Infinite Product for Gamma Function and Infinite Series for Log Gamma Function

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"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life." - John 6:63.

ABSTRACT. I derive an infinite product for gamma function and infinite series for log gamma function.

1. INTRODUCTION

In this paper, I derive the formulas below:

$$\log\Gamma(z)+z-1=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log\left[\frac{(k+z)^{k+z}}{(1+k)^{1+k}}\right]$$

and

$$e^{z-1}\Gamma(z)=\prod_{n=0}^{\infty}\left\{\prod_{k=0}^{n}\left[\frac{(k+z)^{k+z}}{(1+k)^{1+k}}\right]^{(-1)^k\binom{n}{k}}\right\}^{1/(n+1)},$$

that allowed me to write the identity

$$\sqrt{\frac{\pi}{e}}=\prod_{n=0}^{\infty}\left\{\prod_{k=0}^{n}\left[\frac{\left(\frac{1}{2}+k\right)^{\frac{1}{2}+k}}{(1+k)^{1+k}}\right]^{(-1)^k\binom{n}{k}}\right\}^{1/(n+1)}$$

$$=\left(\frac{1}{\sqrt{2}}\right)^{1/1}\cdot\left(\frac{8}{3\sqrt{3}}\right)^{1/2}\cdot\left(\frac{2^4\cdot5^2\cdot\sqrt{5}}{3^6}\right)^{1/3}\cdot\left(\frac{2^{14}\cdot5^7\cdot\sqrt{5}}{3^{13}\cdot7^3\cdot\sqrt{21}}\right)^{1/4}\cdot\left(\frac{2^{40}\cdot5^{10}}{3^{15}\cdot7^{14}}\right)^{1/5}\cdots$$

2. INFINITE PRODUCT REPRESENTATION FOR GAMMA FUNCTION AND INFINITE SERIES FOR LOG GAMMA FUNCTION.

Theorem 2.1. *For* $0<z<1$*, then*

$$\log\Gamma(z)+z-1=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log\left[\frac{(k+z)^{k+z}}{(1+k)^{1+k}}\right] \tag{2.1}$$

and

$$e^{z-1}\Gamma(z)=\prod_{n=0}^{\infty}\left\{\prod_{k=0}^{n}\left[\frac{(k+z)^{k+z}}{(1+k)^{1+k}}\right]^{(-1)^k\binom{n}{k}}\right\}^{1/(n+1)}, \tag{2.2}$$

where $\log\Gamma(z)$ *denotes the log gamma function, log(z) denotes the natural logarithm function,* e^z *denotes the exponential function and* $\Gamma(z)$ *denotes the gamma function.*

Proof. In [1, page 18], we have

$$\psi(u)=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log(u+k). \tag{2.3}$$

Integrating both sides of (2.3) from 1 at z with respect to u, we find

$$\int_1^z\psi(u)\mathrm{d}u=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\int_1^z\log(u+k)\mathrm{d}u. \tag{2.4}$$

I know [2, page 651, formula 6.461] that

$$\int_1^z \psi(u)\mathrm{d}u = \log\Gamma(z). \tag{2.5}$$

On the other hand, I calculate that

$$\begin{aligned}\int_1^z \log(u+k)\mathrm{d}u &= 1-z-(1+k)\log(1+k)+(k+z)\log(k+z)\\ &= 1-z-\log(1+k)^{1+k}+\log(k+z)^{k+z} = 1-z+\log\left[\frac{(k+z)^{k+z}}{(1+k)^{1+k}}\right].\end{aligned} \tag{2.6}$$

From (2.4) at (2.6), it follows that

$$\begin{aligned}\log\Gamma(z) &= \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left\{1-z+\log\left[\frac{(k+z)^{k+z}}{(1+k)^{1+k}}\right]\right\}\\ &= 1-z+\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log\left[\frac{(k+z)^{k+z}}{(1+k)^{1+k}}\right]\\ \Rightarrow \log\Gamma(z)+z-1 &= \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log\left[\frac{(k+z)^{k+z}}{(1+k)^{1+k}}\right].\end{aligned} \tag{2.7}$$

The exponentiation of (2.7) give me

$$e^{z-1}\Gamma(z) = \prod_{n=0}^{\infty}\left\{\prod_{k=0}^{n}\left[\frac{(k+z)^{k+z}}{(1+k)^{1+k}}\right]^{(-1)^k\binom{n}{k}}\right\}^{1/(n+1)},$$

which are the desired results. □

Example 2.2. Set $z = 1/2$ in the Theorem above and encounter

$$\begin{aligned}\sqrt{\frac{\pi}{e}} &= \prod_{n=0}^{\infty}\left\{\prod_{k=0}^{n}\left[\frac{\left(\frac{1}{2}+k\right)^{\frac{1}{2}+k}}{(1+k)^{1+k}}\right]^{(-1)^k\binom{n}{k}}\right\}^{1/(n+1)}\\ &= \left(\frac{1}{\sqrt{2}}\right)^{1/1}\cdot\left(\frac{8}{3\sqrt{3}}\right)^{1/2}\cdot\left(\frac{2^4\cdot 5^2\cdot\sqrt{5}}{3^6}\right)^{1/3}\cdot\left(\frac{2^{14}\cdot 5^7\cdot\sqrt{5}}{3^{13}\cdot 7^3\cdot\sqrt{21}}\right)^{1/4}\cdot\left(\frac{2^{40}\cdot 5^{10}}{3^{15}\cdot 7^{14}}\right)^{1/5}\cdots\end{aligned}$$

REFERENCES

[1] Guillera, Jesús and Sondow, Jonathan, *Double Integrals and Infinite Products for Some Classical Constants via Analytical Continuations of Lerch's Transcendent*, arXiv:math/0506319v3 [math.NT], 5 Aug 2006.

[2] Gradshteyn, I. S. and Ryzhik, I. M., *Tables of Integrals, Series and Products*, Academic Press, 2000.