

Riemann's Functional Equation is Not a Valid Function
and Its Implication on the Riemann Hypothesis

By

Armando M. Evangelista Jr.

armando781973@gmail.com

On

August 28, 2018

ABSTRACT

Riemann's functional equation was formulated by Riemann himself in order to extend the domain of the *zeta function* from the right half-plane into the entire complex plane except at $s = 1$. It also led him to find a real function, so that, at $s = \frac{1}{2} + \omega i$, the real function has zeros for some values of ω . Now, the real function was also related to the zeta function, which in turn has something to do with the distribution of prime numbers. This drove him to develop a formula of relating the zeros of the zeta function to the number of primes given a certain number. Riemann then conjectured that all the zeros of the zeta function are at $s = \frac{1}{2} + \omega i$, which is now known as the **Riemann Hypothesis**. Hence, Riemann's functional equation is the foundation upon which the Riemann Hypothesis is based. But, there is one problem, the function as shall be shown here, suffers from not being able to yield meaningful or valid values, as it should. Also, if one carefully examines how Riemann arrived at his formula, I for one, found it to be unsatisfactory or unconvincing. It is, therefore, the aim of this present work to show, that, if carefully examined, Riemann's functional equation could *not* be a valid function, and consequently, the Riemann Hypothesis crumbles on its claim.

The Riemann's Functional Equation

The Riemann zeta function (or zeta function) is shown below

$$(1) \quad \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots + \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + \omega i,$$

where s is a complex variable with real part σ and imaginary part ω , and i is the imaginary unit, $i = \sqrt{-1}$. It is known that the infinite series in (1) is undefined if $\sigma \leq 0$, conditionally convergent if $0 < \sigma \leq 1$, and absolutely converges if $\sigma > 1$. $\zeta(s)$ is related to the distribution of prime numbers for one obtains from (2) the infinite product,

$$(2) \quad \zeta(s) = \frac{1}{(1-2^{-s})(1-3^{-s})(1-5^{-s})(1-7^{-s}) \cdots (1-p^{-s})} = \prod_p \frac{1}{1-p^{-s}}.$$

This have driven Riemann to developed a formula of relating the supposedly *non-trivial zeros* of $\zeta(s)$ at $s = \frac{1}{2} + \omega i$, with the number of primes given a certain number. A simple inspection of (2) and one can easily conclude that such zeros are nowhere to be found. Where did they come from?

Analytic Continuation

If a function $f_1(s)$ is analytic in domain D_1 and a second function $f_2(s)$ is analytic in domain D_2 :

- (1) $f_2(s) \neq f_1(s)$ for each s in the *intersection* $D_1 \cap D_2$.
- (2) $f_2(s) = f_1(s)$ for each s in the *intersection* $D_1 \cap D_2$.
- (3) If $f_2(s) = f_1(s)$ for each s in the *intersection* $D_1 \cap D_2$ and if $D_1 \subset D_2$ or D_2 is the *union* $D_1 \cup D_2$, then, $f_2(s)$ is the *analytic continuation* of $f_1(s)$ into the second domain D_2 .

(1) simply point to the fact that the two functions though seemingly equal, are not; while (2) shows that they are only equal at their common points. Analytic continuation is only recommended in (3), since $f_1(s)$'s domain will be extended into domain D_2 .

For example, consider the three functions shown below

$$f_1(s) = \frac{1}{1-s} = -\left(\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \cdots + \frac{1}{s^n}\right) = -\sum_{n=1}^{\infty} \frac{1}{s^n}, \quad |s| > 1,$$

$$f_2(s) = \frac{1}{1-s} = 1 + s + s^2 + s^3 + \cdots + s^n = \sum_{n=0}^{\infty} s^n, \quad |s| < 1,$$

$$\text{and, } f_3(s) = \frac{1}{1-s}, \quad s \neq 1,$$

that have similar closed form, $\frac{1}{1-s}$. But, $f_2(s) \neq f_1(s)$ since they are on different domains while $f_1(s) = f_3(s)$ only at their intersection, that is, when their modulus is greater than one. The function $f_3(s)$ occupies the entire domain except at $s = 1$, hence $f_3(s)$ is the analytic continuation of either $f_1(s)$ or $f_2(s)$.

Table1 Some Values for $f_1(s)$, $f_2(s)$, and $f_3(s)$

| s | $f_1(s)$ | $f_2(s)$ | $f_3(s)$ |
|----|-----------|-----------|-----------|
| -2 | 1/3 | undefined | 1/3 |
| -1 | undefined | undefined | 1/2 |
| 0 | undefined | 1 | 1 |
| 1 | undefined | undefined | undefined |
| 2 | -1 | undefined | -1 |
| 3 | -2 | undefined | -2 |

As one can see from some of the values given on Table1, at their intersection, $f_3(s) = f_1(s)$, as expected.

Now, consider the widely known Riemann's functional equation

$$(3) \quad \zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s),$$

where $\Gamma(1-s)$ and $\zeta(1-s)$ are the reflections of the gamma and the zeta functions, respectively. Formula (3) is considered as the analytic continuation of (1) into the entire complex plane except at $s = 1$. The so-called *trivial zeros* of (3) are at each even integer $s = -2n, n \geq 1$.

Table2 Some Values of $\zeta(s)$ for (1) and (3)

| $\zeta(s)$ | Formula (1) | Formula (3) |
|--------------|---|----------------------|
| $\zeta(-2)$ | undefined | 0 |
| $\zeta(-1)$ | undefined | -1/12 |
| $\zeta(0)$ | undefined | -1/2 |
| $\zeta(1/2)$ | $1+1/\sqrt{2}+1/\sqrt{3}+1/\sqrt{4}+\dots = \infty$ | ≈ -1.4603545 |
| $\zeta(1)$ | $1 + 1/2 + 1/3 + 1/4 + \dots = \infty$ | undefined |

| | | |
|------------|------------|--------------|
| $\zeta(2)$ | $\pi^2/6$ | I'm not sure |
| $\zeta(4)$ | $\pi^4/90$ | I'm not sure |

As can be seen from Table2, the two formulas do not yield the same values at their intersection, that is, $(3) \neq (1)$ at $\sigma > 1$. Thus, (3) is not the analytic continuation of (1) into the entire complex plane.

Simple Check for Analytic Continuation

1. Let a function $f_1(s)$ be analytic in domain D_1 , check if $f_1(s) \neq \infty$ for values of s that are not in D_1 . If there is a function $f_2(s)$ that is analytic in domain D_2 equal to $f_1(s)$ at D_1 and $D_1 \subset D_2$, then, $f_2(s)$ is the analytical continuation of $f_1(s)$.
2. If $f_1(s) = \infty$ for all values of s that are not in D_1 , then, it is completely defined by its domain D_1 .
3. If $f_1(s)$ is an integral, perform 1 and 2 *after* the integral has been evaluated, because analytic continuation can not be perform if the integral doesn't exist!

For example, consider the function below

$$f_1(s) = \frac{1}{s-1}, \quad \sigma > 1,$$

for values of $s < 1$, $f_1(s) \neq \infty$, hence, one can analytically continue $f_1(s)$ into $f_2(s)$, if there is one and maybe valid for all s except at $s = 1$. That is,

$$f_2(s) = \frac{1}{s-1}, \quad s \neq 1.$$

Consider the integral

$$f_1(s) = \int_0^{\infty} e^{-(s-1)t} dt.$$

The integral exist if $\sigma > 1$ and its value being $1/(s-1)$ such that

$$f_1(s) = \frac{1}{s-1}, \quad \sigma > 1.$$

The function

$$f_2(s) = \frac{1}{s-1}, \quad s \neq 1.$$

is the analytic continuation of $f_1(s)$ into the domain that is analytic for all s except the origin.

But if one looks at the zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots + \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

one sees that $\zeta(s) = \infty$ if $\sigma \leq 0$, hence, a valid function equal to $\zeta(s)$ at $\sigma \geq 0$ may not exist. Therefore, $\zeta(s)$ could not be extended on the left half-plane. The zeta function is completely defined by (1) on the right half-plane.

Now, consider the integral shown below

$$(4) \quad I(s) = \int_C (-z)^{s-1} e^{-z} dz, \quad \Re(z) > 0, \text{ and } \sigma > 0,$$

formula (4) have to complex quantities z and s , where z is the complex variable and s is a constant relative to z . The integral will converge if the real parts of z and s are greater than zero. Since s is a constant in (4), one can move $(-1)^{s-1}$ outside the integral sign, that is,

$$I(s) = (-1)^{s-1} \int_C z^{s-1} e^{-z} dz,$$

$(-1)^{s-1}$ is a multivalued complex quantity with principal values $(-1)^{-1}(-1)^s = -e^{\pm(\pi is)}$.

Integrating (4) over the Hankel contour: it starts from $+\infty$ towards the circle with a very small radius ρ then goes around the circle counterclockwise once and goes to back $+\infty$.

$$I(s) = (-1)^{s-1} \left(\int_{\infty}^0 x^{s-1} e^{-x} dx + \int_0^{2\pi} (\rho e^{i\theta})^{s-1} e^{-\rho e^{i\theta}} i \rho e^{i\theta} d\theta + \int_0^{\infty} (x e^{2\pi i})^{s-1} e^{-x e^{2\pi i}} dx \right),$$

the second integral above approaches zero as $\rho \rightarrow 0$

$$I(s) = (-1)^{s-1} \left(-\int_0^{\infty} x^{s-1} e^{-x} dx + \int_0^{\infty} (x e^{2\pi i})^{s-1} e^{-x e^{2\pi i}} dx \right) = (-1)^{s-1} \left(\int_0^{\infty} x^{s-1} e^{-x} dx \right) (e^{2\pi is} - 1),$$

and then choosing the principal value of $(-1)^{s-1} = -e^{i\pi s}$,

$$(5) \quad I(s) = 2i \sin(\pi s) \Gamma(s).$$

The integral in (4) is only valid if $\sigma > 0$, but (5) is now valid for negative values of σ due to analytic continuation and has zeros at $s = n$ for integers $n \geq 1$.

Now, consider the contour integral

$$(6) \quad I_1(s) = \int_C \frac{(-x)^{s-1}}{e^x - 1} dx = (-1)^{s-1} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx, \quad x > 0 \text{ and } \sigma > 0,$$

$$I_1(s) = (-1)^{s-1} \left(\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx + \int_0^{2\pi} \frac{(\rho e^{i\theta})^{s-1}}{e^{\rho e^{i\theta}} - 1} \rho i e^{i\theta} d\theta + \int_0^\infty \frac{(xe^{2\pi i})^{s-1}}{e^{xe^{2\pi i}} - 1} dx \right),$$

$$(-1)^{s-1} \left(\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \right) (e^{2\pi i s} - 1),$$

and since

$$(7) \quad \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \zeta(s) \Gamma(s),$$

thus,

$$(8) \quad I_1(s) = 2i \sin(\pi s) \zeta(s) \Gamma(s).$$

Due to the presence of $\zeta(s)$ on (8), $I_1(s)$ is only valid on the right half-plane. By using the reflection formula

$$\Gamma(s) \Gamma(1-s) = \frac{\pi}{\sin(\pi s)},$$

and formulas (6), (7), and (8), one obtains

$$(9) \quad \zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_0^\infty \frac{(-x)^{s-1}}{e^x - 1} dx.$$

Riemann argued that $\zeta(s)$ defined by (9) is now valid over the entire s -plane except at $s = 1$, by simply rearranging formulas (6), (7), and (8)! One must bear in mind that the integral must first be evaluated in order for analytical continuation to be applied. Thus, (9) is simply an identity, that is, the zeta function is equal to itself.

Riemann also claimed that (9) can be evaluated for negative values of s , but $\int_{-\infty}^{\infty} \frac{(-x)^{s-1}}{e^x - 1} dx$ involves the integral

$$\int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx,$$

which will only be valid for positive values of σ and undefined for negative values of σ and $\sigma = 0$.

PROOF :
$$\int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx = \int_0^{\infty} x^{s-1} \sum_{n=1}^{\infty} e^{-nx} dx = \left(\sum_{n=1}^{\infty} n^{-s} \right) \left(\int_0^{\infty} x^{s-1} e^{-x} dx \right)$$

and the integral $\int_0^{\infty} x^{s-1} e^{-x} dx$ is only defined if $\sigma > 0$, that is,

$$\begin{aligned} \int_0^{\infty} x^{s-1} e^{-x} dx &= \int_0^{\infty} x^{s-1} \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^n}{n!} \right) dx \\ &= \left(\frac{x^s}{s} - \frac{x^{s+1}}{(s+1)1!} + \frac{x^{s+2}}{(s+2)2!} - \frac{x^{s+3}}{(s+3)3!} + \frac{x^{s+4}}{(s+4)4!} - \dots + \frac{(-1)^n x^{s+n}}{(s+n)n!} \right) \Bigg|_0^{\infty}, \end{aligned}$$

since, if one substitute negative values of σ on the lower limit of the last expression above, the integral won't exist! Remember that analytic continuation should be perform *after* the integral has been obtained, not *before*. Thus, the integral in (9) is not valid if $\sigma \leq 0$ and (8) is the valid integral on it.

The Misapplication of the Residue Theorem

Riemann applied the Residue Theorem on (6) by assuming the quantities $2\pi ni$ for $n = \pm 1, \pm 2, \pm 3$, and so on, as the *poles* of (6). The function $q(z) = 1/(e^z - 1)$ in (6) is a periodic function of z ,

$$q(z + 2\pi i) = \frac{1}{e^{(z+2\pi i)} - 1} = \frac{1}{e^z - 1},$$

since the exponential function e^z is a periodic function with period $2\pi i$. Hence, the term $2\pi ni$ are the multiple of the *fundamental period* $2\pi i$. It is, therefore, not valid to treat them as the poles of (6). In fact, the only pole of $q(z)$ is at $z = 0$ with residue 1,

$$q(z) = \frac{1}{e^z - 1} = e^{-z} + e^{-2z} + e^{-3z} \dots + e^{-nz} = \sum_{n=1}^{\infty} e^{-nz}, \quad \Re(z) > 0,$$

so that its contour integral on a simple closed path is $2\pi i$, that is,

$$\oint \frac{dz}{e^z - 1} = 2\pi i.$$

Unfortunately, one can not use $z = 0$ on (6) since it will be undefined. Thus, Riemann's determination lead him to apply the Residue Theorem on (6), that is,

$$\int_C \frac{(-z)^{s-1}}{e^z - 1} dz = 2\pi i \left(\sum_{n=1}^{\infty} (-2\pi ni)^{s-1} + (2\pi ni)^{s-1} \right) = 2\pi i \sin\left(\frac{\pi s}{2}\right) 2^s \pi^{s-1} \zeta(1-s),$$

and by equating the last expression to formula (8),

$$(9) \quad 2i \sin(\pi s) \zeta(s) \Gamma(s) = 2\pi i \sin\left(\frac{\pi s}{2}\right) 2^s \pi^{s-1} \zeta(1-s),$$

one obtains the Riemann's functional equation (3),

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s).$$

And by using equations,

$$\Gamma(s) \Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad \text{and} \quad \sqrt{\pi} \Gamma(s) = 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right),$$

one arrives at the equality,

$$(10) \quad \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\left(\frac{1-s}{2}\right)} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

The equality in (10), if it is valid, will mean that the function to the left, say $\Phi(s)$, will be equal to its reflection $\Phi(1-s)$; which means that at $s = \frac{1}{2} + \omega i$, $\Phi(\frac{1}{2} + \omega i)$ is equal to $\Phi(\frac{1}{2} - \omega i)$, and in accordance with the Reflection Principle,

$$\overline{\Phi(s)} = \Phi(\bar{s}),$$

$\Phi(1/2 + \omega i)$ will be a real function. And so, if $\Phi(1/2 + \omega i)$ is real and has zeros for some values of ω , then $\zeta(1/2 + \omega i)$ has also zeros from (10). Since, it is known, that $\Gamma\left(\frac{s}{2}\right)$ has no zeros for all s , Riemann conjectured that all the zeros of $\zeta(s)$ are at $s = 1/2 + \omega i$, which is the famous Riemann Hypothesis.

The function $\Phi(s)$ will have no zeros if it ain't real, because it will be just like $\zeta(s)$ that has no zeros since its modulus is always greater than zero on the right half-plane. The zeros that many are looking for, are most likely the zeros of of the real part of $\Phi(1/2 + \omega i)$, that is,

$$\Re\{\Phi(1/2 + \omega i)\} = 0, \quad \text{for some values of } \omega.$$

Thus, the Riemann Hypothesis relies upon the validity of $\int_c \frac{(-z)^{s-1}}{e^z-1} dz$ being equal to

$$2\pi i \sin\left(\frac{\pi s}{2}\right) 2^s \pi^{s-1} \zeta(1-s), \quad \text{in order for (10) to be true so that } \Phi(1/2 + \omega i) \text{ will be a real}$$

function to have zeros which has the implication of $\zeta(1/2 + \omega i)$ having zeros. I think, I've shown that's not the case, and therefore, Riemann's functional equation could not be a valid function, and the Riemann hypothesis is false.

Conclusions

- (a) The Riemann zeta function is completely defined by (1) on the right half-plane.
- (b) The analytic continuation of an integral function must be performed *after* the integral has been evaluated and *not before*.
- (c) The function $2i \sin(\pi s) \zeta(s) \Gamma(s)$ is a valid contour integral of $\int_C \frac{(-x)^{s-1}}{e^x - 1} dx$ while $\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$ is *not*.
- (d) From (c), the functional equation $\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\left(\frac{1-s}{2}\right)} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$ is not true.
- (e) From (d), $\Phi(1/2 + \omega i)$ is not a real function and has *no* zeros just like the zeta function on the right half-plane.
- (f) From (e), the Riemann hypothesis is, therefore, false.

REFERENCES:

- Riemann, Bernhard (1859). *On the Number of Prime Numbers less than a Given Quantity*.
- Brown, James Ward; Churchill, Ruel V. (1996). *Complex Variables and Applications* (6th ed.). Singapore: McGraw Hill International Editions.

LINKS:

- https://en.wikipedia.org/wiki/Riemann_zeta_function
- https://en.wikipedia.org/wiki/Riemann_hypothesis