\frac{w'}{w} = \frac{\pi'}{\pi} \tag{35}$$

and using (29) with (32) and the fact that $\theta = 2\pi\Theta$, $\theta' = 2\pi'\Theta'$, we find the transformation of the rotation angle in radians

$$\theta' = \frac{\pi'}{\pi}(\theta + \int_0^t w_0 e^{-ct(\lambda\sin\xi+\mu\cos\xi)} dt) = \frac{\pi'}{\pi}\left(\theta + \frac{w_0(1-e^{-ct(\lambda\sin\xi+\mu\cos\xi)})}{c(\lambda\sin\xi+\mu\cos\xi)}\right) \tag{36}$$

The angle of inclination of the signal to the z axis is ξ ($\tan\xi = \frac{\rho}{z}$) and is the same for both observers O, and O'.

Also assume that $\frac{1}{\mu} \le \frac{ce}{w_0}$, the condition needed for $\cos\varphi$ to be real for all ξ.

The $\mathbf{G}$ field which in this case we denote as $\mathbf{G}'$ is given by

$$\mathbf{G}' = \frac{k_G m'}{(\rho^2 + z^2)}\frac{1}{J'}\hat{\mathbf{v}}' \tag{37}$$

where

$$J' = \left(\cos^2 \xi + \frac{\rho \cos\varphi \cos\xi}{cI(\xi,t,\lambda,\mu)} - \frac{\cos\xi U_2(\xi,t)}{I(\xi,t,\lambda,\mu)} \right) \frac{c}{\sqrt{c^2 + w_0^2 \rho^2 e^{-2(\lambda\rho+\mu z)}}} \tag{38}$$

and

$$U_2(\xi,t) = \int_0^t \frac{\partial}{\partial \cos\xi} \left(\sqrt{\frac{1 - w_0^2 t^2 e^{-2ct(\lambda\sin\xi+\mu\cos\xi)}}{1 + w_0^2 t^2 e^{-2ct(\lambda\sin\xi+\mu\cos\xi)}} \frac{\cos^2\xi}{\sin^2\xi}} \right) dt \tag{39}$$

And

$$\hat{\mathbf{v}}' = (v'_\rho, v'_\theta, v'_z) = (\sin\xi\cos\varphi, \sin\xi\sin\varphi, \cos\xi) \tag{40}$$

B.II Observer O'' (the far away not rotating observer)

$$\rho'' = \rho \frac{c}{\sqrt{c^2 + w_0^2 \rho^2 e^{-2(\lambda\rho+\mu z)}}} \tag{41}$$

$$\Theta'' = \Theta + \int_0^t v_0 e^{-ct(\lambda\sin\xi+\mu\cos\xi)} dt = \Theta + \frac{v_0(1 - e^{-ct(\lambda\sin\xi+\mu\cos\xi)})}{c(\lambda\sin\xi + \mu\cos\xi)} = \Theta + \frac{v_0(1-e^{-ct\beta})}{c\beta} \tag{42}$$

Where $\beta = \lambda\sin\xi + \mu\cos\xi$

$$z'' = z \tag{43}$$

$$t'' = t \tag{44}$$

$$\pi'' = \pi \tag{45}$$

$$w_0'' = w_0 \tag{46}$$

$$\theta'' = \theta + \int_0^t w_0 e^{-ct(\lambda\sin\xi+\mu\cos\xi)} dt = \theta + \frac{w_0(1-e^{-ct\beta})}{c\beta} \tag{47}$$

Where φ'' is the angle of deflection of the field signal from the radial direction and is given by

$$\tan\varphi'' = \frac{wt(1 + w^2 t^2 \sin^2\xi)}{1 + c(\lambda\sin\xi + \mu\cos\xi)t^3 w^2 \sin^2\xi} = \frac{w\sqrt{\rho^2+z^2}}{c} \frac{1 + \frac{w^2\rho^2}{c^2}}{1 + \frac{w^2\rho^2(\lambda\rho+\mu z)}{c^2}} \tag{48}$$

Where $w = w_0 e^{-ct(\lambda\sin\xi+\mu\cos\xi)} = w_0 e^{-ct\beta}$,

The inclination of the path of the signal with respect to the z axis is given by

$$\tan\xi'' = \tan\xi \frac{wt}{\sqrt{1 + w^2t^2\sin^2\xi}} \sqrt{1 + \frac{(1 + c\beta t^3 w^2 \sin^2\xi)^2}{w^2t^2(1 + w^2t^2\sin^2\xi)^2}} = \tan\xi \frac{\sqrt{w^2t^2(1+w^2t^2\sin^2\xi)^2 + (1 + c\beta t^3 w^2\sin^2\xi)^2}}{(1 + w^2t^2\sin^2\xi)^{\frac{3}{2}}}$$

(49)

Where $\tan\xi = \rho / z$. Equation (49) can also be written as

$$\tan\xi'' = \tan\xi \frac{\sqrt{w^2 \frac{\rho^2+z^2}{c^2}(1 + w^2\frac{\rho^2}{c^2})^2 + (1 + (\lambda\rho+\mu z) w^2 \frac{\rho^2}{c^2})^2}}{(1 + w^2 \frac{\rho^2}{c^2})^{\frac{3}{2}}} \tag{50}$$

The **G** field, which in this case we denote as $\mathbf{G}''$, is given by

$$\mathbf{G}'' = -\frac{k_G m'}{(\rho^2+z^2)} \frac{(c^2 + w_0^2 \rho^2 e^{-2(\lambda\rho+\mu z)})^{\frac{3}{2}}}{c(c^2 + \lambda w_0^2 \rho^3 e^{-2(\lambda\rho+\mu z)})} \hat{\mathbf{v}}'' \tag{51}$$

with $\hat{\mathbf{v}}'' = (v''_\rho, v''_\theta, v''_z) = (\sin\xi''\cos\varphi'', \sin\xi''\sin\varphi'', \cos\xi'')$

The field for O'' forms a "barrier" like the no slippage case when w_0 is very big. By barrier we mean a maximum of the magnitude of the **G** field along with a sideway turn of the signals emitted by the rotating body. In this case the radial distances may be shrunk to sub-atomic (*microcosmos*) levels.

3 Force between two rotating bodies with parallel axes of rotation

We have already seen how a rotating point body A creates a $\mathbf{G}$ field around it. What is the force felt by another point body B that is also rotating and vice versa? In what direction does the force point? Is it symmetrical for A and B? These are the questions to be dealt here.

For rotation without slippage, we will examine both the $\mathbf{G}'$ field and $\mathbf{G}''$ field. For rotation with slippage we will only consider $\mathbf{G}''$ and only briefly comment on $\mathbf{G}'$.

First, we need to clarify some matters about observers. Up till now the nearby but not rotating observer O' was placed at the origin, where the rotating point mass is located. What if another observer O'_2 stationary with respect to O' is placed at some distance from the origin but not far away like observer O'' ? Length measurements denoted by, s, or angles like θ, will not be affected because he is stationary with respect to O'. However, O'_2 is subject to the gravity field of body A, while O', being at the center of the rotating body A, is not. This will affect the clock of O'_2, which will run slower. (see for example Møller [1]). If we denote by t'_{NG} the time of the No Gravity observer (O') and by t'_2 the time of O'_2 then the rate of clock will be altered by the factor $\frac{dt'_2}{dt'_{NG}}$. Other quantities of interest such as angular velocity, w, velocity, υ, will be altered as follows:

$w'_2 = \frac{d\theta'_2}{dt'_2} = \frac{d\theta'_2}{dt'_{NG}} \frac{dt'_{NG}}{dt'_2}$ but $\theta'_2 = \theta'_{NG}$ because angles are spatial measurements, thus

$$w'_2 = \frac{d\theta'_{NG}}{dt'_{NG}} \frac{dt'_{NG}}{dt'_2} = w'_{NG} \frac{dt'_{NG}}{dt'_2}$$

Similarly, for velocity, $\upsilon'_2 = \frac{ds'_2}{dt'_2} = \frac{ds'_{NG}}{dt'_{NG}} \frac{dt'_{NG}}{dt'_2} = \upsilon'_{NG} \frac{dt'_{NG}}{dt'_2}$. This also holds when $\upsilon = c$ the speed of light. Thus, the ratio $\frac{w}{c}$, or $\frac{\upsilon}{c}$ remain invariant between the two observers.

Consequently the angle of deflection φ given by (8), will remain invariant since it only depends on w/c

Further , $\frac{d\rho_2'}{dt_2'} = c_2' \sin\xi \cos\varphi = c_{NG}' \frac{dt_{NG}'}{dt_2'} \sin\xi \cos\varphi = \frac{d\rho_{NG}'}{dt_2'}$. It follows that ρ_2' and ρ_{NG}' differ by a constant. But at $t_2' = 0$ they are equal to each other. Therefore, $\rho_2' = \rho_{NG}'$ and thus ρ' is invariant as expected since it is a spatial measurement.
On account of (5) π is also invariant between observers O_2' and O' .
However, $I(\xi, t)$ is not invariant and therefore, the magnitude of the field **G** will differ but not its direction, which remains invariant.
Repeating the above exercise we see that the findings hold for case B.I, where we have the nearby observers but rotation with slippage. Therefore, in the following we will not require that observer O' is necessarily located at the origin of the axis at A.
Following the same reasoning for cases A.II and B.II, for the far away observer, we find that all the above quantities plus the magnitude of the **G** field remain invariant whether the observer is under gravity or not.

3.1 The $\mathbf{F}'$ force for rotation without slippage (observer O', case A.I)

First we need some notation. Body A has rotating mass m_A' as seen by observer O' (m_A' is calculated in [3] from the stationary mass m_A), angular velocity of rotation w_A and body B has m_B' (calculated the same way as m_A') and w_B respectively. Also let the distance between the bodies A, B as observed by observer O' (who stands within the **G** fields created by A and B) be $\rho_{AB}'(w_A, w_B)$ to remind us that it is a function of w_A and w_B, and let for the moment the planes of rotation of the two bodies be the same so that their distance in the z direction is zero ($z = 0$) for simplicity. Later we relax this restriction. Let also A and B be within each other's reach of the respective **G** field.
Suppose for the moment that $w_B = 0$. Then the signals traveling from A will reach B with an angle of deflection with respect to the line joining the two bodies as seen by O', equal to φ_A.[[Recall that from (8) $\cos\varphi_A\big]_{z=0} = \frac{c}{\sqrt{c^2 + w_A^2 \rho_{AB}'(0,0)^2}}$ or

$\tan\varphi_A\big]_{z=0} = \frac{w_A \rho_{AB}'(0,0)}{c}$, where $\rho_{AB}'(0,0)$ is the length of the path of the signal that travels from A to B, which is the same as the straight line from A to B, when there is no rotation i.e. when $w_A = w_B = 0$.]]
The force $\mathbf{F}_{AB}'$ that body B will feel as seen by observer O', will be defined as the product of the $\mathbf{G}_{AB}'$ field (the field due to A as it is felt by B) with the mass of body B. This is justified by the idea that the number of signals of the field that body B intercepts will be proportional to its observed mass, m_B'. This can be written as

$$\mathbf{F}_{AB}' = \mathbf{G}_{AB}' m_B' \tag{52}$$

Now we will let body B rotate (along with the space around it) so that $w_B \neq 0$. To visualize the situation look at Figure 1. The case of no rotation is shown in Figure 1 (a).

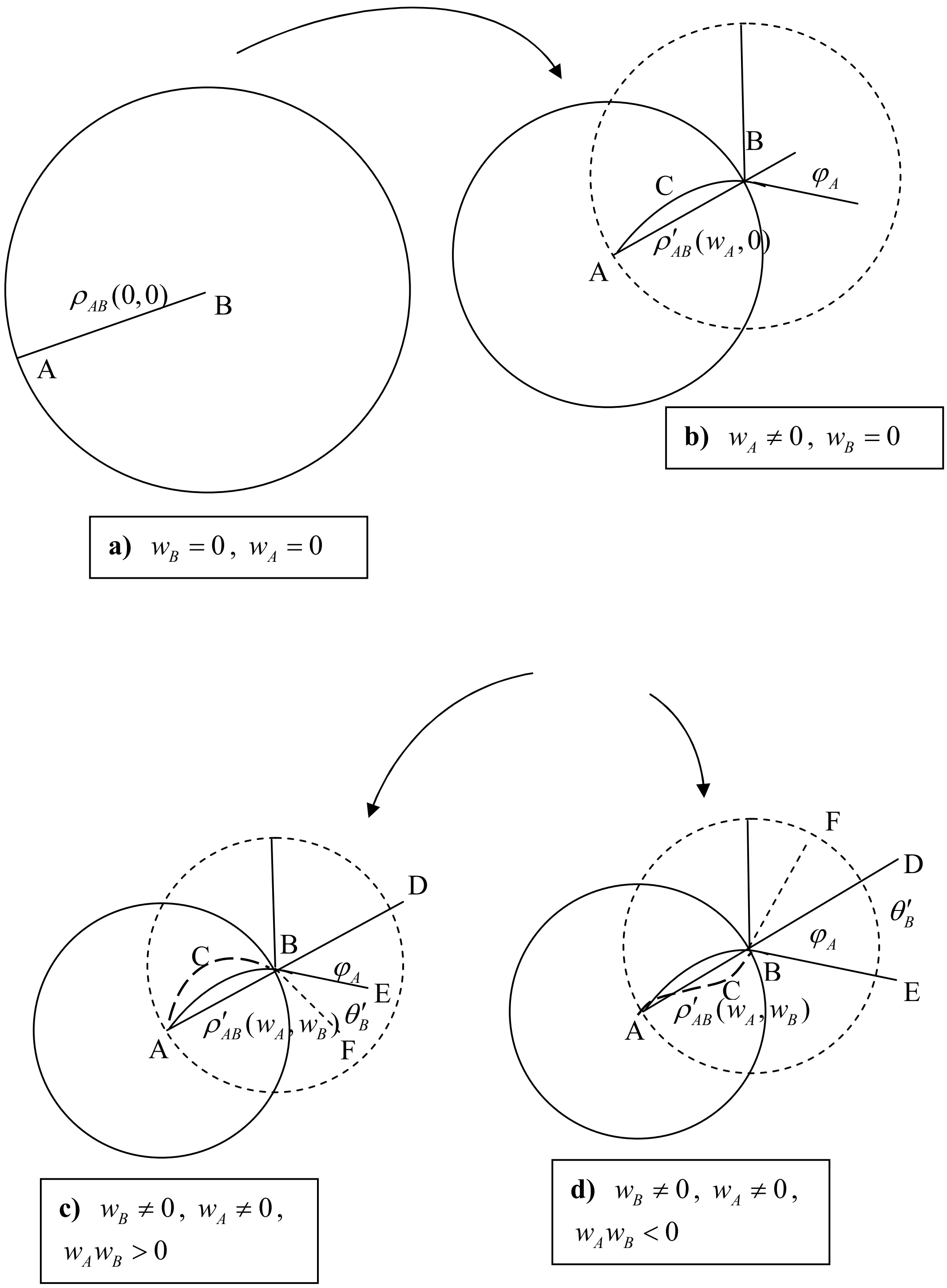


Figure 1 (a) Neither body rotates . The straight line distance which coincides with the path of the signal is AB and its length is $\rho_{AB}(0,0)$ **(b)** Only A rotates. The curved line ACB indicates the path of a signal from A towards B through C as seen by observer O' . The length of the path ACB is again $\rho_{AB}(0,0)$ but the straight line for observer O' is shorter: $\rho'_{AB}(w_A,0) = AB$ **(c)** A and B rotate in the same direction **(d)** A and B rotate in opposite directions. The dashed line from ACB in cases (c) and (d) shows the path of the signal from A to B (whose length is again $\rho_{AB}(0,0)$) as seen by observer O'. The straight line from A to B for that observer is $\rho'_{AB}(w_A, w_B) = AB$

Look at Figure 1(b), where disc A is rotating and disc B is stationary. According to an observer O_B at B the path of the signal follows the curved path ACB. Suppose now that body B and the frame, where it stands at the origin, starts rotating with angular velocity w_B, and let O_B rotate with B, while a second observer O'_B on top of O_B does not rotate with B. Observer O_B will continue to see the signal follow the same curved path ACB . But O'_B will see the path of the signal change as the result of the addition of the two rotations of space (of A and B). It will be more curved if B rotates in the same direction as A, and less curved if it rotates in the opposite direction. For the more curved case see the dashed path ACB from A to B in Figure 1(c). For the less curved case, when it rotates in the opposite direction of w_A see the dashed path ACB from A to B in Figure 1(d). In fact, $O_B{}'$, will see the angle of deflection φ_A ($=\measuredangle DBE$) at point B increase by $\theta'_B = w'_B t_{AB}$, where $t_{AB} = \frac{\rho'_{AB}(0,0)}{c}$. Note that O'_B does not necessarily have to be located at B. He is a type O' observer as we explained above.

So the final deflection will be $\varphi_A + \theta'_B$. But in fact θ'_B is measured at B, which is zero distance from B, and therefore, there is no effect from the rotation of B. Thus $\theta'_B = \theta_B$ and $w'_B = w_B$. (See Figure 1 (c) $\measuredangle EBF = \theta'_B$ and $\varphi_A + \theta'_B = \measuredangle DBF$, where φ_A is given by $\tan\varphi_A = \frac{w_A \rho'_{AB}(0,0)}{c}$). In the following we may use θ_B instead of θ'_B since they are equal. If B is rotating in the opposite direction, the final deflection angle $\varphi_A + \theta'_B$ will be the result of subtraction (since θ'_B is negative) and will look as in Figure 1(d) where φ_A $=\measuredangle DBE$, $\theta'_B = \measuredangle EBF$ and $\varphi_A + \theta'_B = \measuredangle FBD$. Recall that observer O' lies within the extend of both **G** fields, that of A and that of B, which for the outside far away observer O'' extend up to the cylinder of radius $\frac{c}{w_A}$ around the axis of rotation of body A and the cylinder of radius $\frac{c}{w_B}$ around the axis of rotation of body B.

The straight line distance between A and B according to observer O' is $\rho'_{AB}(w_A, w_B)$. In order to relate $\rho'_{AB}(w_A, w_B)$ to $\rho'_{AB}(0,0)$ we must imagine that O_B has a straight trans parent rod of length $\rho'_{AB}(w_A, 0)$ whose one end he hold and it points in the radial direction from B, through which a signal is send, whenever its free end passes from A. Then O' will see a curve from A to B being traveled by the signal and the radius being contracted to $\rho'_{AB}(w_A, w_B)$ according to (1) at $z = 0$. So in effect we apply (1) at $z = 0$ twice:

$$\frac{w_B \rho'_{AB}(w_A, 0)}{c} = \sinh \frac{w_B \rho'_{AB}(w_A, w_B)}{c} \tag{53}$$

$$\frac{w_A\rho'_{AB}(0,0)}{c}=\sinh\frac{w_A\rho'_{AB}(w_A,0)}{c} \tag{54}$$

From which we arrive at

$$\rho'_{AB}(0,0)=\frac{c}{w_A}\sinh\left[\frac{w_A}{w_B}\sinh\frac{w_B\rho'_{AB}(w_A,w_B)}{c}\right] \tag{55}$$

We may remark here that when $-2\le x\le 2$ then $\sinh x\approx x$ and therefore, when $-2\le\frac{w_B\rho_{AB}(w_A,w_B)}{c}\le 2$ and $-2\le\frac{w_A\rho_{AB}(w_A,w_B)}{c}\le 2$ then $\rho'_{AB}(0,0)\simeq\rho'_{AB}(w_A,w_B)$.

Observe that the subscript AB indicates the direction of the signal from body A to body B. $\rho'_{AB}(w_A,w_B)$ is not symmetric in the direction of the signal. It is also not symmetric in w_A,w_B. But it is symmetric if both A,B and w_A,w_B are interchanged. This means that since observer O' will see the same distance, ($\rho'_{AB}(w_A,w_B)=\rho'_{BA}(w_A,w_B)$) the corresponding distances that the light signals travel will be different ie., $\rho'_{AB}(0,0)\neq\rho'_{BA}(0,0)$. This, in turn, means that a signal traveling from A to B will travel a different distance ($\rho'_{AB}(0,0)$) than the signal traveling from B to A ($\rho'_{BA}(0,0)$) as observer O' sees them.
By symmetrical arguments the signals that arrive to A from B will have an angle of deflection equal to $\varphi_B+\theta'_A$ where $\theta'_A=w'_At_{BA}$, $\tan\varphi_B=\frac{w_B\rho'_{BA}(0,0)}{c}$, and $t_{BA}=\frac{\rho'_{BA}(0,0)}{c}$
By the same arguments as for observers at B, $\theta'_A=\theta_A$, $w'_B=w_B$ and hence we will not use the prime. The magnitude of the force is again given by (52) but the direction is changed. Thus,

$$\left|\mathbf{F}'_{BA}\right|=\left|\mathbf{G}'_{BA}\right|m'_A \tag{56}$$

where $\left|\mathbf{G}'_{BA}\right|$ is given by (16), while the direction is given by the deflection from the straight line $\rho'_{BA}(w_B,w_A)$ by the total deflection $\varphi_B+\theta_A$.

We are ready now to relax the condition that $z=0$ (recall that z is the distance between A and B projected on the z axis).
For observer O' the field created by A is limited by $|z|<\frac{c}{w_A}$
The force $\mathbf{F}'_{AB}$ perceived by body B that is due to the $\mathbf{G}'_{AB}$ field created by body A is given by (56) and $\mathbf{G}'_{AB}$ is given by (11) and combining them,

$$\mathbf{F}'_{AB}=-\frac{k_G m'_A m'_B}{\left(z^2+\frac{\rho'^2_{AB}(0,0)z^2w_A^4U(w_A,\xi_{AB},t_{AB})}{I(w_A,\xi_{AB},t_{AB})(\rho'^2_{AB}(0,0)+z^2)}+\frac{\rho'^2_{AB}(0,0)\cos\varphi_A\sqrt{\rho'^2_{AB}(0,0)+z^2}}{cI(w_A,\xi_{AB},t_{AB})}\right)\frac{c}{\sqrt{c^2+w_A^2\rho'^2_{AB}(0,0)}}}\hat{\mathbf{v}}'_{AB} \tag{57}$$

Where we have denoted $I(\xi_{AB},t_{AB})$ as $I(w_A,\xi_{AB},t_{AB})$ and similarly for $U(w_A,\xi_{AB},t_{AB})$ to remind us that it also depends on w_A. Also in (57) for economy of space we denoted

$\xi_{AB}(0,0) = \xi_{AB}$ and $t_{AB}(0,0) = t_{AB}$. The magnitude of the field $|\mathbf{G}'_{AB}|$ given by (11) remains unaffected by the rotation of B, because it gives the magnitude of the field at point B, where the rotation of B has no effect. It only affects the number of signals per unit volume at positive distance from B (not at zero distance from B). This is why (11) still holds. Only the direction $\hat{\mathbf{v}}'_{AB}$ of the signals of the field as they fall on body B changes because the angle of deflection changes as we will explain below. The curved distance $\rho'_{AB}(0,0)$ is the projection on the plane of rotation or horizontal plane (i.e. the plane that is perpendicular to the axes of rotation of the bodies A, B) of the path that the field signal follows to go from A to B, which is curved and possibly winding around A and around B. The total distance that the signals travel from A to B is $\sqrt{\rho'^2_{AB}(0,0) + z^2}$, while $\xi_{AB}(0,0)$ is the angle of inclination, which the signals from A make with the z axis ($\tan\xi_{AB}(0,0) = \dfrac{\rho'_{AB}(0,0)}{z}$).

Also the angle of deflection of the signals arriving at B will increase from φ_A to $\varphi_A + \theta_B$, where from (8) we know that $\tan\varphi_A = \dfrac{w_A \rho'_{AB}(0,0)}{\sin\xi_{AB}(0,0)\sqrt{c^2 - w_A^2 z^2}}$, while

$$\theta'_B = w'_B t_{AB}(0,0) = \theta_B = w_B t_{AB} \tag{58}$$

and

$$t_{AB}(0,0) = \frac{\sqrt{\rho'^2_{AB}(0,0) + z^2}}{c} = \frac{z}{c\cos\xi_{AB}(0,0)} \tag{59}$$

The direction of the signals of the field is represented by the unit vector $\hat{\mathbf{v}}'_{AB}$, which (when projected on the horizontal plane) gives the total angle of deflection $\varphi_A + \theta_B$ of the signals from the projection of the straight line between A and B on the horizontal plane. Specifically,

$$\hat{\mathbf{v}}'_{AB} = (\sin\xi_{AB}(0,0)\cos(\varphi_A + \theta_B), \sin\xi_{AB}(0,0)\sin(\varphi_A + \theta_B), \cos\xi_{AB}(0,0)) \tag{60}$$

for the cylindrical components (ρ, θ, z).

Further, to determine $\rho'_{AB}(0,0)$ from the observed $\rho'_{AB}(w_A, w_B)$, we apply the same reasoning as for the two dimensional case ($z = 0$) that we used above and apply the transformation (1) twice,

$$\rho'_{AB}(w_A, 0) = c\sin\xi_{AB}(0,0) I(w_A, \xi_{AB}(0,0), t_{AB}(0,0)) \tag{61}$$

$$\rho'_{AB}(w_A, w_B) = c\sin\xi_{AB}(w_A, 0) I(w_B, \xi_{AB}(w_A, 0), t_{AB}(0,0)) \tag{62}$$

$$\tan\xi_{AB}(0,0) = \frac{\rho'_{AB}(0,0)}{z} \tag{63}$$

$$\tan\xi_{AB}(w_A, 0) = \frac{\rho'_{AB}(w_A, 0)}{z} \tag{64}$$

$$t_{AB}(0,0) = \frac{\sqrt{\rho'^2_{AB}(0,0) + z^2}}{c} \tag{65}$$

Substituting,(65) and (64) in (62) we have an equation relating $\rho'_{AB}(w_A, w_B)$ to $\rho'_{AB}(w_A, 0)$. While substituting (65) and (63) in (61) we have an equation relating

$\rho'_{AB}(w_A,0)$ to $\rho'_{AB}(0,0)$. Thus from $\rho'_{AB}(w_A,w_B)$ we may find $\rho'_{AB}(w_A,0)$ and then $\rho'_{AB}(0,0)$.

Finally, from (1), (5) and (9)

$$\frac{w'_B\rho'_{AB}(w_A,w_B)}{w_B\rho'_{AB}(w_A,0)}=\frac{c}{\sqrt{c^2+w_B^2\rho'_{AB}(w_A,0)^2)}} \tag{66}$$

From which w'_B may be determined and then θ'_B found using (58). But in our case the latter is not needed since we measure θ'_B at zero distance from B and hence as we said $\theta'_B=\theta_B$ and $w'_B=w_B$.

We may observe that the force in (57) is asymmetric in A and B both in direction and magnitude.

3.2 The $\mathbf{F}''$ force for rotation without slippage (observer O'' case A.II)

Observer O'' lies outside both $\mathbf{G}''$ cylinders of the bodies A, B. That is he is beyond the cylinder of radius $\frac{c}{w_A}$ with axis of rotation that of body A and cylinder with radius $\frac{c}{w_B}$ with axis of rotation that of body B. But bodies A and B lie within each other's G-cylinder. The angle of deflection that is due only to the rotation of A is by (24) given by $\tan\varphi''_A=\underbrace{\frac{w_A\sqrt{\rho''^2_{AB}(0,0)+z^2}}{c}}_{w_At_{AB}}\left(1+\frac{w_A^2\rho''_{AB}(0,0)^2\sin^2\xi_{AB}(0,0)}{c^2}\right)$. This angle of deflection will be increased because of the rotation of B by $\theta''_B=w_Bt_{AB}=\theta_B$ where $t_{AB}=\frac{\sqrt{\rho''^2_{AB}(0,0)+z^2}}{c}$ and the total deflection will be $\varphi''_A+\theta''_B$. Observe that $\theta''_B=\theta_B$ and that we may, therefore, omit the double prime for θ'' below. The angle of inclination of the signal velocity vector from the z axis for observer O is $\tan\xi_{AB}(0,0)=\tan\xi''_{AB}(0,0)=\frac{\rho''_{AB}(0,0)}{z}$ where z is the distance between A and B projected on the z axis.

The radial distance ρ''_{AB} is given by (17) and applying it twice we have,

$$\rho''_{AB}(w_A,w_B)=\rho''_{AB}(w_A,0)\frac{c}{\sqrt{c^2+w_B^2\rho''^2_{AB}(w_A,0)}} \tag{67}$$

$$\rho''_{AB}(w_A,0)=\rho''_{AB}(0,0)\frac{c}{\sqrt{c^2+w_A^2\rho''^2_{AB}(0,0)}} \tag{68}$$

Substitute (68) in (67) to find

$$\rho''_{AB}(w_A,w_B)=\frac{c\rho''_{AB}(0,0)}{\sqrt{c^2+(w_A^2+w_B^2)\rho''^2_{AB}(0,0)}} \tag{69}$$

which is symmetric in A, B. From the observed radial distance between A, B, which is represented by $\rho''_{AB}(w_A,w_B)$ one can use (69) to find the radial component (the projection on the horizontal plane) of the length of the path of the signal from A to B which is given by $\rho''_{AB}(0,0)$. In fact we may solve to find,

$$\rho''_{AB}(0,0)=\frac{c\rho''_{AB}(w_A,w_B)}{\sqrt{c^2-(w_A^2+w_B^2)\rho''^2_{AB}(w_A,w_B)}} \tag{70}$$

This value may be used to determine t_{AB} , $\xi''_{AB}(0,0)$ and $\tan\varphi''_A$ which is given by (24)

$$\tan\varphi''_A = w_A t_{AB}(1+w_A^2 t_{AB}^2 \sin^2\xi''_{AB}(0,0)) \tag{71}$$

While from (25)

$$\tan\xi''_{AB}(w_A,0)=\tan\xi''_{AB}(0,0)\frac{\sqrt{1+w_A^2t_{AB}^2(1+w_A^2t_{AB}^2\sin\xi''_{AB}(0,0))^2}}{(1+w_A^2t_{AB}^2\sin^2\xi''_{AB}(0,0))^{\frac{3}{2}}} \tag{72}$$

Now the force exerted on B is given by,

$$\mathbf{F}''_{AB}=\mathbf{G}''_{AB}m'_B \tag{73}$$

And substituting for $\mathbf{G}''_{AB}$ from (26) we obtain,

$$\mathbf{F}''_{AB}=-\frac{k_G m'_A m'_B\left(c^2+w_A^2\rho''^2_{AB}(0,0)\right)^{\frac{3}{2}}}{c^3(\rho''^2_{AB}(0,0)+z^2)}\hat{\mathbf{v}}''_{AB} \tag{74}$$

This can also be expressed in terms of $\rho''_{AB}(w_A,w_B)$ using (70). The unit vector of the direction of the signals of the field is given by

$$\hat{\mathbf{v}}''_{AB}=(\sin\xi''_{AB}(w_A,w_B)\cos(\varphi''_A+\theta_B),\sin\xi''_{AB}(w_A,w_B)\sin(\varphi''_A+\theta_B),\cos\xi''_{AB}(w_A,w_B)) \tag{75}$$

The unprimed quantities refer to observer O. As we explained in the previous section the ratio w/c , remains invariant under the presence of a gravitational field and the same is true for the quantities *wt,* and *ct*. Thus, for the far away observer the angles φ''_A, θ''_B , ξ''_{AB} and the distance ρ''_{AB} , will not depend on whether he or the other stationary observers at A or B or some other location, are under the influence of a gravity field and in particular of the **G** field created by the other rotating body.

Still we need to show how $\tan\xi''_{AB}(w_A,w_B)$ is determined so that the unit vector in (75) is well defined. Imagine a light signal starting from B is send through a transparent rod that has an angle $\xi''_{BA}(w_A,w_B)$ to the z axis and rotates with B having its one end fixed at B. After describing a curved path, the signal arrives at A when both B and A are rotating. We need to satisfy the following :

From (69)

$$\rho''_{AB}(w_A,w_B)=\frac{c\rho''_{AB}(0,0)}{\sqrt{c^2+(w_A^2+w_B^2)\rho''^2_{AB}(0,0)}}=\frac{ct_{AB}\sin\xi''(0,0)}{\sqrt{1+(w_A^2+w_B^2)t_{AB}^2\sin^2\xi''(0,0)}} \tag{76}$$

Where $\rho''_{AB}(w_A,w_B)=\rho''_{BA}(w_A,w_B)$, $\rho''_{AB}(0,0)=\rho''_{BA}(0,0)$, $t_{AB}=t_{BA}=\dfrac{\sqrt{\rho''_{AB}(0,0)^2+z^2}}{c}$

Letting $\upsilon_{C,BA}$ be the speed of the signal on the curved path from B to A we must have the speed in the z direction be unaffected by rotations :

$$\upsilon_{C,BA}\cos\xi''_{BA}(w_A,w_B)=c\cos\xi''_{BA}(0,0) \tag{77}$$

Where $\cos\xi''_{BA}(0,0)=\dfrac{z}{\sqrt{\rho''_{AB}(0,0)^2+z^2}}=\cos\xi''_{AB}(0,0)$

The tangential speed of the signal must equal $w_B\rho''_{BA}(w_A,w_B)$:

$$\upsilon_{C,BA}\sin\xi''_{BA}(w_A,w_B)\sin\varphi''_{BA}=w_B\rho''_{BA}(w_A,w_B) \tag{78}$$

The radial speed of the signal must equal to the rate of increase of the radial distance:

$$\frac{d\rho''_{BA}(w_A,w_B)}{dt} = \upsilon_{C,BA}\sin\xi''_{BA}(w_A,w_B)\cos\varphi''_{BA} \tag{79}$$

Where φ''_{BA} is the angle of deflection of the signal from the radial from B to A projected on the plain at $z=0$.

Solving the above we start with (76) and we find

$$\frac{d\rho''_{BA}(w_A,w_B)}{dt} = \frac{c\sin\xi_{BA}(0,0)}{(1+(w_A^2+w_B^2)t_{BA}^2\sin\xi_{BA}^2(0,0))^{\frac{3}{2}}} \tag{80}$$

Using this and dividing (78) by (79) and applying (76), we find

$$\tan\varphi''_{BA} = \frac{w_B\rho''_{BA}(w_A,w_B)}{\frac{d\rho''_{BA}(w_A,w_B)}{dt}} = w_B t_{BA}(1+(w_A^2+w_B^2)t_{BA}^2\sin\xi''^2_{BA}(0,0)) \tag{81}$$

Dividing (78) by (77) we find

$$\tan\xi''_{BA}(w_A,w_B) = \frac{w_B\rho''_{BA}(w_A,w_B)}{c\cos\xi''_{BA}(0,0)\sin\varphi''_{BA}} \tag{82}$$

And using (76) and (81) we obtain,

$$\tan\xi''_{BA}(w_A,w_B) = \frac{w_B t_{BA}\sin\xi''_{BA}(0,0)}{c\cos\xi''_{BA}(0,0)\sqrt{1+(w_A^2+w_B^2)t_{BA}^2\sin^2\xi''_{BA}(0,0)}}\sqrt{1+\cot^2\varphi''_{BA}} \tag{83}$$

Or

$$\tan\xi''_{BA}(w_A,w_B) = \tan\xi''_{BA}(0,0)\frac{\sqrt{1+w_B^2t_{BA}^2(1+(w_A^2+w_B^2)t_{BA}^2\sin\xi''_{BA}(0,0))^2}}{(1+(w_A^2+w_B^2)t_{BA}^2\sin^2\xi''_{BA}(0,0))^{\frac{3}{2}}} \tag{84}$$

Finally from (77)

$$\upsilon_{C,BA} = \frac{c\cos\xi''_{BA}(0,0)}{\cos\xi''_{BA}(w_A,w_B)} = c\cos\xi''_{BA}(0,0)\sqrt{1+\tan\xi''^2_{BA}(0,0)\frac{(1+w_B^2t_{BA}^2(1+(w_A^2+w_B^2)t_{BA}^2\sin\xi''_{BA}(0,0))^2)}{(1+(w_A^2+w_B^2)t_{BA}^2\sin^2\xi''_{BA}(0,0))^3}}$$

(85)

From (84) we can find for a signal that travels the opposite way, from A to B:

$$\tan\xi''_{AB}(w_A,w_B) = \tan\xi''_{AB}(0,0)\frac{\sqrt{1+w_A^2t_{AB}^2(1+(w_A^2+w_B^2)t_{AB}^2\sin\xi''_{AB}(0,0))^2}}{(1+(w_A^2+w_B^2)t_{AB}^2\sin^2\xi''_{AB}(0,0))^{\frac{3}{2}}} \tag{86}$$

by changing the subscript BA to AB since $\xi''_{AB}(0,0)=\xi''_{BA}(0,0)$, $t_{AB}=t_{BA}$. Thus, $\xi''_{AB}(w_A,w_B)$ is found and the unit vector in (75) is determined as required.

3.3 The $\mathbf{F}'$ force for rotation with slippage (observer O', case B.I)

In this case we should follow the same steps as for case A.I and after we determine, $\rho'_{AB}(0,0)$, we calculate $\theta_B = \int_0^{t_{AB}} w_{B0}e^{-ct\beta_{AB}}dt = \frac{w_{B0}}{c\beta_{AB}}(1-e^{-ct_{AB}\beta_{AB}})$, where

$t_{AB} = \frac{\sqrt{\rho_{AB}(0,0)^2+z^2}}{c}$, $\beta_{AB} = \lambda\sin\xi_{AB}(0,0)+\mu\cos\xi_{AB}(0,0)$

However, calculations are more difficult because of the exponential in the angular velocity.

3.4 The $\mathbf{F}''$ force for rotation with slippage (observer O'', case B.II)

From (41), and setting $w_A = w_{A0} e^{-ct_{AB}\beta_{AB}}$, $w_B = w_{B0} e^{-ct_{AB}\beta_{AB}}$, where

$\beta_{AB} = \lambda \sin \xi''_{AB}(0,0) + \mu \cos \xi''_{AB}(0,0)$ and $t''_{AB} = t_{AB} = \dfrac{\sqrt{\rho''_{AB}(0,0)^2 + z^2}}{c}$ we have,

$$\rho''_{AB}(w_A,0) = \rho''_{AB}(0,0) \frac{c}{\sqrt{c^2 + w_{A0}^2 \rho''^2_{AB}(0,0) e^{-2(\lambda\rho''_{AB}(0,0)+\mu z)}}} \tag{87}$$

$$\rho''_{AB}(w_A,w_B) = \rho''_{AB}(w_A,0) \frac{c}{\sqrt{c^2 + w_{B0}^2 \rho''^2_{AB}(w_A,0) e^{-2(\lambda\rho''_{AB}(0,0)+\mu z)}}} \tag{88}$$

From which $\rho''_{AB}(0,0)$ is determined. Observe that $\rho''_{AB}(w_A,w_B) = \rho''_{BA}(w_A,w_B)$. The force is given by,

$$\mathbf{F}''_{AB} = \mathbf{G}''_{AB} m'_B = -\frac{k_G m'_A m'_B}{(\rho''_{AB}(0,0)^2 + z^2)} \frac{(c^2 + w_{A0}^2 \rho''_{AB}(0,0)^2 e^{-2(\lambda\rho''_{AB}(0,0)+\mu z)})^{\frac{3}{2}}}{c(c^2 + \lambda w_{A0}^2 \rho''_{AB}(0,0)^3 e^{-2(\lambda\rho''_{AB}(0,0)+\mu z)})} \hat{\mathbf{v}}''_{AB} \tag{89}$$

where the unit vector is given by,

$$\hat{\mathbf{v}}''_{AB} = (\sin \xi''_{AB}(w_A,w_B)\cos(\varphi''_A + \theta''_B), \sin \xi''_{AB}(w_A,w_B)\sin(\varphi''_A + \theta''_B), \cos \xi''_{AB}(w_A,w_B)) \tag{90}$$

Using simpler notation $t''_{AB} = t_{AB} = \dfrac{\sqrt{\rho''_{AB}(0,0)^2 + z^2}}{c}$ and

$\tan \xi''_{AB}(0,0) = \tan \xi_{AB}(0,0) = \tan \xi_{AB} = \dfrac{\rho''_{AB}(0,0)}{z}$ we find $\tan \varphi''_A$ from (48).

$$\tan \varphi''_A = \frac{w_A t_{AB}(1 + w_A^2 t_{AB}^2 \sin^2 \xi)}{1 + c\beta_{AB} t_{AB}^3 w_A^2 \sin^2 \xi} = \frac{w_A \sqrt{\rho''_{AB}(0,0)^2 + z^2}}{c} \frac{1 + \dfrac{w_A^2 \rho''_{AB}(0,0)^2}{c^2}}{1 + \dfrac{w_A^2 \rho''_{AB}(0,0)^2 (\lambda \rho''_{AB}(0,0) + \mu z)}{c^2}} \tag{91}$$

Then we calculate $\theta''_B = \int_0^{t''_{AB}} w_{B0} e^{-ct\beta_{AB}} dt = \theta_B = \int_0^{t_{AB}} w_{B0} e^{-ct\beta_{AB}} dt = \dfrac{w_{B0}}{c\beta_{AB}}(1 - e^{-ct_{AB}\beta_{AB}})$

From (49)

$$\tan \xi''_{AB}(w_A,0) = \tan \xi_{AB}(0,0) \frac{\sqrt{(1 + w_A^2 t_{AB}^2 \sin^2 \xi_{AB}(0,0))^2 + (1 + c\beta_{AB} t_{AB}^3 w_A^2 \sin^2 \xi_{AB}(0,0))^2}}{(1 + w_A^2 t_{AB}^2 \sin^2 \xi_{AB}(0,0))^{\frac{3}{2}}} \tag{92}$$

To determine $\tan \xi''_{AB}(w_A,w_B)$, which is needed to determine the unit vector in (89) we follow a similar approach to that for the case A.II above,
The only difference is in the calculation of the derivative of the radial distance,

$$\frac{d\rho''_{BA}(w_A,w_B)}{dt} = \frac{c \sin \xi_{BA}(0,0)(1 + c\beta_{BA} t_{BA}^3 (w_A^2 + w_B^2)\sin \xi_{BA}^2(0,0))}{(1 + (w_A^2 + w_B^2) t_{BA}^2 \sin \xi_{BA}^2(0,0))^{\frac{3}{2}}} \tag{93}$$

Where we observe that $\beta_{AB} = \beta_{BA}$, $t_{AB} = t_{BA}$,
$\sin \xi_{BA}(0,0) = \sin \xi_{AB}(0,0) = \sin \xi''_{AB}(0,0) = \sin \xi''_{BA}(0,0)$.

After some calculation as in A.II we find for a signal traveling from B to A, while both A and B are rotating,

$$\tan\varphi''_{BA} = \frac{w_B \rho''_{BA}(w_A, w_B)}{\frac{d\rho''_{BA}(w_A, w_B)}{dt}} = \frac{w_B t_{BA}(1+(w_A^2+w_B^2)t_{BA}^2 \sin\xi''^2_{BA}(0,0))}{1+c\beta_{BA}t_{BA}^3(w_A^2+w_B^2)\sin\xi''^2_{BA}(0,0)} \tag{94}$$

$$\tan\xi''_{BA}(w_A, w_B) = \tan\xi''_{BA}(0,0)\frac{w_B t_{BA}}{\sqrt{1+(w_A^2+w_B^2)t_{BA}^2 \sin\xi''^2_{BA}(0,0)}}\sqrt{1+\frac{(1+c\beta_{BA}t_{BA}^3(w_A^2+w_B^2)\sin^2\xi''_{BA}(0,0))^2}{w_B^2 t_{BA}^2(1+(w_A^2+w_B^2)t_{BA}^2 \sin\xi''^2_{BA}(0,0))^2}} \tag{95}$$

Also written as

$$\tan\xi''_{BA}(w_A, w_B) = \tan\xi''_{BA}(0,0)\frac{\sqrt{w_B^2 t_{BA}^2(1+(w_A^2+w_B^2)t_{BA}^2 \sin\xi''^2_{BA}(0,0))^2+(1+c\beta_{BA}t_{BA}^3(w_A^2+w_B^2)\sin^2\xi''_{BA}(0,0))^2}}{(1+(w_A^2+w_B^2)t_{BA}^2\sin\xi''^2_{BA}(0,0))^{\frac{3}{2}}} \tag{96}$$

$$\upsilon_{C,BA} = \frac{c\cos\xi''_{BA}(0,0)}{\cos\xi''_{BA}(w_A, w_B)} \tag{97}$$

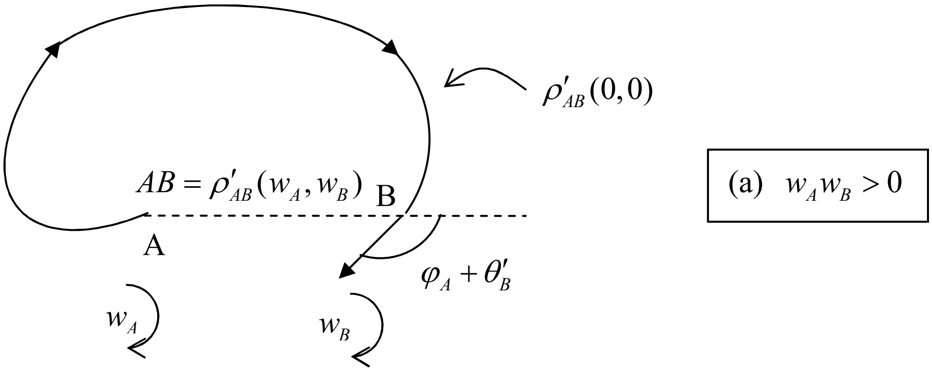
It follows that for a signal traveling from A to B when both A and B are rotating the angle of inclination to the z axis is,

$$\tan\xi''_{AB}(w_A, w_B) = \tan\xi''_{AB}(0,0)\frac{\sqrt{w_A^2 t_{AB}^2(1+(w_A^2+w_B^2)\sin\xi''^2_{AB}(0,0))^2+(1+c\beta_{AB}t_{AB}^3(w_A^2+w_B^2)\sin^2\xi''_{AB}(0,0))^2}}{(1+(w_A^2+w_B^2)t_{AB}^2\sin\xi''^2_{AB}(0,0))^{\frac{3}{2}}} \tag{98}$$

And we substitute in (90) to determine the direction of $\hat{\mathbf{v}}''_{AB}$, which points opposite to the direction of the force.

4 Visualization of the signals' path for the $\mathbf{G}'$ and $\mathbf{G}''$ field and the attractive-repulsive effect

In Figure 1 (d) and 1 (c) the dashed path from A to B shows how the signal travels from A to B for the case of rotations in the same direction and in the opposite direction. Let us expand on that in Figure 2.



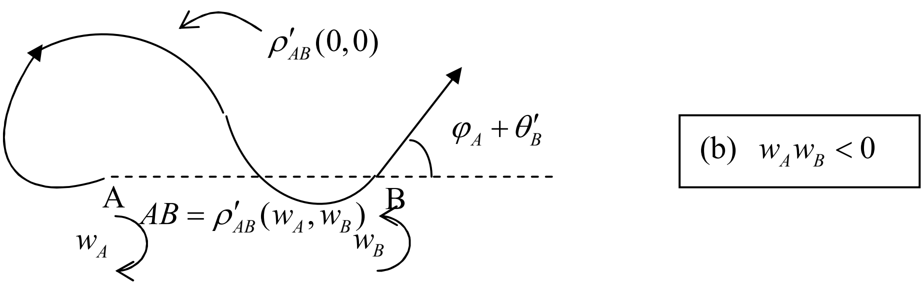
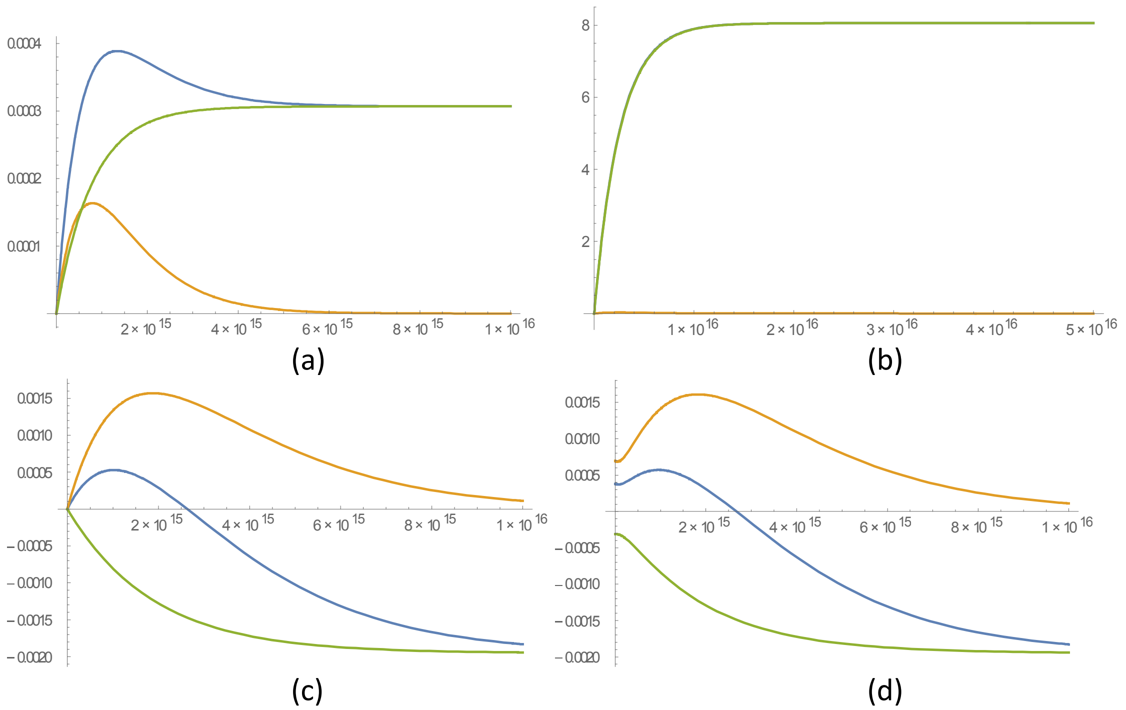


Figure 2 Assuming that bodies A,B are both within the reach of each other's G field**,** the signals' path from body A to body B is shown by the curved (or possibly winding) path with length . The straight line distance observed by observer O', is $\rho'_{AB}(w_A, w_B)$ or $\rho''_{AB}(w_A, w_B)$ for observer O''. (**a**) When they are rotating in the same direction the angle of deflection φ_A increases to $\varphi_A + \theta'_B$ or ($\varphi_A + \theta''_B$) producing a repulsive field. (**b**) When they are rotating in the opposite direction the angle of deflection φ_A decreases to $\varphi_A + \theta'_B$ or ($\varphi_A + \theta''_B$) (since $\theta'_B < 0$) producing an attractive field for B. Recall that $\theta_B = \theta'_B = \theta''_B$

For rotation with or without slippage, the signals of a **G** field produced by a rotating body A will fall on a body B with angle of deflection $\varphi_A \leq \frac{\pi}{2}$. If body B is rotating in the same direction as A (Figure 2(a)), the angle of deflection will increase and for appropriate range of values of w_B the resulting angle of deflection $\varphi_A + \theta_B$ will initially result in a repulsive field experienced by B. If the angular velocity w_B of rotation continues to increase θ_B will increase resulting in winding around B. This will give ranges of w_B for which we will have alternating repulsive and attractive forces experienced by B. Similar observations hold for case of Figure 2(b) where $w_B < 0$ has opposite sign to w_A. As we increase $|w_B|$, the angle of deflection decreases to $\varphi_A + \theta_B$ (because $\theta_B < 0$), resulting initially in an attractive force as experienced by B. But as

$|w_B|$ continues to increase we will have winding of θ_B, resulting in ranges of w_B for which body B will feel alternating a repulsive and attractive force.
For values of w_B that do not cause winding of θ_B, we may talk simply about attractive and repulsive force experienced by B depending on whether the rotation is in the same or opposite direction.
Similar remarks hold for $\mathbf{G}''$, $\rho''_{AB}(w_A, w_B)$, θ_B, φ''_A and for rotation with slippage.
Figure 3 presents in blue the plot of the total angle of deflection for rotation with slippage, $\varphi''_{ATot} \triangleq \varphi''_A + \theta_B$ in radians versus $\rho = \rho''_{AB}(0,0)$ for various cases both for macrocosmos and microcosmos. (Recall from [3] that macrocosmos is when $\dfrac{w_{A0}\rho''_{AB}(0,0)}{c} << e^{\lambda\rho''_{AB}(0,0)+\mu z}$, and microcosmos when $\dfrac{w_{A0}\rho''_{AB}(0,0)}{c} >> e^{\lambda\rho''_{AB}(0,0)+\mu z}$). In orange we see angle $\varphi''_A = \arctan\left(\dfrac{w_A\sqrt{\rho''_{AB}(0,0)^2 + z^2}}{c} \dfrac{1+\dfrac{w_A^2\rho''_{AB}(0,0)^2}{c^2}}{1+\dfrac{w_A^2\rho''_{AB}(0,0)^2(\lambda\rho''_{AB}(0,0)+\mu z)}{c^2}}\right)$

and in green is $\theta_B = \dfrac{w_{B0}}{c\beta_{AB}}(1-e^{-ct_{AB}\beta_{AB}})$. Figures 3 (a), (b), (c), (d), (e) are examples of macrocosmos, while (f), (g), (h) are from microcosmos. It is possible to have examples, where the total angle of deflection is attractive for all distance or alternating attractive – repulsive depending on the distance from A (the origin).



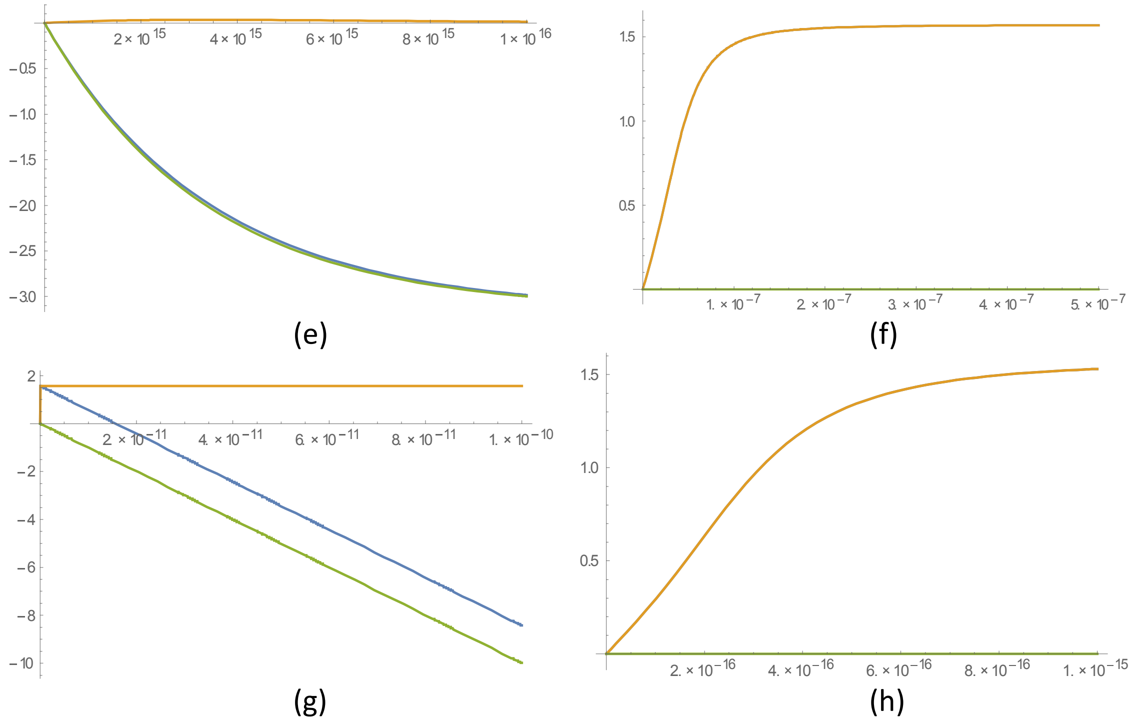


Figure 3 The graph of the total angle of deflection $\varphi''_{ATot} = \varphi''_A + \theta_B$ (blue) vs ρ. Also the graph of φ''_A in orange and θ_B in green. (a), (b), (c), (d), (e) are macrocosmos cases while (f), (g), (h) are microcosmos examples. In particular, (**a**) $w_{A0} \approx 2.16*10^{-10}$ rad/s, $w_{B0} \approx 1.64*10^{-10}$ rad/s, $\lambda \approx 1.68*10^{-15}$ m^{-1}, $z \approx 1.27*10^{14}$ m, , the total angle of deflection remains positive for all ρ but does not exceed $\pi/2$. (**b**) $w_{A0} \approx 9.26*10^{-9}$ rad/s, $w_{B0} \approx 9.4*10^{-7}$ rad/s, $\lambda \approx 3.89*10^{-16}$ m^{-1}, $z \approx 0$ m. The total deflection angle spans values from 0 to 8 in radians. Therefore, for ($0, \pi/2$) it is attractive, for $(\pi/2, 3\pi/2)$ it is repulsive, fot $(3\pi/2, 5\pi/2)$ it is attractive and for $(5\pi/2, 8)$ it is repulsive. (**c**) $w_{A0} \approx 6.82*10^{-10}$ rad/s, $w_{B0} \approx -3.12*10^{-10}$ rad/s, $\lambda \approx 5.27*10^{-16}$ m^{-1}, $z \approx 0$ m.
Body B rotates with negative sign. The total angle of deflection takes both positive and negative values depending on the distance but remains within $(\pi/2, -\pi/2)$ and hence it is always attractive. (**d**) This is the same as (c) only $z = 3.01*10^{14}$ m. (**e**) $w_{A0} \approx 8.52*10^{-9}$ rad/s, $w_{B0} \approx -2.84*10^{-7}$ rad/s, $\lambda \approx 3*10^{-16}$ m^{-1}, $z \approx 0$ m. The total angle of deflection spans $(0, -\pi/2)$ where it is attractive and $(-\pi/2, -3)$ where it is repulsive. (**f**) $w_{A0} \approx 5.7*10^{15}$ rad/s, $w_{B0} \approx -1*10^{15}$ rad/s, $\lambda \approx 1*10^{-10}$ m^{-1}, $z \approx 0$ m. The total angle of deflection spans $(0, \pi/2)$ where it is attractive, (**g**) $w_{A0} \approx 3*10^{23}$ rad/s, $w_{B0} \approx -3*10^{19}$ rad/s, $\lambda \approx 1*10^{-4}$ m^{-1}, $z \approx 0$ m. The total angle of deflection spans (-2,8) as the distance varies from 0 to 10^{-10}. The space where the total angle of deflection takes values is segmented to $(2, \pi/2)$ where it is repulsive, $(\pi/2, -\pi/2)$ where it is attractive, $(-\pi/2, -3\pi/2)$ where it is repulsive, $(-3\pi/2, -5\pi/2)$ where it is attractive, $(-5\pi/2, 8)$ where it is repulsive. (**h**) $w_{A0} \approx 8.4*10^{23}$ rad/s, $w_{B0} \approx -3*10^{24}$ rad/s, $\lambda \approx 5*10^{-9}$ m^{-1}, $z \approx 0$ m. For distances from 0 to 10^{-15} m the total

angle of deflection is virtually equal to the angle of deflection that is due to the rotation of body A only and varies from 0 to $\pi/2$ (attractive).

5 Interaction of two spinning bodies with axes not parallel

Until now we have assumed that the axes of rotation of the two bodies were parallel. This was done for simplicity and in order to understand the problem better by approaching it stepwise. Now we will look at the general situation, when the axes of rotation of the two bodies are not parallel. Figure 4 (a) and (b) shows the setup.

We imagine a body A rotating around axis Z_1. The plane perpendicular to Z_1 at A is called the plane of rotation and is denoted as PL1. Similarly a body B rotates around an axis Z_2 and the plane of rotation is PL2. PL1and PL2 intersect at XX' with angle ϕ. A signal from A to B travels a curved path. The tangent to this curve at B is extended tangentially to some point F. The straight line from A to B is extended to E. Let a plane PL1' parallel to PL1 pass through point B (PL1' is not shown in Figure 4(a)). The projection of E on PL1' is E_1 (not shown) and the projection on PL2 is E_2.The projection of F on PL1' is F_1 (not shown) and on PL2 is F_2. Angle $\measuredangle E_1BF_1 = \varphi_1$ and angle $\measuredangle F_2BE_2 = \varphi_2$. Also angle $\measuredangle Z_1'BF = \xi_{F1}$ (not shown, where Z_1' is a line parallel to axis Z_1 passing through B) and $\measuredangle Z_2BF = \xi_{F2}$. Further, we observe that starting from A with cylindrical coordinates, point B is at height z_1 and radial distance ρ_1, while starting from B, point A is at height z_2 and radial distance ρ_2. The plane that contains Z_2 and passes through A, contains also E (since A, B lie on this plane and E lies on the extension of AB). This plane crosses XX' at Q. AL is drawn perpendicular to XX'. The angle $\measuredangle LAQ = u_1$ is the azimuth angle of B with respect to the cylindrical coordinate system that has origin at A and axis Z_1. The azimuth angle is measured looking down from Z_1 counterclockwise starting from AL (the perpendicular from A to XX'). The angle $\measuredangle AQL = \gamma_{E1} = 90° - u_1$

Now our strategy is as follows. We let body B not rotate for the moment and we transform $\varphi_{1,}\xi_{F1,}\rho_{1,}z_1, u_1$,to $\varphi_{2,}\xi_{F2,}\rho_{2,}z_2, u_2$. After that we will let B rotate and see how $\varphi_2, \xi_{F2,}\rho_{2,}z_2, u_2$ are changed by the rotation.

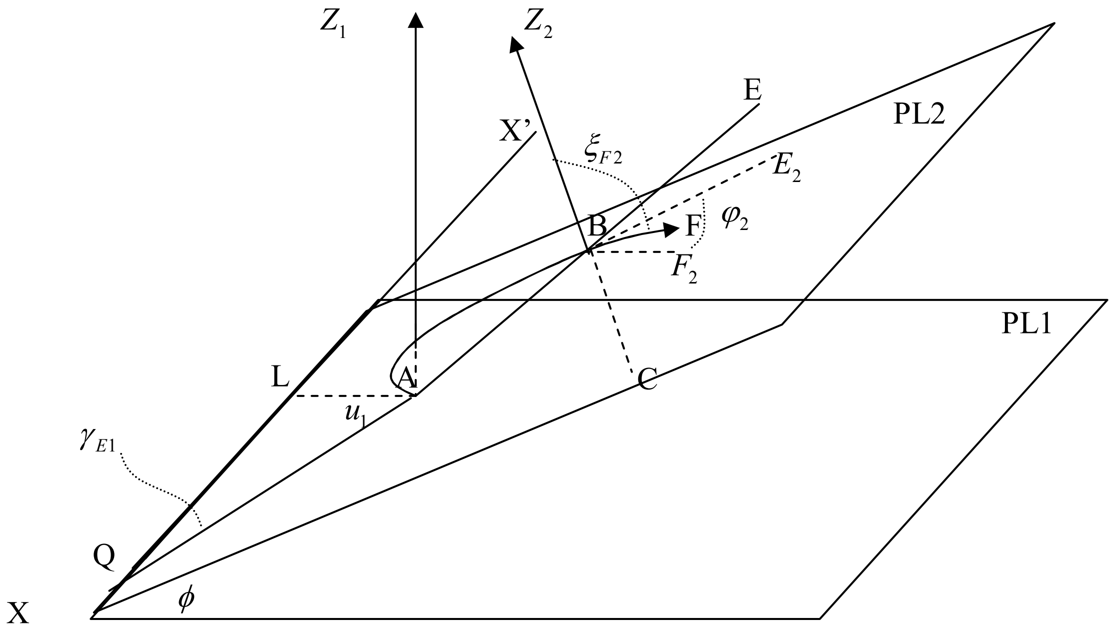


Figure 4(a) Body A rotates with axis of rotation Z_1 and plane of rotation PL1. Body B has axis of rotation Z_2 and plane of rotation PL2. The signal that travels from A to B is shown by the curve that passes through A and B. Taking the tangent at B we extend it to F. AB is the straight line from A to B which is extended to E. Angle $\measuredangle Z_2BF = \xi_{F2}$. The projections of E , F on PL2 are E_2, F_2, respectively. Angle $\measuredangle E_2BF_2 = \varphi_2$, the angle of deflection on PL2. Angle $\measuredangle LAQ = u_1$ is the azimuth angle of Z_2 with respect to cylindrical coordinate system Z_1 with origin at A. The angle between PL1 and PL2 is ϕ.

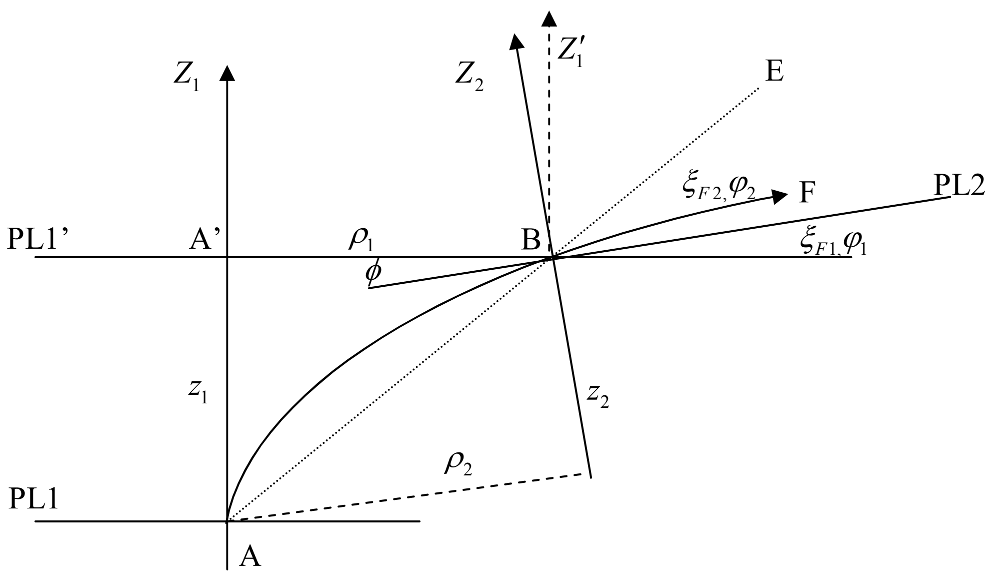


Figure 4(b) This a schematic side view of a section of Figure 4(a). The tangent to the signal path at B and the straight line from ABE form an angle in space with vertex at B. The projection of this angle **(** $\measuredangle EBF$ **)** on PL1 forms the angle φ_1 while the projection on PL2 forms the angle φ_2 **.** Also the tangent to the curve (A,B) at B forms an angle ξ_{F1} with Z_1' **(** $\measuredangle Z_1'BF$ **),** while the same forms an angle ξ_{F2} with PL2 **(** $\measuredangle Z_2BF$ **).** Further, starting from A with cylindrical coordinates ,B is at height z_1 and radial distance ρ_1 and azimuth angle u_1**.** While starting from B with cylindrical coordinates the point A is at a height z_2 and radial distance ρ_2 and azimuth angle u_2**.** PL1' is a plane parallel to PL1 that passes through B .

We will use two Lemmas from Geometria proven in Appendix A to show how $\varphi_2, \xi_{F2,} \rho_{2,} z_2$ are related to $\varphi_1, \xi_{F1}, \rho_1, z_1$. Draw plane PL1'parallel to PL1 passing through body B. The planes PL2 and PL1' refer us to Lemma 1 of Appendix A. We know that $\measuredangle E_1BX = \gamma_{E1}$ and we call $\measuredangle E_2BX = \gamma_{E2}$, $\measuredangle F_1BX = \gamma_{F1}$ and $\measuredangle F_2BX = \gamma_{F2}$. We also know that

$$\gamma_{E1} - \gamma_{F1} = \varphi_1 \tag{99}$$

$$\gamma_{E2} - \gamma_{F2} = \varphi_2 \tag{100}$$

$$\gamma_{E1} = \frac{\pi}{2} - u_1 \tag{101}$$

And from Lemma 1, noting that $\xi_1 = \frac{\pi}{2} - x_1$ and $\xi_2 = \frac{\pi}{2} - x_2$

$$\tan \gamma_{E2} = (\sin \phi \frac{\cot \xi_{E1}}{\sin \gamma_{E1}} + \cos \phi) \tan \gamma_{E1} \tag{102}$$

$$\tan\gamma_{F2} = (\sin\phi\frac{\cot\xi_{F1}}{\sin\gamma_{F1}} + \cos\phi)\tan\gamma_{F1} \quad (103)$$

$$\tan\xi_{E1} = \frac{\rho_1}{z_1} \quad (104)$$

$$\cos\xi_{E2} = \cos\phi\cos\xi_{E1} - \sin\xi_{E1}\sin\gamma_{E1}\sin\phi = \cos\phi\frac{\rho_1}{\sqrt{\rho_1^2 + z_1^2}} - \sin\phi\frac{z_1}{\sqrt{\rho_1^2 + z_1^2}}\cos u_1 \quad (105)$$

$$\cos\xi_{F2} = \cos\phi\cos\xi_{F1} - \sin\xi_{F1}\sin\gamma_{F1}\sin\phi = \cos\phi\cos\xi_{F1} - \sin\xi_{F1}\cos(u_1 + \varphi_1)\sin\phi \quad (106)$$

Using (102) , (101), (104) we find $\tan\gamma_{E2}$
We use (103) and (99) to find $\tan\gamma_{F2}$. From these we may find φ_2 since $\gamma_{E2} - \gamma_{F2} = \varphi_2$
Now we use Lemma 2 of Appendix A to find

$$z_2 = -z_1\cos\phi - \rho_1\cos u_1\sin\phi \quad (107)$$

$$\rho_2^2 = \rho_1^2(1 - \cos^2 u_1\sin^2\phi) + z_1^2\sin^2\phi - 2z_1\rho_1\sin\phi\cos\phi\cos u_1 \quad (108)$$

$$\sin u_2 = -\frac{\rho_1}{\rho_2}\sin u_1 \quad (109)$$

Finally, we allow body B to rotate around Z_2 and use : For rotation without slippage, use either (61) to (65) to determine ρ_2', or (67) to (70), to determine ρ_2'' from ρ_2. Similarly, for rotation without slippage use (87) and (88) to determine ρ_2''. For example, to determine ρ_2'' for rotation without slippage,

$$\rho''_{2.AB}(w_A, w_B) = \rho''_{2.AB}(w_A, 0)\frac{c}{\sqrt{c^2 + w_B^2\rho''^2_{2.AB}(w_A, 0)}} \quad (110)$$

$$\rho''_{1.AB}(w_A, 0) = \rho''_{1.AB}(0,0)\frac{c}{\sqrt{c^2 + w_A^2\rho''^2_{1.AB}(0,0)}} \quad (111)$$

And the procedure is: Start from $\rho''_{1.AB}(0,0)$ use (111) to find $\rho''_{1.AB}(w_A, 0)$. Then use (107), (108), (109) to change coordinates and determine $\rho''_{2.AB}(w_A, 0)$ and then use (110) to finally get $\rho''_{2.AB}(w_A, w_B)$**.** Similarly, for $\rho'_{2.AB}(w_A, w_B)$**,** $z'_{2.AB}(w_A, w_B)$**,** $z''_{2.AB}(w_A, w_B)$
Also, the angle of deflection $\varphi_{2.AB}$, will be increased by $\theta_B' = w_B' t_{AB}$ and $\theta_B'' = w_B t_{AB}$, respectively where $t_{AB} = \frac{\sqrt{\rho'^2_{1.AB}(0,0) + z^2_{1.AB}(0,0)}}{c}$.

6 Summary

The force acting on a body rotating within the **G** field created by another rotating body is in general not central and not symmetric. It depends not only on the **G** field created by the other rotating body but also on its own rotation not only in magnitude but also in direction. As a consequence, depending on the angular velocity and the distance between the bodies, we may vary the direction and magnitude of the force and thus make it attractive or repulsive. The force is calculated for the non rotating observer O' as well as for the far away observer O'' for the cases of rotation with and without slippage and are valid both for macrocosmos and microcosmos. Finally, we use geometry to show how we may calculate the force, when the axes of rotation of the two interacting bodies are not parallel.

7 References

1. Møller C., *The Theory of Relativity*, Oxford, 1952
2. Pechlivanides P. *On Rotating Frames and the Relativistic contraction of the Radius,* http://vixra.org/abs/1411.0229, 2014
3. Pechlivanides P. *The Relativistic Field of a Rotating Body,* http://vixra.org/abs/1703.0213, 2017

Appendix A

Lemmas from Geometria

Lemma 1
Let two planes PL1 and PL2 intersect along a line XX' and the angle of intersection be ϕ. Draw a line from a point A on XX' to any point C. Let the projection of AC on PL1 be AB. and the projection of AC on PL2 be AD. Call angles $\measuredangle CAB = x_1$, $\measuredangle CAD = x_2$. Draw a plane through C vertical to XX'. Let it cross XX' at E. Call angles $\measuredangle EAB = \gamma_1$, $\measuredangle EAD = \gamma_2$, $\measuredangle CEB = \vartheta_1$, $\measuredangle CED = \vartheta_2$. Then

(1)

$$\cos x_1 \cos\gamma_1 = \cos x_2 \cos\gamma_2 \tag{A.112}$$

(2)

$$\tan\vartheta_1 = \frac{\tan x_1}{\sin\gamma_1}, \tan\vartheta_2 = \frac{\tan x_2}{\sin\gamma_2} \tag{A.113}$$

(3)

$$\sin x_2 = \cos x_1 \sin\gamma_1 \frac{\sin\vartheta_2}{\cos\vartheta_1}, \sin x_1 = \cos x_2 \sin\gamma_2 \frac{\sin\vartheta_1}{\cos\vartheta_2} \tag{A.114}$$

(4)

$$\tan\gamma_2 = \tan\gamma_1 \frac{\cos\vartheta_2}{\cos\vartheta_1} \tag{A.115}$$

Proof:

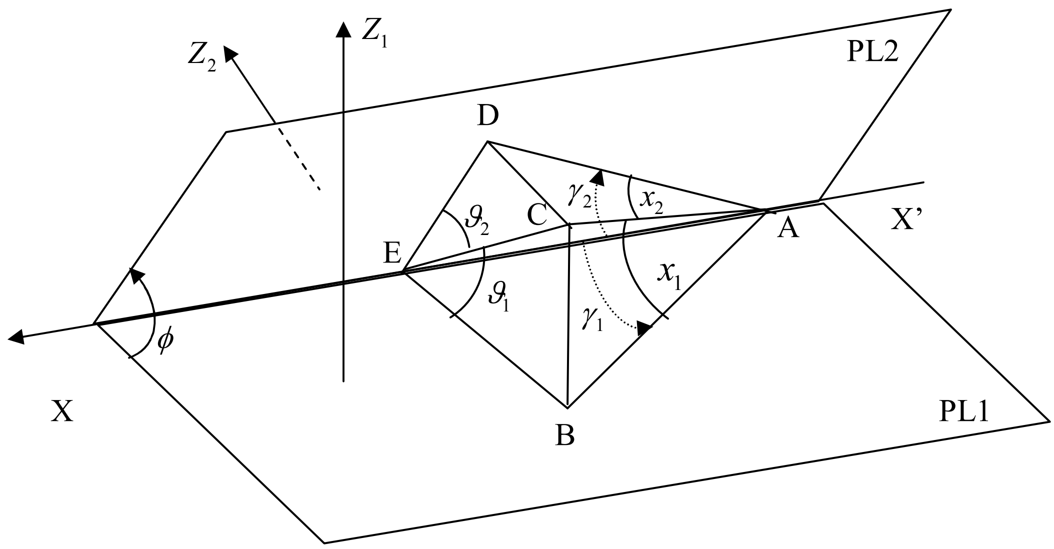


Figure A-1

The plane perpendicular to XX' that passes through C will pass through D, B (because D,B are the projections of C on the two planes respectively). (see Figure A-1)

(1) To prove the first equation observe that triangle CAB is orthogonal at B, triangle ABE is orthogonal at E and also triangle CDA is orthogonal at B, triangle DEA is orthogonal at E. Hence,

$$AE = AC\cos x_1 \cos\gamma_1 = AC\cos x_2 \cos\gamma_2$$

From which the equation to be proved follows

(2) To prove the second equation

$$\tan\vartheta_1 = \frac{CB}{BE} = \frac{AC\sin x_1}{AC\cos x_1 \sin\gamma_1} = \frac{\tan x_1}{\sin\gamma_1}$$

And

$$\tan\vartheta_2 = \frac{CD}{DE} = \frac{AC\sin x_2}{AC\cos x_2 \sin\gamma_2} = \frac{\tan x_2}{\sin\gamma_2}$$

(3) To prove the third equation

$$\sin x_2 = \frac{CD}{AC} = \frac{CE\sin\vartheta_2}{AC} = \frac{BE\sin\vartheta_2}{AC\cos\vartheta_1}$$

But $BE = AC\cos x_1 \sin\gamma_1$, hence,

$$\sin x_2 = \frac{\cos x_1 \sin\gamma_1 \sin\vartheta_2}{\cos\vartheta_1}$$

The second equation of (3) follows by symmetrical arguments

(4) To prove the fourth equation

$$\tan\gamma_2 = \frac{ED}{AE}$$
$$ED = EC\cos\vartheta_2$$
$$AE = AB\cos\gamma_1$$
$$\sin\gamma_1 = \frac{EB}{AB}$$
$$EB = EC\cos\vartheta_1$$

From the above equations use the first four to solve for $\tan\gamma_2$ and find

$$\tan\gamma_2 = \frac{EC}{EB}\cos\vartheta_2\tan\gamma_1$$

And use the fifth to substitute for $\frac{EC}{EB}$ and obtain,

$$\tan\gamma_2 = \frac{\cos\vartheta_2}{\cos\vartheta_1}\tan\gamma_1$$

(QED)

Lemma 1 tells us how to find the projection angles of a line on Plane 2 when we know the projection angles on Plane 1 and the angle between the planes.

Discussion

The angles ϑ_1 and ϑ_2 are related to ϕ. In fact, the formulas in (3) and (4) of Lemma 1 can be further manipulated and expressed in terms of ϕ . To do this we must look at the two planes from X towards X' and make some definitions about orientations.

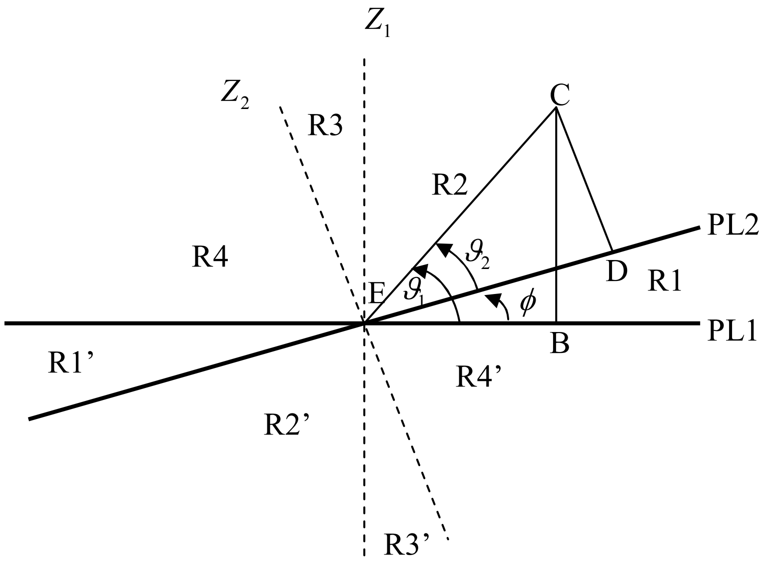


Figure A-2

Given a plane PL1 we draw a vertical pointing in the Z direction and call it Z_1 and the same for PL2 and Z_2. Given a line of intersection of PL1 and PL2 we call it XX' and define a direction towards X. In Figure C-2, XX' is vertical to the paper surface and X is on our side). The angle ϕ of intersection of PL1 and PL2 is zero when Z_1 and Z_2 are parallel both pointing in the same direction and the half planes PL1 and PL2 coincide. Angle ϕ is measured counterclockwise as we look from X towards X' starting from PL1 and ending at PL2.(see Figure A-2).

We draw a plane X_1 through the intersection of PL1 and PL2 (XX') that is vertical to PL1 (X_1 contains Z_1). We also draw a plane X_2 through the intersection of PL1 and PL2 (XX') that is vertical to PL2 (X_2 contains Z_2). This way space is divided in eight regions: R1, R1', R2, R2',R3, R3', R4, R4'(see Figure C-2)

Observe that varying x_1, γ_1 so that $-\frac{\pi}{2} \le x_1 \le \frac{\pi}{2}$ and $0 \le \gamma_1 \le 2\pi$ spans the surface of a sphere. In fact, x_1, γ_1 are the angles used in spherical coordinates. The same is true for x_2, γ_2, where $x_1 = 0$ when AC lies on PL1 and it is positive in the positive side of Z_1; and γ_1 is measured counterclockwise looking down from Z_1, starting from AX towards AB. Similarly, $x_2 = 0$ when AC lies on PL2 and it is positive on the positive side of Z_2, while γ_2 is measured counterclockwise as we look down from Z_2 on PL2 and starting from AX towards AD.

Also, $\vartheta_1 = 0$ when AC lies on PL1 where $-\frac{\pi}{2} \le \vartheta_1 \le \frac{\pi}{2}$ being positive on the positive side of Z_1, and $\vartheta_2 = 0$ when AC lies on PL2 where $-\frac{\pi}{2} \le \vartheta_2 \le \frac{\pi}{2}$ being positive on the positive side of Z_2

We summarize all this in the following table

Region	ϑ_1, x_1	ϑ_2, x_2	γ_1	γ_2	$\vartheta_1 - \vartheta_2$
R1	+	-	$0 \le \gamma_1 \le \pi$	$0 \le \gamma_2 \le \pi$	$\vartheta_1 - \vartheta_2 = \phi$
R1'	-	+	$\pi \le \gamma_1 \le 2\pi$	$\pi \le \gamma_2 \le 2\pi$	$\vartheta_1 - \vartheta_2 = -\phi$
R2	+	+	$0 \le \gamma_1 \le \pi$	$0 \le \gamma_2 \le \pi$	$\vartheta_1 - \vartheta_2 = \phi$
R2'	-	-	$\pi \le \gamma_1 \le 2\pi$	$\pi \le \gamma_2 \le 2\pi$	$\vartheta_1 - \vartheta_2 = -\phi$
R3	+	+	$\pi \le \gamma_1 \le 2\pi$	$0 \le \gamma_2 \le \pi$	$\vartheta_1 + \vartheta_2 = \phi$
R3'	-	-	$0 \le \gamma_1 \le \pi$	$\pi \le \gamma_2 \le 2\pi$	$\vartheta_1 + \vartheta_2 = -\phi$
R4	+	+	$\pi \le \gamma_1 \le 2\pi$	$\pi \le \gamma_2 \le 2\pi$	$\vartheta_1 - \vartheta_2 = -\phi$
R4'	-	-	$0 \le \gamma_1 \le \pi$	$0 \le \gamma_2 \le \pi$	$\vartheta_1 - \vartheta_2 = \phi$

Table A-1 The + sign in columns for (ϑ_1, x_1) and (ϑ_2, x_2) indicates that the quantities are non-negative, while the – sign that they are non-positive.

(a) When C lies in R1 equation (A.114) of Lemma 1 becomes

$-\sin x_2 = \cos x_1 \sin \gamma_1 \dfrac{-\sin \vartheta_2}{\cos \vartheta_1}$ but in R1 $\vartheta_1 - \vartheta_2 = \phi$ and hence,

$$\sin x_2 = \cos x_1 \sin \gamma_1 \frac{\cos\phi \sin\vartheta_1 - \sin\phi\cos\vartheta_1}{\cos\vartheta_1} = \cos x_1 \sin\gamma_1 (\cos\phi \tan\vartheta_1 - \sin\phi)$$

Using (A.113) of Lemma 1 which in R1 becomes $\tan\vartheta_1 = \dfrac{\tan x_1}{\sin\gamma_1}$ we obtain,

$$\sin x_2 = \cos\phi \sin x_1 - \cos x_1 \sin\gamma_1 \sin\phi \qquad \text{(A.116)}$$

And similarly,

$$\sin x_1 = \cos\phi \sin x_2 + \cos x_2 \sin\gamma_2 \sin\phi \qquad \text{(A.117)}$$

Observe here that if we replace ϕ by -ϕ to indicate the reverse transformation then (A.117) becomes symmetric to (A.116). Namely,

$$\sin x_1 = \cos\phi \sin x_2 - \cos x_2 \sin\gamma_2 \sin\phi \qquad \text{(A.118)}$$

Also (A.115) becomes

$$\tan\gamma_2 = (\sin\phi \frac{\tan x_1}{\sin\gamma_1} + \cos\phi)\tan\gamma_1 \qquad \text{(A.119)}$$

And

$$\tan\gamma_1 = (\cos\phi + \sin\phi \frac{-\tan x_2}{\sin\gamma_2})\tan\gamma_2 = (\cos\phi - \sin\phi\frac{\tan x_2}{\sin\gamma_2})\tan\gamma_2 \qquad \text{(A.120)}$$

Again if we replace ϕ by -ϕ to indicate the reverse transformation we end up with a relation symmetrical to (A.119)

If we repeat the calculations for the remaining regions R1’, R2, R2’, R3, R3’, R4, R4’, we find that the same relations (A.116), (A.117), (A.118), (A.119), (A.120) continue to hold in all regions.

Lemma 2
Let two planes PL1 and PL2 intersect along line XX' with an angle $0 \le \phi \le 90°$. Let a line Z_1, perpendicular to PL1, that crosses it at point A and a line Z_2 perpendicular to PL2 that crosses it at point B. Draw a line from B parallel to Z_1 that crosses PL1 at G. Draw the line AG and extend it until it crosses XX' at F. Call the line segments AG= ρ_1 and the line segment BG= z_1. Draw a line from A perpendicular to XX' that crosses it at L. Call the angle $\measuredangle LAG = u$ (it is measured counterclockwise looking from Z_1 down on PL1, starting from AL and ending on AG). Draw a line from A perpendicular to Z_2 that crosses it at I. Call BI= z_2 and AI= ρ_2.
Then
(1)

$$z_2 = -z_1 \cos\phi - \rho_1 \cos u \sin\phi \tag{A.121}$$

(2)

$$\rho_2^2 = \rho_1^2(1-\cos^2 u \sin^2\phi) + z_1^2 \sin^2\phi - 2z_1\rho_1 \sin\phi\cos\phi\cos u \tag{A.122}$$

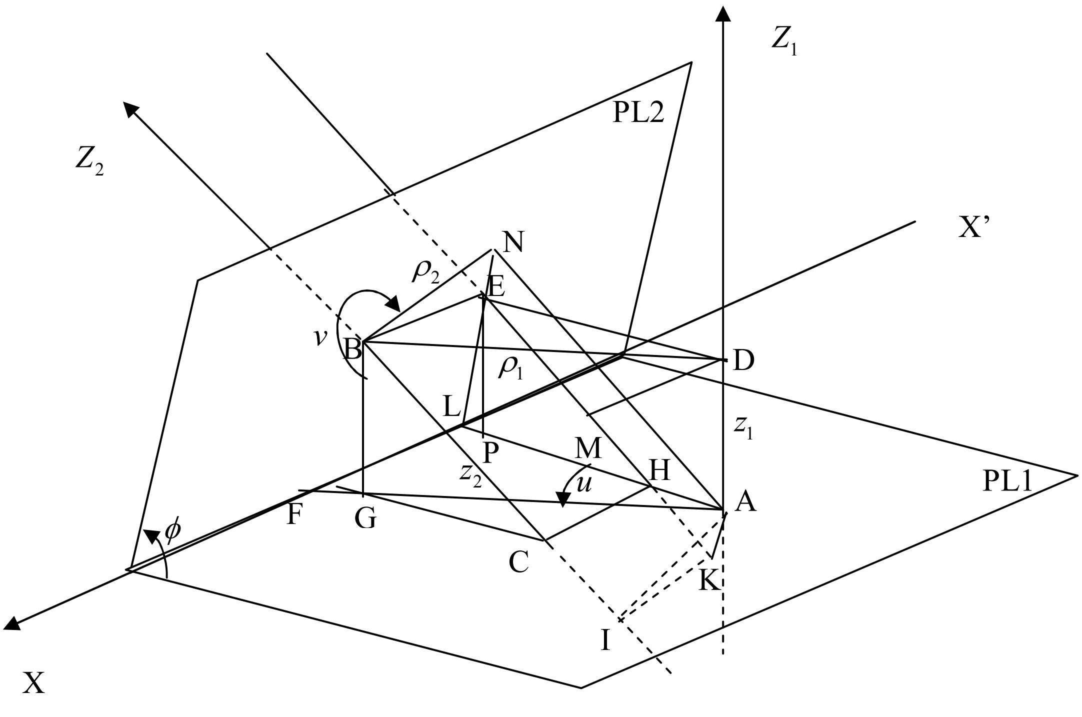


Figure A-3

Proof (using Figure A-3)
Draw a line from B parallel to XX'. Let the plane that passes through Z_1 and is perpendicular to XX' cross the previously drawn line at E. Draw ED parallel to AL. Draw a line from E perpendicular to PL1 that crosses it at P.
The plane defined by BGC is perpendicular to PL1 and PL2 because BC lies on Z_2 which is perpendicular on PL2 and BG was drawn perpendicular to PL1. Therefore,

the plane defined by BGC is perpendicular to XX' and therefore it is parallel to the plane defined by EDLA . In fact EDAP is an orthogonal parallelogram and AD=BG. Draw a line through E parallel to Z_2. It will cross LA at some point H. Draw a line from A perpendicular to the line defined by EH, and let it cross it at K. Also draw a line from D perpendicular to EH and let it cross it at M.
The plane defined by AKI is perpendicular to both Z_2 and its parallel line EMHK. Therefore, EBIK is an orthogonal parallelogram and therefore, BI=EK, or
$z_2 = EM + MK$
Observe that by construction BDAG is orthogonal parallelogram and therefore angle $\measuredangle LAG = \measuredangle BDE = u$.
But $EM = \rho_1 \cos u \sin\phi$ because triangle BED is orthogonal and angle $\measuredangle BDE = u$ and also triangle DEM is orthogonal and angle $\measuredangle DEM = \frac{\pi}{2} - \phi$. To show that angle $\measuredangle DEM = \frac{\pi}{2} - \phi$ observe that angle $\measuredangle EDM = \phi$ because its sides are perpendicular to lines (Z_1, and EMHK) that are perpendicular to the two planes (PL1, PL2) that cross with angle ϕ.
Also MK is the projection of AD (which is equal to z_1) on line EMHK which was drawn parallel to Z_2.The angle between the two line EMHK and Z_1 is ϕ because each is perpendicular to the two plane PL1 and PL2 that cross at angle ϕ. Hence
$MK = z_1 \cos\phi$
Gathering things together we obtain $|z_2| = z_1 \cos\phi + \rho_1 \cos u \sin\phi$. But z_2 lies in the negative semi axis of Z_2 and therefore we may write $z_2 = -z_1 \cos\phi - \rho_1 \cos u \sin\phi$ which proves (A.121)
To prove (A.122) observe that $AB^2 = \rho_1^2 + z_1^2 = \rho_2^2 + z_2^2$ and solve for ρ_2^2 using(A.121). (QED)

Discussion 1
Note that ρ_1, z_1 are the cylindrical coordinates of point B with respect to origin at A and axis Z_1, while ρ_2, z_2 are the cylindrical coordinates of A with respect to origin at B and axis Z_2. The azimuth angle is u for Z_1 and it is measured counterclockwise (as we look down from Z_1) starting from the plane that contains Z_1 and is vertical to PL2. In a similar fashion we define the azimuth angle for the cylindrical coordinate system Z_2 as v measured counterclockwise (as we look down from Z_2) starting from the plane that contains Z_2 and is vertical to PL1, (see Figure A-3).
The plane that passes through Z_2 and is vertical to PL1 includes CG . Let its extension cross XX' at Q and draw QB. Draw LE and extend it to some point N so that triangle BEN is parallel and equal to triangle AKI. Then $\rho_2 = AI = BN$ and $\measuredangle NBE = \frac{3\pi}{2} - v$
We can easily see now that

$BE = GP = \rho_1 \sin u$ and also $BE = \rho_2 \cos(\frac{3\pi}{2} - v) = -\rho_2 \sin v$ and therefore

$$\rho_1 \sin u = -\rho_2 \sin v \qquad \text{(A.123)}$$

Further, we observe that $LE = LN - EN$ where

$LN = \frac{|z_2|}{\tan\varphi}$, $EN = \rho_2 \sin(\frac{3\pi}{2} - v) = -\rho_2 \cos v$, $z_1 = PE \sin\varphi$. Substituting above we obtain,

$z_1 = \frac{|z_2|}{\tan\varphi} \sin\phi - \rho_2 \cos v \sin\phi = |z_2| \cos\phi - \rho_2 \cos v \sin\phi$ and since z_2 is non positive

$$z_1 = -z_2 \cos\phi - \rho_2 \cos v \sin\phi \qquad \text{(A.124)}$$

Discussion 2

In the proof of Lemma 2 we assumed that $0 \le \phi \le 90°$. If we define ϕ to be measured counterclockwise starting when the two half planes PL1, PL2 coincide and we look from X towards X', while the positive half axes Z_1, Z_2 coincide when $\phi = 0$, Then Lemma 2 continues to hold for all $0 \le \phi \le 360°$. This is the same convention that we used for Lemma 1. Finally, Lemma 2 tells us how to change from one cylindrical coordinate system with coordinates (ρ_1, z_1, u) to another with coordinates (ρ_2, z_2, v)